Contents

CHAPTER 1

The compact-open topology

In this first chapter we will discuss a useful topology on sets of transformations, the so-called *compact-open* topology. This topology which will play a crucial role in the remaining part of these notes.

We assume that the reader understands the material of basic courses in algebra, general topology, calculus and linear algebra. Concepts such as group, field, homomorphism, isomorphism, kernel, range, topological space, metric space, Banach space, Hilbert space, compact space, product space, Tychonoff product topology, continuous function, etc., should be familiar. For all undefined notions see e.g., HEWITT and ROSS [?], SCHAEFER [?], DUGUNDJI [?], ENGELKING [?], NAGATA [?] and VAN MILL [?].

1.1. Spaces and groups

Our primary interest is in *separable metrizable spaces.* This is why we do not bother usually to state our theorems in their most general forms for this would only distract the reader from the essential parts of our considerations. We therefore call a separable metrizable space simply a *space* from now on. Some of the topological spaces that we consider are not separable and metrizable however. For example, if X is not compact then the Banach space $C^*(X)$ is not separable. We find it convenient to ignore this inconsistency.

(A) Spaces. The separable metrizable spaces can be characterized in general topological terms, as follows. A topological space X is separable and metrizable if and only if X is a regular T_1-space with a countable base. This is called *Urysohn's Metrization Theorem.* If X is separable and metrizable then X is regular and T_1 for obvious reasons and has a countable base. Conversely, if X is regular and T_1 and has a countable base then it is paracompact and hence normal, being Lindelöf, so by Urysohn's Lemma for normal spaces there are enough real-valued continuous functions to separate closed sets. It is then easy to show that X can be embedded in the *Hilbert cube* $Q = \prod_{n=1}^{\infty}[-1, 1]_n$

and hence is separable and metrizable. (See ENGELKING [**?**, Theorem 4.2.9] for more details.)

A space X is called *homogeneous* if for all $x, y \in X$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$. So, a space X is homogeneous if from the topological standpoint all points in X behave in the same way. The 1-sphere is homogeneous, and so is Q (this is Keller's Theorem, see VAN MILL [**?**, 1.6.6] for details).

A space X is called *topologically complete* or *Polish* if it admits an admissible complete metric. Observe that a closed subspace of a topologically complete space is topologically complete as well. It can be shown that every G_δ-subset of a Polish space is Polish. So $\mathbb{P}$ is Polish. It can also be shown that every Polish subspace A of a space X is a G_δ-subset of X.

A space X is a *Baire space* if the intersection of countably many dense open subsets of X is again dense. Or, equivalently, if the union of any countable family of nowhere dense subsets of X has empty interior. It is well-known and very often used fact that a Polish space is Baire. This is called the *Baire Category Theorem*. So $\mathbb{Q}$ is not Polish.

Exercise 1.1.1. *Suppose that S and T are dense Polish subspaces of some space X. Prove that $S \cap T \neq \emptyset$.*

Exercise 1.1.2. *Give an example of a Baire space that is not Polish.*

For any space X we let ϱ denote an admissible metric on X. For notational conventions, see §**??**.

(B) Topological groups. A group G which is also a topological space, is called a *topological group* if the function $G \times G \to G$ defined by

$$(x, y) \mapsto x \cdot y^{-1}$$

is continuous. This easily implies that a *translation* on G, i.e., a function from G into G of the form $x \mapsto x \cdot a$ or of the form $x \mapsto a \cdot x$ for some fixed $a \in G$, is a homeomorphism. So every topological group G is homogeneous: if $u, v \in G$ then the translation $x \mapsto (vu^{-1})x$ maps u onto v. Observe that if a translation has a fixed-point then it must be the identity. This implies that not all homogeneous spaces have the structure of a topological group. Simply observe that Q is homogeneous, but has the fixed-point property (this follows from Brouwer's Theorem, see VAN MILL [**?**, 2.4.6]).

We say that the topology on G and the group structure on G are *compatible*. Observe that if G is a group then its group structure is always compatible with the discrete topology on G. Let $G = \mathbb{Q}$ with its standard group structure $\cdot$, and topology τ generated by the collection

$$\{[q, r) \cap \mathbb{Q} : q, r \in \mathbb{Q}, q < r\}.$$

Then $(\mathbb{Q}, \tau)$ is a separable metrizable space (it is in fact homeomorphic to $\mathbb{Q}$ with its standard topology), but τ and $\cdot$ are clearly incompatible.

If G is a group, and $A, B \subseteq G$, then

$$AB = \{ab : a \in A, b \in B\}, \quad A^{-1} = \{a^{-1} : a \in A\}.$$

We call A *symmetric* if $A = A^{-1}$. Observe that $A \cap A^{-1}$ is symmetric.

Observe that the neutral element e of a topological group G has a neighborhood base $(U_n)_n$ consisting of symmetric open neighborhoods of G such that $U_{n+1}^2 \subseteq U_n$ for every n.

Topological groups G and H are called *isomorphic* if there is an algebraic isomorphism $\pi\colon G \to H$ that is also a homeomorphism.

If G_n is a topological group for every n, then the product topology makes the product group $G = \prod_{n=1}^{\infty} G_n$ a topological group.

(C) Zero-dimensional spaces. A space X is *zero-dimensional* if it has a base of clopen sets. It is easy to see that a space X is zero-dimensional if and only if every open cover of X can be refined by a clopen partition. The subspaces $\mathbb{R}$ that are zero-dimensional are easily characterized: a subspace $X \subseteq \mathbb{R}$ is zero-dimensional if and only if $\mathbb{R} \setminus X$ is dense. So the rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{P}$, and the Cantor set C are zero-dimensional. These spaces have simple and useful topological characterizations.

Exercise 1.1.3. *Let X be a space. Then*

(1) $X \approx \mathsf{C}$ *if and only if X is compact, zero-dimensional, nonempty, and has no isolated points.*
(2) $X \approx \mathbb{P}$ *if and only if X is Polish, nonempty, zero-dimensional and nowhere compact.*
(3) $X \approx \mathbb{Q}$ *if and only if X is countable, nonempty, and has no isolated points.*

Observe that $\mathbb{Q}$ is subgroup of $\mathbb{R}$, hence is a topological group. Exercise 1.1.3(1) shows that C is homeomorphic to $\{0,1\}^\infty$, which is a topological group, the operation being coordinatewise addition modulo 2. Also $\mathbb{P}$ is a topological group. Simply observe that Exercise 1.1.3(2) shows that $\mathbb{P}$ is homeomorphic to the topological group $\mathbb{Z}^\infty$.

In §**??** we will present topological characterizations of $\mathbb{Q} \times \mathsf{C}$, $\mathbb{Q} \times \mathbb{P}$ and $\mathbb{Q}^\infty$ that are in the same spirit.

A zero-dimensional space X is called *strongly homogeneous* provided that all of its nonempty clopen subsets are homeomorphic.

Exercise 1.1.4. *Prove that a strongly homogeneous zero-dimensional space is homogeneous.*

Observe that Exercise 1.1.3 shows that $\mathbb{Q}$, $\mathbb{P}$ and C are strongly homogeneous. The space $\mathbb{Z} \times \mathsf{C}$ is a zero-dimensional topological group which is not strongly homogeneous (it contains a clopen copy of the compact space C which is not homeomorphic to the entire space). VAN ENGELEN [?] proved the highly nontrivial result that among the homogeneous zero-dimensional absolute Borel sets, $\mathbb{Z} \times \mathsf{C}$ is the only space that is not strongly homogeneous. In §?? we will present an example of a zero-dimensional subgroup of $\mathbb{R}$ that is not strongly homogeneous. By van Engelen's result, such a subgroup cannot be Borel.

1.2. The compact-open topology

In this section we will define and study a useful topology on sets of transformations, the so-called *compact-open* topology, which will play a crucial role in the remaining part of this book.

(A) The compact-open topology. If X and Y are spaces then $C(X,Y)$ denotes the collection of all continuous functions from X to Y. If $Y = \mathbb{R}$ then $C(X)$ abbreviates $C(X, \mathbb{R})$.

It is convenient to endow $C(X,Y)$ with a useful topology. If $K \subseteq X$ and $U \subseteq Y$, then

$$[K,U] = \{f \in C(X,Y) : f[K] \subseteq U\}.$$

The collection

$$\{[K,U] : K \subseteq X \text{ compact}, U \subseteq Y \text{ open}\}$$

serves as a subbase for a topology on $C(X,Y)$. This topology is the so-called *compact-open topology* on $C(X,Y)$ and plays a central role in this book. From now on we will always endow $C(X,Y)$ with the compact-open topology.

The topology on $C(X,Y)$ can be described more economically, as follows.

Exercise 1.2.1. *Let X and Y be spaces, and let $\mathcal{S}$ be a subbase for Y.*

(1) *The collection*

$$\{[K,S] : K \subseteq X \text{ compact}, S \in \mathcal{S}\}$$

is a subbase for $C(X,Y)$.

(2) *Let $\mathcal{F}$ be a collection of compact sets in X with the following property: For every compact A in X with neighborhood U in X there is a finite $\mathcal{G} \subseteq \mathcal{F}$ with $A \subseteq \bigcup \mathcal{G} \subseteq U$. Then the collection*

$$\{[F,S] : F \in \mathcal{F}, S \in \mathcal{S}\}$$

is a subbase for $C(X,Y)$.

Lemma 1.2.2. *Let X and Y be spaces. If A is closed in X and B is closed in Y then $[A,B]$ is closed in $C(X,Y)$.*

Proof. Assume that $f \notin [A, B]$. Then for some $a \in A$, $f(a) \notin B$. Hence

$$C(X, Y) \setminus [A, B] = \bigcup_{a \in A} [\{a\}, Y \setminus B]$$

is open. □

Corollary 1.2.3. *Let X and Y be spaces. Then $C(X, Y)$ is regular, T_1 and has a countable base if X is locally compact.*

Proof. Let f and g be arbitrary distinct elements of $C(X, Y)$. Pick $x \in X$ such that $f(x) \neq g(x)$. Then $[\{x\}, Y \setminus \{f(x)\}]$ is a neighborhood of g which does not contain f. So $C(X, Y)$ is T_1.

To prove that $C(X, Y)$ is regular, let $f \in [K, U]$ with $K \subseteq X$ compact and $U \subseteq Y$ open. There is an open $W \subseteq Y$ such that $f[K] \subseteq W \subseteq \overline{W} \subseteq U$. Then $f \in [K, W]$ and the closure of $[K, W]$ is contained in $[K, \overline{W}] \subseteq [K, U]$ by Lemma 1.2.2. This is clearly as required.

Let $\mathcal{B}$ be a countable base for Y which is closed under finite unions. In addition, let $\mathcal{A}$ be a countable base for X which is closed under finite unions, while moreover $\overline{A}$ is compact for every $A \in \mathcal{A}$ (here we use that X is locally compact). We claim that the collection

$$\{[\overline{A}, B] : A \in \mathcal{A}, B \in \mathcal{B}\}$$

is a (countable) subbase for $C(X, Y)$. To prove this, let $K \subseteq X$ be compact, and $U \subseteq Y$ be open. For an arbitrary $f \in [K, U]$, first pick $B \in \mathcal{B}$ such that $f[K] \subseteq B \subseteq U$. It is clear that this is possible since $f[K]$ is compact and $\mathcal{B}$ is closed under finite unions. Next, pick $A \in \mathcal{A}$ such that $K \subseteq A \subseteq \overline{A} \subseteq f^{-1}[B]$. Then $f \in [\overline{A}, B] \subseteq [K, U]$. □

So by Urysohn's Metrization Theorem (Page 1), it follows that $C(X, Y)$ is separable and metrizable provided that X is locally compact. The local compactness assumption is essential, as the following result shows.

Proposition 1.2.4. *Let X be a space. The following statements are equivalent:*

(1) *$C(X, \mathbb{R})$ has a countable base,*
(2) *$C(X, \mathbb{R})$ is first countable,*
(3) *X is locally compact.*

Proof. For (3) ⇒ (1), we can use Corollary 1.2.3. Since (1) ⇒ (2) is trivial, it suffices to prove (2) ⇒ (3). Striving for a contradiction, assume that X is not locally compact at x. Let $f \colon X \to \mathbb{R}$ be the constant function with value 0, and let $\{[K_n, U_n] : n \in \mathbb{N}\}$ be a local subbase at f in $C(X, \mathbb{R})$. Observe that

$$0 \in \bigcap_{n=1}^{\infty} U_n. \tag{1}$$

If $n \in \mathbb{N}$ then the compact set $\bigcup_{i=1}^{n} K_i$ is not a neighborhood of x, so we may pick

$$x_n \in B(x, 1/n) \setminus \bigcup_{i=1}^{n} K_i. \tag{2}$$

Put $K = \{x\} \cup \{x_n : n \in \mathbb{N}\}$. Then K is compact, and $[K, (-1, 1)]$ is a neighborhood of f in $C(X, \mathbb{R})$. There consequently is an $N \in \mathbb{N}$ such that

$$f \in \bigcap_{i=1}^{N} [K_i, U_i] \subseteq [K, (-1, 1)]. \tag{3}$$

We claim that $K \subseteq \bigcup_{i=1}^{N} K_i$. If this is not true, then we may pick $p \in K \setminus \bigcup_{i=1}^{N} K_i$. Let $g\colon X \to \mathbb{R}$ be a continuous function having the properties $g\big[\bigcup_{i=1}^{N} K_i\big] = 0$ and $g(p) = 1$. Then $g \in \bigcap_{i=1}^{N} [K_i, U_i]$ by (1), hence $g(p) \in (-1, 1)$ by (3). This is a contradiction. Hence $K \subseteq \bigcup_{i=1}^{N} K_i$, but this contradicts (2). □

From now on, our main interest will be in the spaces $C(X, Y)$ for locally compact X.

Exercise 1.2.5. *Let X and Y be spaces. For every $y \in Y$ let $f_y \colon X \to Y$ be the constant function with value y. Then $\mathcal{F} = \{f_y : y \in Y\}$ is a closed subset of $C(X, Y)$, and the function $y \mapsto f_y$ is a homeomorphism $Y \to \mathcal{F}$.*

Corollary 1.2.6. *If X is a space then $C(\{pt\}, X) \approx X$.*

(B) Continuity of composition. Let X, Y and Z be spaces. If $f \in C(X, Y)$ and $g \in C(Y, Z)$ then $g \circ f \in C(X, Z)$. It is natural to investigate whether the function

$$C(X, Y) \times C(Y, Z) \to C(X, Z); \qquad (f, g) \overset{T}{\mapsto} g \circ f$$

is continuous.

Lemma 1.2.7. *Let X, Y and Z be spaces with Y locally compact. Then $T\colon C(X, Y) \times C(Y, Z) \to C(X, Z)$ is continuous.*

Proof. Let $T(f, g) = g \circ f \in [K, U]$, where $K \subseteq X$ is compact, $U \subseteq Z$ is open, $g \in C(Y, Z)$ and $f \in C(X, Y)$. Since $g^{-1}[U]$ is open, $f[K]$ is compact and $f[K] \subseteq g^{-1}[U]$, there is an open neighborhood V of $f[K]$ in Y such that $f[K] \subseteq V \subseteq \overline{V} \subseteq g^{-1}[U]$ and $\overline{V}$ is compact (here we use that Y is locally compact). Then clearly,

$$T\big[[K, V] \times [\overline{V}, U]\big] \subseteq [K, U],$$

which is as required. □

The local compactness of Y is essential in Lemma 1.2.7, see Example 1.2.9 below.

(C) The evaluation function. If X and Y are spaces, then the function $\omega\colon C(X,Y)\times X \to Y$ defined by $\omega(f,x) = f(x)$ is called the *evaluation function* of $C(X,Y)$.

Lemma 1.2.8. *Let X and Y be spaces with X locally compact. Then $\omega\colon C(X,Y)\times X \to Y$ is continuous.*

Proof. Consider the composition function

$$T\colon C(\{\mathrm{pt}\},X)\times C(X,Y) \to C(\{\mathrm{pt}\},Y).$$

Then T is continuous by Lemma 1.2.7. If we identify $C(\{\mathrm{pt}\},X)$ with X and $C(\{\mathrm{pt}\},Y)$ with Y it follows that T corresponds to the function $(x,f)\mapsto f(x)$. Hence ω is continuous, being the composition of the change of coordinates

$$C(X,Y)\times X \to X\times C(X,Y)$$

and T. □

The local compactness of X is essential in this result as well as in Lemma 1.2.8.

Example 1.2.9. There are spaces X and Y such that $\omega\colon C(X,Y)\times X \to Y$ is not continuous.

Proof. Let $X = \mathbb{Q}$ and $Y = \mathbb{I}$. Let $f \in C(\mathbb{Q},\mathbb{I})$ be the constant function with value 0. We claim that ω is not continuous at $(f,0)$. Let $U = \mathbb{I}\setminus\{1\}$. Then $(f,0)\in \omega^{-1}[U]$. We will prove that $\omega^{-1}[U]$ is not a neighborhood of $(f,0)$. To prove this, take an arbitrary basic neighborhood of $(f,0)$. It has the form

$$\bigcap_{i=1}^{n}[K_i,U_i]\times V,$$

where $K_1,\ldots,K_n\subseteq \mathbb{Q}$ are compact, $U_1,\ldots,U_n\subseteq\mathbb{I}$ are open, and $V\subseteq\mathbb{Q}$ is open. Since $\mathbb{Q}$ is nowhere locally compact, there is an element

$$q\in V\setminus\bigcup_{i=1}^{n}K_i.$$

Let $g\colon \mathbb{Q}\to\mathbb{I}$ be a Urysohn function such that $g(q)=1$ and $g[\bigcup_{i=1}^n K_i]=0$. Since $0\in\bigcap_{i=1}^n U_i$, it follows that $g\in\bigcap_{i=1}^n[K_i,U_i]$, and $q\in V$. But $\omega(g,q) = g(q) = 1$ and hence $(g,q)\notin\omega^{-1}[U]$. □

(D) Products. We study function spaces of products.

Let X, Y and Z be spaces. If $\alpha\colon X \times Y \to Z$ is continuous, then define $\hat{\alpha}\colon X \to C(Y,Z)$ by

$$\big(\hat{\alpha}(x)\big)(y) = \alpha(x,y) \qquad (x \in X, y \in Y).$$

If $x \in X$ is fixed, then the function $Y \to Z$ defined by $y \mapsto \alpha(x,y)$ is evidently continuous. Hence $\hat{\alpha}$ is well-defined.

Conversely, if $\hat{\alpha}\colon X \to C(Y,Z)$ is continuous, then we define $\alpha\colon X \times Y \to Z$ by

$$\alpha(x,y) = \big(\hat{\alpha}(x)\big)(y) \qquad (x \in X, y \in Y).$$

It is not clear whether α is continuous.

Theorem 1.2.10. *Let X, Y and Z be spaces.*

(1) *If $\alpha\colon X \times Y \to Z$ is continuous then $\hat{\alpha}\colon X \to C(Y,Z)$ is continuous.*
(2) *If $\hat{\alpha}\colon X \to C(Y,Z)$ is continuous, and if Y is locally compact, then $\alpha\colon X \times Y \to Z$ is continuous.*

Proof. For (1), let $\hat{\alpha}(x) \in [K,U]$, where $K \subseteq Y$ is compact and $U \subseteq Z$ is open. Then for every $y \in K$, $\big(\hat{\alpha}(x)\big)(y) \in U$, i.e., $\alpha(x,y) \in U$. This implies that $\{x\} \times K$ is contained in the open set $\alpha^{-1}[U]$. By compactness of K there is an neighborhood V of x in X such that $V \times K \subseteq \alpha^{-1}[U]$. Then clearly, $V \subseteq \hat{\alpha}^{-1}\big[[K,U]\big]$, which completes the proof.

For (2), consider the composition of functions

$$X \times Y \xrightarrow{\hat{\alpha}\times 1_Y} C(Y,Z) \times Y \xrightarrow{\omega} Z.$$

Here ω is the evaluation function. The first function is evidently continuous, so also is the second by Lemma 1.2.8 (here we use that Y is locally compact). The combined map is therefore continuous, and an easy calculation yields that it is α. □

Now let X, Y and Z be spaces. For every $\alpha \in C(X \times Y, Z)$ we have by Theorem 1.2.10 that $\hat{\alpha} \in C\big(X, C(Y,Z)\big)$. It is a natural question whether this assignment is continuous.

Theorem 1.2.11. *Let X, Y and Z be spaces with Y locally compact. The map $\alpha \mapsto \hat{\alpha}$ is a homeomorphism from $C(X \times Y, Z)$ onto $C\big(X, C(Y,Z)\big)$.*

Proof. By Theorem 1.2.10, $\alpha \mapsto \hat{\alpha}$ is bijective. A subbasis for $C\big(X, C(Y,Z)\big)$ consists of all sets of the form $\big[A, [B,W]\big]$, where $A \subseteq X$, $B \subseteq Y$ are compact and $W \subseteq Z$ is open (Exercise 1.2.1(1)).

Claim 1. The collection

$$\{[A \times B, V] : A \subseteq X, B \subseteq Y \text{ compact}, V \subseteq Z \text{ open}\}$$

is a subbase for $C(X \times Y, Z)$.

Proof. If $K \subseteq X \times Y$ is compact and U is an open neighborhood of K then there is a finite collection $\mathcal{F}$ of subsets of $X \times Y$ of the form $A \times B$, where $A \subseteq X$ and $B \subseteq Y$ are compact, such that $K \subseteq \bigcup \mathcal{F} \subseteq U$. So we are done by Exercise 1.2.1(2). ♢

Now observe that if $A \subseteq X$, $B \subseteq Y$ are compact, and $W \subseteq Z$ is open, then

$$\alpha \in [A \times B, W] \iff \hat{\alpha} \in \big[A, [B, W]\big],$$

which proves the theorem. □

Observe that the homeomorphism $D(\alpha) = \hat{\alpha}$ in Theorem 1.2.11 is explicitly given by the formula

$$\big((D(\alpha))(x)\big)(y) = \alpha(x, y) \qquad (x \in X, y \in Y).$$

Corollary 1.2.12. *Let X, Y and Z be spaces, with both X and Y locally compact. Then the function $E\colon C\big(X, C(Y, Z)\big) \to C\big(Y, C(X, Z)\big)$ defined by*

$$\big((E(f))(y)\big)(x) = \big(f(x)\big)(y) \qquad (y \in Y, x \in X)$$

is a homeomorphism.

Proof. Simply observe that by Theorem 1.2.11 applied twice,

$$C\big(X, C(Y, Z)\big) \approx C(X \times Y, Z) \approx C(Y \times X, Z) \approx C\big(Y, C(X, Z)\big).$$

The simple calculation that the composition of these homeomorphisms is given explicitly by the function E is left to the reader. □

(E) A metric for $C(X, Y)$. It is convenient to have an explicit formula for a metric that generates the topology of $C(X, Y)$. In view of Proposition 1.2.4, we can only hope for such a formula if X is locally compact.

So let X be locally compact. We can write X as $\bigcup_{i=1}^{\infty} U_i$, where the U_i are open, $\overline{U}_i \subseteq U_{i+1}$ and $\overline{U}_i$ is compact for every i. Now let Y be any space, and let ϱ be an admissible metric on Y. For $f, g \in C(X, Y)$, and $n \in \mathbb{N}$, define

$$(*) \qquad d_n(f, g) = \min\left[\frac{1}{n}, \sup\left\{\varrho\big(f(x), g(x)\big) : x \in \overline{U}_n\right\}\right].$$

Finally, put

$$(**) \qquad d(f, g) = \sup_n d_n(f, g).$$

Observe that the definition of d depends on the choice of the metric ϱ and the sequence $(U_n)_n$. That this is irrelevant, is shown in the next result.

Exercise 1.2.13. *Prove that d is a metric on $C(X, Y)$.*

Proposition 1.2.14. *d generates the topology of $C(X, Y)$.*

Proof. Let $K \subseteq X$ be compact, and $U \subseteq Y$ be open. For $f \in [K, U]$ we will find $\varepsilon > 0$ such that if $d(f, g) < \varepsilon$ then $g \in [K, U]$. If such an $\varepsilon > 0$ does not exist, then for every $n \in \mathbb{N}$ we can find $g_n \in C(X, Y)$ and $x_n \in K$ such that $d(f, g_n) < 1/n$ and $g_n(x_n) \notin U$. Since K is compact, we may assume without loss of generality that there exists $x \in K$ such that $x_n \to x$. Pick $N \in \mathbb{N}$ such that $K \subseteq U_N$. For $n \geq N$ we have $d_n(f, g_n) \leq d(f, g_n) < 1/n$, hence

$$\sup\left\{\varrho\big(f(y), g_n(y)\big) : y \in \overline{U}_n\right\} < 1/n.$$

But then

$$\varrho\big(f(x_n), g_n(x_n)\big) < 1/n$$

for every $n \geq N$. Since $f(x_n) \to f(x)$, this implies that $g_n(x_n) \to f(x)$. Hence $f(x) \notin U$ since $g_n(x_n) \notin U$ for every n, and $Y \setminus U$ is closed. This is a contradiction.

Conversely, take arbitrary $f \in C(X, Y)$ and $\varepsilon > 0$. Pick $m \in \mathbb{N}$ such that $1/m < \varepsilon$. Since $\overline{U}_m$ is compact, there is a cover $\mathcal{F}$ of $\overline{U}_m$ by closed subsets of $\overline{U}_m$ such that for every $F \in \mathcal{F}$ we have $\operatorname{diam} f[F] < 1/2m$. Let V_F for every $F \in \mathcal{F}$ be an open neighborhood of $f[F]$ such that $\operatorname{diam} V_F < 1/m$. Put

$$\mathcal{A} = \bigcap_{F \in \mathcal{F}} [F, V_F].$$

Then $\mathcal{A}$ is an open neighborhood of f in $C(X, Y)$. Pick an arbitrary $g \in \mathcal{A}$. We claim that $d(f, g) < \varepsilon$. To see that this is true, observe that

$$g[F] \cup f[F] \subseteq V_F$$

for every $F \in \mathcal{F}$, hence $\varrho\big(f(x), g(x)\big) < 1/m$ for every $x \in \overline{U}_m$. This gives us by compactness of the $\overline{U}_i$'s that

$$d_i(f, g) = \min\left[\frac{1}{i}, \sup\left\{\varrho\big(f(x), g(x)\big) : x \in \overline{U}_i\right\}\right] < 1/m.$$

for every $i \leq m$ (here we use that $\overline{U}_i \subseteq U_m$ for every $i < m$). Observe that if $n > m$ then

$$d_n(f, g) \leq 1/n < 1/m.$$

From this we conclude that

$$d(f, g) = \sup_n d_n(f, g) \leq 1/m < \varepsilon,$$

as required. $\square$

The formulas $(*)$ and $(**)$ can be simplified considerably if X is compact. For if X is compact, then $X = U_n$ for certain n, and so for $m \geq n$ we have

$$d_m(f, g) = \min\left[\frac{1}{m}, \sup\left\{\varrho\big(f(x), g(x)\big) : x \in X\right\}\right].$$

By the same reasoning as in the proof of Proposition 1.2.4 we get that the formula

$$\hat{\varrho}(f, g) = \sup\left\{\varrho\big(f(x), g(x)\big) : x \in X\right\}$$

defines an admissible metric on $C(X,Y)$. If X is compact, this metric will be used on $C(X,Y)$ from now on. The topology on $C(X,Y)$ that is generated by $\hat{\varrho}$ is called the topology of *uniform convergence*.

(F) Limits of continuous functions. It is a natural question to ask when $C(X,Y)$ is Polish. This basic question will be studied now.

Let X and (Y,ϱ) be spaces. For all $f,g \in C(X,Y)$ put

$$\hat{\varrho}(f,g) = \sup\{\varrho\big(f(x),g(x)\big) : x \in X\}.$$

Observe that $\hat{\varrho}(f,g) \in [0,\infty]$. We saw above that $\hat{\varrho}$ metrizes $C(X,Y)$ if X is compact.

If $X = Y = \mathbb{N}$ with the Euclidean metric ϱ, f is the identity and g is the function $n \mapsto 2n$, then clearly $\hat{\varrho}(f,g) = \infty$. As a consequence, $\hat{\varrho}$ need not be a metric on $C(X,Y)$, it is some sort of an 'extended' metric. It is convenient to adopt some of the terminology of metrics for $\hat{\varrho}$. For example, we call a sequence $(f_n)_n$ in $C(X,Y)$ *Cauchy* if for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\hat{\varrho}(f_n,f_m) < \varepsilon$ for all $n,m \geq N$. This is of course nothing but the ordinary definition of a Cauchy sequence in a metric space.

Exercise 1.2.15. *Let X and (Y,ϱ) be spaces. Assume that $X = \bigcup_{n=1}^{\infty} U_n$, where $\overline{U}_n \subseteq U_{n+1}$ and U_n is open for every n. Let $(f_n)_n$ be a sequence in $C(X,Y)$ such that for every $x \in X$, $\lim_{n\to\infty} f_n(x)$ exists. Assume moreover that for every m, the sequence $(f_n \restriction U_m)_n$ is $\hat{\varrho}$-Cauchy. Then the function $f\colon X \to Y$ defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous and the sequence $(f_n)_n$ converges to f in $C(X,Y)$.*

Corollary 1.2.16. *Let X and (Y,ϱ) be spaces with X locally compact. Then the metric d for $C(X,Y)$ defined on Page 10 is complete provided that ϱ is complete. As a consequence, $C(X,Y)$ is Polish if and only if Y is Polish.*

Proof. Assume first that ϱ is complete. Let $(U_n)_n$ be the sequence of open subsets of X that we used for the construction of d. Let $(f_n)_n$ be a d-Cauchy sequence in $C(X,Y)$.

Claim 1. For every $N \in \mathbb{N}$, $(f_n \restriction U_N)_n$ is $\hat{\varrho}$-Cauchy.

Proof. Let $\varepsilon > 0$. We may assume without loss of generality that $\varepsilon < 1/N$. Pick $N_1 \in \mathbb{N}$ such that $d(f_n,f_m) < \varepsilon$ for all $n,m \geq N_1$. Then for an arbitrary $x \in U_N$ we have

$$\min\left[\frac{1}{N}, \varrho\big(f_n(x),f_m(x)\big)\right] \leq d_N(f_n,f_m) \leq d(f_n,f_m) < \varepsilon < \frac{1}{N}$$

for all $n,m \geq N$. As a consequence, $\varrho\big(f_n(x),f_m(x)\big) < \varepsilon$ for all $n,m \geq N_1$ and all $x \in X$. $\square$

Now take an arbitrary $x \in X$, say $x \in U_N$ for certain $N \in \mathbb{N}$. By Claim 1, $\big(f_n(x)\big)_n$ is ϱ-Cauchy, hence $\lim_{n\to\infty} f_n(x)$ exists by the completeness of ϱ.

Let $f(x)$ denote that limit. Then f is continuous, hence belongs to $C(X, Y)$, and $f_n \to f$ in $C(X, Y)$ by Exercise 1.2.15. Hence $C(X, Y)$ is Polish.

Conversely, if $C(X, Y)$ is Polish then so is Y since $C(X, Y)$ contains a closed homeomorph of Y (Exercise 1.2.5). □

By a similar reasoning, one gets the following result.

Corollary 1.2.17. *Let X and (Y, ϱ) be spaces with X compact. Then $\hat{\varrho}$ is complete if ϱ is complete.*

(G) Function spaces. Function spaces yield interesting examples of topological groups. Let G be a topological group, and let X be a space. Define a binary operation $\star$ on $C(X, G)$ by

$$(f \star g)(t) = f(t) \cdot g(t) \quad (t \in X).$$

We say that $\star$ is the *standard group operation* on $C(X, G)$.

Theorem 1.2.18. *Let X be a space and G a topological group. Then the standard operation on $C(X, G)$ is a group operation making $C(X, G)$ a topological group.*

Proof. That $\star$ is a group operation on $C(X, G)$ is easily proved and is left as an exercise to the reader. We claim that the function $\varphi\colon C(X, G) \times C(X, G) \to C(X, G)$ defined by

$$\varphi(f, g) = f \star g^{-1}$$

is continuous. To prove this, let $U \subseteq C(X, G)$ be open, and take $(f, g) \in \varphi^{-1}[U]$. Then $f \star g^{-1} \in U$, hence there are for some n compact sets $K_1, \ldots, K_n$ in X and open sets $V_1, \ldots, V_n$ in G such that

$$f \star g^{-1} \in \bigcap_{i=1}^{n} [K_i, V_i] \subseteq U.$$

Fix $i \leq n$ for a moment. For every $t \in K_i$ we have $f(t) \cdot g(t)^{-1} \in V_i$. Since G is a topological group, there are open neighborhoods A_t and B_t of $f(t)$ and $g(t)$, respectively, such that $A_t \cdot B_t^{-1} \subseteq V_i$. Now, since both f and g are continuous, there is a closed neighborhood W_t of t in X such that for all $s \in W_t$ we have $f(s) \in A_t$ and $g(s) \in B_t$. By compactness of K_i, there is a finite subset F of K_i such that the collection $\{W_t : t \in F\}$ covers K_i. For every $t \in F$ put $K_t = W_t \cap K_i$. So

$$f \in \mathcal{F}_i = \bigcap_{t \in F} [K_t, A_t], \quad g \in \mathcal{G}_i = \bigcap_{t \in F} [K_t, B_t]$$

and $\mathcal{F}_i$ and $\mathcal{G}_i$ are neighborhoods of f and g, respectively. If $f' \in \mathcal{F}_i$ and $g' \in \mathcal{G}_i$ and $t' \in K_i$ then pick $t \in F$ such that $t' \in K_t$ and conclude that

$$(f' \star (g')^{-1})(t') = f'(t') \cdot g'(t')^{-1} \in A_t \cdot B_t^{-1} \subseteq V_i.$$

This proves that $\mathcal{F}_i \times \mathcal{G}_i$ is a neighborhood of (f, g) which is mapped by φ onto a subset of $[K_i, V_i]$.

So $\mathcal{F} = \bigcap_{i=1}^n \mathcal{F}_i$ and $\mathcal{G} = \bigcap_{i=1}^n \mathcal{G}_i$ are neighborhoods of f and g, respectively, such that

$$\varphi[\mathcal{F} \times \mathcal{G}] \subseteq \bigcap_{i=1}^n [K_i, V_i] \subseteq U,$$

as required. □

So by applying Theorem 1.2.18 and Corollary 1.2.16 we get:

Corollary 1.2.19. *If X is locally compact and G is a Polish group then $C(X, G)$ endowed with the standard group operation is a Polish group.*

This result will have interesting applications §**??**. Corollary 1.2.19 will be used there to prove that certain global extension properties of a Polish group can be localized.

Let L be a (real) linear space. If X is a space then $C(X, L)$ is a topological group by Theorem 1.2.18. But even more is true. If $f \in C(X, L)$ and $\lambda \in \mathbb{R}$ then λf denotes the function $(\lambda f)(x) = \lambda f(x)$. So this multiplication by scalars and the standard group operation on $C(X, L)$ show it is a vector space. These operations on $C(X, L)$ are also called *standard*. As to be expected, the standard operations transform $C(X, L)$ into a linear space.

Theorem 1.2.20. *Let X be a space and L a linear space. Then $C(X, L)$ with its standard vector space structure is a linear space.*

The proof of this result is entirely similar to the proof of Theorem 1.2.18 and is left as an exercise to the reader.

1.3. Banach spaces

We shall present some important examples of function spaces that are Banach spaces.

(A) The spaces $C^*(X)$. For a nonempty compact space X we let $C(X)$ abbreviate $C(X, \mathbb{R})$. Obviously, $C(X)$ is a vector space; addition of functions and scalar multiplication are defined pointwise. If $f \in C(X)$ then define its norm, $\|f\|$, by

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

(Observe that by compactness of X this supremum is attained. That is: there is an element $x \in X$ such that $\|f\| = |f(x)|$.) It is easily seen that $\|\cdot\| \colon C(X) \to [0, \infty)$ is indeed a norm; it is called the *sup-norm* on $C(X)$. Consequently, the function

$$\hat{\varrho}(f, g) = \|f - g\| \qquad (*)$$

is a metric on $C(X)$. It metrizes the compact-open topology on $C(X)$, as was shown on Page 11, and is complete by Corollary 1.2.17. We consequently conclude that $C(X)$ is a Banach space.

If X is not compact then $C(X)$ contains unbounded functions, and so the formula

$$\|f\| = \sup\{|f(x)| : x \in X\}$$

does not define a norm on $C(X)$. By considering the subset $C^*(X)$ of $C(X)$ consisting of all *bounded* functions, this problem does not occur; $C^*(X)$ endowed with the sup-norm is a Banach space for the same reasons $C(Y)$ is for compact Y.

The topology defined here on $C^*(X)$ is called the *topology of uniform convergence.*

We say that a normed space L is *isometrically isomorphic* to a normed space E if there is there is a linear isomorphism $\alpha\colon L \to E$ such that

$$\|\alpha(x) - \alpha(y)\| = \|x - y\|$$

for all $x, y \in L$. We will prove in Corollary 1.3.5 below that every Banach space is isometrically isomorphic to a closed linear subspace of $C(\mathbb{I})$. Hence $C(\mathbb{I})$ is a 'universal' space for Banach spaces. In order to prove this, we need to present some basic material on Banach spaces

(B) Functionals. Let B be a (real) Banach space. The symbol I_B denotes the identity operator on B. If B and E are Banach spaces then $\mathcal{L}(B, E)$ is the Banach space of all bounded linear transformations of $T\colon B \to E$, with norm

$$\|T\| = \sup\{\|Tx\| : x \in B, \|x\| \leq 1\}.$$

Observe that $\mathcal{L}(B, E)$ is a subset of the product space E^B, but that its Banach space topology is different from the subspace topology it inherits from E^B endowed with the Tychonoff product topology. The *dual* B^* of B is the Banach space $\mathcal{L}(B, \mathbb{R})$. If $x \in B \setminus \{\underline{0}\}$ then there is an element $f \in B^*$ such that $f(x) = \|x\|$ and $\|f\| = 1$. This is a well-known consequence of the Hahn-Banach Theorem (see e.g., Brown and Page [**?**, Corollary 5.3.5]).

Let $T \in \mathcal{L}(B, B)$. For every $f \in B^*$ we have that the composition

$$B \xrightarrow{T} B \xrightarrow{f} \mathbb{R}$$

clearly belongs to B^*. Let $T^*\colon B^* \to B^*$ be defined by $T^* f = f \circ T$. Hence

$$T^* f(x) = f(Tx)$$

for every $x \in B$. We say that T^* is the *adjoint* of T. Observe that the adjoint of the identity on B is the identity on B^*. In formula: $I_B^* = I_{B^*}$.

The following lemma is left as an exercise to the reader.

Lemma 1.3.1. *Let B be a Banach space. If $T, S \in \mathcal{L}(B, B)$ then*

(1) $T^* \in \mathcal{L}(B^*, B^*)$,
(2) $\|T^*\| = \|T\|$,
(3) $(S \circ T)^* = S^* \circ T^*$,
(4) *if* $S \neq T$ *then* $S^* \neq T^*$.

Let B be a Banach space. For every $x \in X$, define $x^* \colon B^* \to B^*$ by

$$x^*(f) = f(x).$$

It is evident that x^* is well-defined. Notice that if $x, y \in B$ are distinct then so are x^* and y^*. Simly let $f \in X^*$ be such that $f(x) \neq f(y)$. Then $x^*(f) \neq y^*(f)$, hence $x^* \neq y^*$.

Lemma 1.3.2. *Let B be a Banach space. Then*

(1) *For every $x \in B$, the function x^* is a bounded functional on B^*.*
(2) *If $D \subseteq \{x \in B : \|x\| \leq 1\}$ is dense then the collection $\{d^* : d \in D\}$ separates the points of B^*.*

Proof. That x^* is linear is trivial. That it is bounded, follows from the following trivial observation. Take an arbitrary $f \in X^*$ such that $\|f\| \leq 1$. Then

$$|x^*(f)| = |f(x)| \leq \|f\| \cdot \|x\| \leq \|x\|.$$

For (2), let $f, g \in X^*$ be distinct. Then for some $x \in B$, $f(x) \neq g(x)$. Since both f and g are linear, we have $x \neq \underline{0}$. But then for $y = {}^{x}\!/_{\|x\|}$ we also have $f(y) \neq g(y)$. Since $\|y\| = 1$ and D is dense in the closed unit ball of B, there is an element $d \in D$ such that $f(d) \neq g(d)$. But then $d^*(f) \neq d^*(g)$. □

(C) The weak* topology on B^*. Let B be a Banach space. The *weak* topology* on B^* is the topology determined by the continuous functionals

$$\{x^* : x \in B\}$$

on B^*. Since these functionals separate points in B^* (Lemma 1.3.2), the weak* topology is well-defined. Observe that B^* with the weak* topology is nothing but B^* with the topology it inherits from the product $\mathbb{R}^B$ with the Tychonoff product topology. Hence B^* with the weak* topology is a locally convex linear space.

Theorem 1.3.3. *Let B be a Banach space and let $\Sigma^* = \{f \in X^* : \|f\| \leq 1\}$ be the unit ball of X^* with the topology induced by the weak* topology. Then Σ^* is a compact, metrizable and convex subspace of $\mathbb{R}^B$.*

Proof. The convexity of Σ^* is trivial. In addition, its compactness is easily demonstrated. If $f \in \Sigma^*$ then

$$|f_x| = |f(x)| \leq \|f\| \cdot \|x\| \leq \|x\|.$$

From this we conclude that Σ^* is contained in the compact subspace

$$P = \prod_{x\in B} [-\|x\|, \|x\|] \tag{1}$$

of $\mathbb{R}^B$. So for the compactness of Σ^*, all we need to prove is that it is closed in $\mathbb{R}^B$. To this end, let $\xi \in \mathbb{R}^B \setminus \Sigma^*$.

Case 1. ξ is not linear

Then there are $x, y \in B$ and $\lambda, \mu \in \mathbb{R}$ such that

$$\xi(\lambda x + \mu y) \neq \lambda\xi(x) + \mu\xi(y),$$

i.e.,

$$\xi_{\lambda x+\mu y} \neq \lambda\xi_x + \mu\xi_y.$$

There are open neighborhoods U, V and W of $\xi_{\lambda x+\mu y}, \xi_x$ and ξ_y in $\mathbb{R}$ such that if $u \in U, v \in V$ and $w \in W$ are arbitrary, then

$$u \neq \lambda v + \mu w.$$

But then

$$\{\eta \in \mathbb{R}^B : \eta_{\lambda x+\mu y} \in U, \eta_x \in V, \eta_y \in W\}$$

is an open neighborhood of ξ in $\mathbb{R}^B$ which misses B^*.

So we may assume without loss of generality that ξ is linear.

Case 2. There is an $x \in X$ such that $|\xi_x| > \|x\|$.

Then by (1) it follows that the set $\{\eta \in \mathbb{R}^B : |\eta_x| > \|x\|\}$ is an open neighborhood of ξ in $\mathbb{R}^B$ which misses Σ^*.

So we may assume without loss of generality that $|\xi_x| \leq \|x\|$ for all $x \in B$. But this implies that $\xi\colon B \to \mathbb{R}$ is continuous and has norm at most 1 , i.e., $\xi \in \Sigma^*$. This is a contradiction.

All there remains to prove is that Σ^* is metrizable. For this, let $D \subseteq \{x \in B : \|x\| \leq 1\}$ be a countable dense set. Consider the projection

$$\pi\colon \mathbb{R}^B \to \mathbb{R}^D.$$

By Lemma 1.3.2(2), $\pi \restriction \Sigma^*$ is one-to-one. Hence the already established compactness of Σ^* gives us that $\pi \restriction \Sigma^*$ is an embedding. □

Corollary 1.3.4. *Let B be a Banach space and let*

$$\Sigma^* = \{f \in X^* : \|f\| \leq 1\}$$

be the unit ball of X^* with the topology induced by the weak topology. If $D = \{d_n : n \in \mathbb{N}\}$ is dense in $\{x \in B : \|x\| \leq 1\}$ then the formula*

$$\varrho(f, g) = \sum_{n=1}^{\infty} 2^{-n}|f(d_n) - g(d_n)|$$

defines an admissible metric for Σ^*.

Proof. In the proof of Theorem 1.3.3 we proved that the projection $\pi\colon \mathbb{R}^B \to \mathbb{R}^D$ is one-to-one on Σ^*. So π embeds Σ^* in the compact space

$$\prod_{n=1}^{\infty} [-\|d_n\|, \|d_n\|].$$

Since $\|d_n\| \le 1$ for every n, we are done (Page **??**). □

Theorem 1.3.3 has an interesting corollary.

Corollary 1.3.5. *Let B be a Banach space. Then B is isometrically isomorphic to a closed linear subspace of $C(\mathbb{I})$.*

Proof. Consider the unit ball Σ^* in B^* endowed with the weak* topology. Then Σ^* is by Lemma 1.3.2 a compact convex subspace of the countable infinite product of lines. There consequently is a continuous surjection $\alpha\colon \mathbb{I} \to \Sigma^*$ (van Mill [**?**, Corollary 4.2.32]). For $x \in B$ define $\varphi_x \in C(\mathbb{I})$ by

$$\varphi_x(t) = \alpha(t)(x) \quad (t \in \mathbb{I}).$$

We claim that the function $x \mapsto \varphi_x$ is a linear isometry from B onto a closed linear subspace of $C(\mathbb{I})$.

To prove that φ_x is linear, simply observe that for all $x, y \in B$, $\lambda, \mu \in \mathbb{R}$ and $t \in \mathbb{I}$ we have

$$\begin{aligned}\varphi_{\lambda x+\mu y}(t) &= \alpha(t)(\lambda x + \mu y)\\ &= \lambda\alpha(t)(x) + \mu\alpha(t)(y)\\ &= \lambda\varphi_x(t) + \mu\varphi_y(t).\end{aligned}$$

That $x \mapsto \varphi_x$ is an isometry follows equally easily. Indeed, since $x \mapsto \varphi_x$ is linear, it suffices to prove that it preserves the norm of every vector. Now if $x \in B$ and $t \in \mathbb{I}$ then

$$|\varphi_x(t)| = |\alpha(t)(x)| \le \|\alpha(t)\| \cdot \|x\| \le \|x\|.$$

This shows that $\|\varphi_x\| \le \|x\|$. In addition, if $x \in B$ then there is an element $f \in \Sigma^*$ such that $f(x) = \|x\|$. Pick $t \in \mathbb{I}$ with $\alpha(t) = f$. We then have

$$\varphi_x(t)| = |\alpha(t)(x)| = |f(x)| = \|x\|,$$

which proves that $\|x\| \le \|\varphi_x\|$.

To prove that the range of $x \mapsto \varphi_x$ is closed is left as an exercise to the reader. □

So we conclude that $C(\mathbb{I})$ is a ‘universal’ space for all Banach spaces in the same way as Q is ‘universal’ for spaces (see Page 2). We will prove in §**??** that there is a ‘universal’ topological group, i.e., a topological group containing an isomorphic copy of any topological group.

1.4. Groups of homeomorphisms

In this section we will continue to present interesting examples of function spaces. In fact, we will discuss the central topic in this book here for the first time: groups of homeomorphisms.

Let X and Y be spaces and $\mathcal{H}(X, Y)$ denote the subspace

$$\{h \in C(X, Y) : h \text{ is a homeomorphism}\}.$$

of $C(X, Y)$. If $X = Y$ then for $\mathcal{H}(X, X)$ we shall simply write $\mathcal{H}(X)$. The space $\mathcal{H}(X)$ has a standard group structure since if $f, g \in \mathcal{H}(X)$ then so is $f \circ g^{-1}$. As usual, $\mathcal{H}(X)$ is called the *(auto)homeomorphism group* of X.

It will be convenient to introduce the following notation. If X is a space and $A \subseteq X$ then $\mathcal{H}(X|A)$ denotes the subgroup

$$\{h \in \mathcal{H}(X) : h \restriction A = 1_A\}$$

of $\mathcal{H}(X)$.

A topology τ on $\mathcal{H}(X)$ is called *admissible* if it makes $\mathcal{H}(X)$ a topological group (with its standard group structure). We will investigate whether the compact-open topology on $\mathcal{H}(X)$ is admissible.

It is a classic theorem of Arens [**?**] that if a space X is noncompact, locally compact, and locally connected, then $\mathcal{H}(X)$ is a topological group because the compact-open topology coincides with the topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$, where αX is the Alexandroff one-point compactification of X. We improve on this result in Proposition 1.4.1. Recall that a *continuum* is a compact connected space.

Proposition 1.4.1. *Let X be a locally compact space.*

1. *The function $\mathcal{H}(X) \times \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $(f, g) \mapsto f \circ g$ is continuous.*
2. *If X is compact then the function $f \mapsto f^{-1}$ is continuous on $\mathcal{H}(X)$.*
3. *If every point in X has a neighborhood that is a continuum then the compact-open topology on $\mathcal{H}(X)$ coincides with the group topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$.*

So $\mathcal{H}(X)$ with its standard group operation is a topological group if X is locally connected (and locally compact) or X is compact.

Proof. For (1), apply Lemma 1.2.7.

For (2), simply observe that for $K, U \subseteq X$ with K compact and U open, we have

$$f \in [K, U] \iff f^{-1} \in [X \setminus U, X \setminus K],$$

which is as required.

We now prove (3). The topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$ is generated by the subbasis

$$\{[K, O] : K \text{ compact and } O \text{ open in } X\} \cup \{[F, X\backslash K] \colon F \text{ closed}, K \text{ compact in } X\}.$$

Let F be closed in X, and let O be the complement of a compact subset K of X. We show that $[F, O]$ is open in the compact-open topology. Let f be an arbitrary element of $[F, O]$, and consider the compactum $f^{-1}(K)$. Select for each x in $f^{-1}(K)$ a neighbourhood C_x of x in X that is a continuum. By compactness we can find a finite subset A of $f^{-1}(K)$ such that

$$f^{-1}(K) \subseteq \bigcup_{a \in A} \operatorname{int} C_a.$$

Define the compactum $C = \bigcup_{a \in A} C_a$ and construct in the same way a compact neighbourhood C' of C. Consider the obviously open subset

$$U = [C' \cap F, O] \cap [\operatorname{Fr} C', f(X \setminus C)] \cap \bigcap_{a \in A} [\{a\}, f(\operatorname{int} C_a)]$$

of $\mathcal{H}(X)$ in the compact-open topology. It is clear that f belongs to U, and now we need verify only that U is a subset of $[F, O]$.

Assume to the contrary that h lies in $U \setminus [F, O]$. Thus there is an x in F with $h(x)$ in $K = X \setminus O$. Since $h \in U \subset [C' \cap F, O]$, we see that x cannot lie in C'. Note that

$$x \in h^{-1}(K) \subseteq h^{-1}(f(C)),$$

hence x lies in $h^{-1}(f(C_a))$ for some a in A. Since

$$h \in U \subseteq [\{a\}, f(\operatorname{int} C_a)]$$

we see that a lies in $h^{-1}(f(C_a))$. Observe that $h^{-1}(f(C_a))$ is a continuum that connects the point a inside C' with x outside C', hence there is a y in $h^{-1}(f(C_a)) \cap \operatorname{Fr} C'$. Since $U \subseteq [\operatorname{Fr} C', f(X \setminus C)]$, we infer that $h(y) \notin f(C)$, which contradicts the fact that $y \in h^{-1}(f(C_a))$. The proof is complete. $\square$

Interestingly, $\mathcal{H}(X)$ need not be a topological group if X is locally compact, see Example 1.4.2 for an example demonstrating this. As far as we know, a satisfactory characterization of the locally compact spaces X for which the compact-open topology on $\mathcal{H}(X)$ is admissible, is not known.

Example 1.4.2. If $X = \mathbb{N} \times \mathsf{C}$ then the compact-open topology on $\mathcal{H}(X)$ is not admissible.

Proof. Pick an arbitrary element $p \in X$ with compact clopen neighborhood U. Put

$$B = \{h \in \mathcal{H}(X) : h^{-1} \in [\{p\}, U]\}.$$

We claim that B is not open in $\mathcal{H}(X)$ which clearly suffices. Striving for a contradiction, assume that B is open. Since $1_X \in B$, there are for some

n compact subsets $K_1, \dots, K_n$ in X and open subsets $V_1, \dots, V_n$ in X such that

$$1_X \in \bigcap_{i=1}^{n} [K_i, V_i] \subseteq B.$$

Observe that $K_i \subseteq V_i$ so we may pick a compact and clopen C_i such that $K_i \subseteq C_i \subseteq V_i$ for every i. Then

$$1_X \in \bigcap_{i=1}^{n} [C_i, C_i] \subseteq \bigcap_{i=1}^{n} [K_i, V_i] \subseteq B.$$

Let E be a compact clopen neighborhood of p such that $E \subseteq C_i$ if $p \in C_i$ and $E \cap C_i = \emptyset$ if $p \notin C_i$. Let $p' \in E \setminus \{p\}$ and partition $E \setminus \{p'\}$ into the disjoint nonempty clopen sets $E_1, E_2, \dots$. We may assume without loss of generality that $p \in E_1$. Now let $\{E_0, E_{-1}, E_{-2}, \dots\}$ be a discrete collection nonempty clopen compact sets in X such that

$$\Big(U \cup \bigcup_{i=1}^{n} C_i\Big) \cap \bigcup_{i=0}^{\infty} E_{-i} = \emptyset.$$

For every $i \in \mathbb{Z}$ let

$$f_i \colon E_i \to E_{i+1}$$

be an arbitrary homeomorphism. Define $h \colon X \to X$ as follows:

$$h(z) = \begin{cases} f_i(z) & (z \in E_i), \\ z & (\text{otherwise}). \end{cases}$$

It is easy to see that h is a homeomorphism. In addition, $h^{-1}(p) \in E_0$, so $h^{-1}(p) \notin U$, i.e., $h \notin B$. Fix $i \le n$. If $p \notin C_i$ then h restricts to the identity on C_i, hence $h \in [C_i, C_i]$. If $p \in C_i$ then $h \restriction C_i \setminus E$ is the identity and $h[E] \subseteq E \subseteq C_i$. From this we conclude that

$$h \in \bigcap_{i=1}^{n} [C_i, C_i] \subseteq B,$$

which is a contradiction. □

An inspection of the proof of Example 1.4.2(1) shows that it works since $\mathbb{N} \times \mathsf{C}$ is zero-dimensional and has no isolated points. For $\mathbb{N}$ with the discrete topology the proof breaks down. In fact, the compact-open topology on $\mathcal{H}(\mathbb{N})$ *is* admissible since $\mathbb{N}$ is locally connected (Proposition 1.4.1(2)). In fact, since compact sets in $\mathbb{N}$ are finite, the compact-open topology on $\mathcal{H}(\mathbb{N})$ coincides with the topology of pointwise convergence. So $\mathcal{H}(\mathbb{N})$ with the compact-open topology is the subspace

$$\mathbb{S}_\infty = \{\pi \in \mathbb{N}^{\mathbb{N}} : \pi \text{ is a permutation}\}.$$

of the Baire space $\mathbb{N}^{\mathbb{N}}$ (see Exercise 1.5.2 below).

It is a little disappointing that the compact-open topology on $\mathcal{H}(X)$ only works well if X is compact or if X is locally compact and locally connected. For that reason we will almost exclusively deal with the groups $\mathcal{H}(X)$ for spaces X that are compact or locally compact and locally connected. It is however possible to endow $\mathcal{H}(X)$ for locally compact X with an interesting admissible topology. Indeed, think of X as subspace of its one-point compactification $\alpha X = X \cup \{\infty\}$. Since each homeomorphism $f\colon X \to X$ can be extended to a homeomorphism $\alpha f\colon \alpha X \to \alpha X$, we can identify $\mathcal{H}(X)$ and the closed subgroup

$$\{h \in \mathcal{H}(\alpha X) : h(\infty) = \infty\}$$

of $\mathcal{H}(\alpha X)$. The subspace topology $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$ is clearly admissible (and natural). When we endow $\mathcal{H}(X)$ with this topology, we will denote it by $\mathcal{H}_\alpha(X)$.

(A) Completeness properties of $\mathcal{H}(X)$. It is convenient to have an admissible metric for $\mathcal{H}(X)$ which is defined in terms of properties of X. We will construct such metrics. As we will see, there are even natural *complete* admissible metrics for $\mathcal{H}(X)$.

An admissible metric ϱ of a topological group G is called *left invariant* provided that for all $a, x, y \in G$,

$$\varrho(ax, ay) = \varrho(x, y).$$

Define *right invariant* metric similarly. An admissible metric ϱ on G is called *invariant* if it is both left- and right invariant. See §**??** for more information on these concepts.

Observe that if ϱ is an admissible left invariant metric on G then

$$\sigma(x, y) = \varrho(x^{-1}, y^{-1}) \quad (x, y \in G)$$

is an admissible right invariant metric on G and vice versa. This is so since the function $x \mapsto x^{-1}$ is a homeomorphism of G.

Exercise 1.4.3. *Let X be a compact space with admissible metric ϱ. Then $\hat{\varrho}$ is a right invariant admissible metric for $\mathcal{H}(X)$. In addition, the function σ defined by*

$$\sigma(f, g) = \hat{\varrho}(f, g) + \hat{\varrho}(f^{-1}, g^{-1})$$

is a complete admissible metric for $\mathcal{H}(X)$ (but not necessarily left- or right invariant).

Remark 1.4.4. In Example 1.4.7 below we will show that $\hat{\varrho}$ need not be left invariant, and that σ is in general neither left- nor right invariant. In addition, in Exercise 1.5.2 we will present an example of a Polish group not admitting an admissible complete left invariant metric. NOG IETS VERDER UITSPITTEN.

Corollary 1.4.5. *If X is compact then $\mathcal{H}(X)$ is Polish and hence a Baire space. If X is locally compact then $\mathcal{H}_\alpha(X)$ is Polish and hence a Baire space.*

Proof. Simply observe that $\mathcal{H}_\alpha(X)$ is a closed subgroup of $\mathcal{H}(\alpha X)$. □

The question naturally arises whether $\mathcal{H}(X)$ is not only Polish, but even compact, or locally compact. We will come back to this in §**??**. For the moment, the following result suffices.

Exercise 1.4.6. *Prove that $\mathcal{H}(\mathsf{C})$ is not locally compact.*

That $\mathcal{H}(X)$ has a right invariant metric, is quite interesting. This raises the question which topological groups have such a metric. We will come back to this in §**??**, where it will be shown that this is true for *all* topological groups.

Example 1.4.7. Let ϱ denote the euclidean metric on $\mathbb{I}$. Then the metric $\hat{\varrho}$ is not left invariant and

$$\sigma(f,g) = \hat{\varrho}(f,g) + \hat{\varrho}(f^{-1},g^{-1})$$

is neither left- nor right invariant.

Proof. Indeed, for every $n \in \mathbb{N}$ define $f_n \in \mathcal{H}(\mathbb{I})$ by the requirements that $f_n(0) = 0$, $f_n(1/2) = 1/n$, $f_n(1) = 1$, and f_n is linear on the intervals $[0, 1/2]$ and $[1/2, 1]$. The sequence $(f_n)_n$ is evidently $\hat{\varrho}$-Cauchy. Observe that from $f_n^{-1}(1/n) = 1/2$ and $f_n^{-1}(1/2n) = 1/4$ we have

$$\hat{\varrho}(f_n^{-1}, f_{2n}^{-1}) \geq |f_n^{-1}(1/2n) - f_{2n}^{-1}(1/2n)| = |1/4 - 1/2| = 1/4. \tag{1}$$

This implies that $\hat{\varrho}$ is not left invariant. Simply observe that by Exercise 1.4.3 we have $\hat{\varrho}(f_n f_n^{-1}, f_n f_{2n}^{-1}) = \hat{\varrho}(1_{\mathbb{I}}, f_n f_{2n}^{-1}) = \hat{\varrho}(f_{2n}, f_n) \to 0$.

We now turn our attention to σ. By (1),

$$\sigma(f_n, f_{2n}) = \hat{\varrho}(f_n, f_{2n}) + \hat{\varrho}(f_n^{-1}, f_{2n}^{-1}) \geq 1/4 \tag{2}$$

and by Exercise 1.4.3,

$$\begin{aligned} \sigma(f_n f_{2n}^{-1}, f_{2n} f_{2n}^{-1}) &= \hat{\varrho}(f_n f_{2n}^{-1}, f_{2n} f_{2n}^{-1}) + \hat{\varrho}(f_{2n} f_n^{-1}, f_{2n} f_{2n}^{-1}) \\ &= \hat{\varrho}(f_n, f_{2n}) + \hat{\varrho}(f_{2n} f_n^{-1}, 1_{\mathbb{I}}) \\ &= \hat{\varrho}(f_n, f_{2n}) + \hat{\varrho}(f_{2n}, f_n) \\ &= 2\hat{\varrho}(f_n, f_{2n}) \\ &\longrightarrow 0 \quad (n \to \infty). \end{aligned} \tag{3}$$

So σ is not right invariant. The fact that σ is not left invariant follows easily as well. First observe that $\sigma(f,g) = \sigma(f^{-1}, g^{-1})$ for all $f, g \in \mathcal{H}(\mathbb{I})$. This shows by (3) that

$$\sigma(f_{2n} f_n^{-1}, f_{2n} f_{2n}^{-1}) \longrightarrow 0 \quad (n \to \infty)$$

while by (2),

$$\sigma(f_n^{-1}, f_{2n}^{-1}) = \sigma(f_n, f_{2n}) \geq 1/4.$$

So we are done. □

(B) The Inductive Convergence Criterion. An important application of the completeness of $\mathcal{H}(X)$ is the so-called 'Inductive Convergence Criterion' which is a very useful tool in homeomorphism theory.

Theorem 1.4.8 (The Inductive Convergence Criterion)**.** *Let X be compact, and for each $n \in \mathbb{N}$, let $h_n \in \mathcal{H}(X)$. If for each n we have $\hat{\varrho}(h_{n+1}, h_n) < 2^{-n}$ and*

$$\hat{\varrho}(h_{n+1}, h_n) < 3^{-n} \cdot \min\big\{ \min\{\varrho(h_i(x), h_i(y) : \varrho(x,y) \geq 1/n\} : 1 \leq i \leq n\big\}$$

then $h = \lim_{n\to\infty} h_n$ is a homeomorphism of X.

Proof. Since $\hat{\varrho}(h_{n+1}, h_n) < 2^{-n}$, and $\hat{\varrho}$ is complete (Corollary 1.2.17), the function $h = \lim_{n\to\infty} h_n$ exists and is continuous. Since each h_n is surjective, and X is compact, it is easily seen that h is surjective. It suffices to prove that h is one-to-one. To this end, suppose that $x, y \in X$, $x \neq y$, and take n so large that $\varrho(x,y) \geq 1/n$. Let $\varepsilon = \varrho\big(h_n(x), h_n(y)\big)$. Then for each $k \geq n$,

$$\hat{\varrho}(h_{k+1}, h_k) < 3^{-k} \cdot \varepsilon$$

since $1 \leq n \leq k$ and $\varrho(x,y) \geq 1/n \geq 1/k$. From this it follows that

$$\varrho\big(h_n(x), h(x)\big) < \sum_{k=n}^{\infty} 3^{-k} \cdot \varepsilon \leq 1/2\varepsilon.$$

Similarly, $\varrho\big(h_n(y), h(y)\big) < 1/2\varepsilon$. Since $\varepsilon = \varrho\big(h_n(x), h_n(y)\big)$, this evidently gives us that $h(x) \neq h(y)$. ☐

Let X be a compact space and let $(h_n)_n$ be a sequence in $\mathcal{H}(X)$. It is clear that for each $n \in \mathbb{N}$ the function

$$f_n = h_n \circ \cdots \circ h_1$$

belongs to $\mathcal{H}(X)$. If $f = \lim_{n\to\infty} f_n$ exists then it will be denoted by

$$\lim_{n\to\infty} h_n \circ \cdots \circ h_1$$

and is called the *infinite left product* of the sequence $(h_n)_n$. It is easy to find conditions on the sequence $(h_n)_n$ which ensure that $\lim_{n\to\infty} h_n \circ \cdots \circ h_1$ exists and belongs to $\mathcal{H}(X)$. Indeed, first observe that if $f, g, h \in \mathcal{H}(X)$ then

$$\hat{\varrho}(h \circ f, g \circ f) = \hat{\varrho}(h, g).$$

This implies that

$$\hat{\varrho}(h_{n+1} \circ h_n \circ \cdots \circ h_1, h_n \circ \cdots \circ h_1) = \hat{\varrho}(h_{n+1}, 1_X).$$

This means by Theorem 1.4.8 that if the sequence $\big(\hat{\varrho}(h_n, 1_X)\big)_n$ converges rapidly to 0 then the infinite left product of the sequence $(h_n)_n$ exists and is a homeomorphism of X.

It turns out that we are never interested in the precise speed at which $(\hat{\varrho}(h_n, 1))_n$ converges. We are always in the pleasant situation that the sequence $(h_n)_n$ is defined inductively and that we are able to choose the next homeomorphism 'sufficiently close' to the identity. This simplifies life considerably.

We illustrate the power of the Inductive Convergence Criterion by using it to prove results on countable dense homogeneity. Other applications will follow later.

A space X is called *countable dense homogeneous* provided that for all countable and dense subsets $D, E \subseteq X$ there exists an element $f \in \mathcal{H}(X)$ with $f[D] = E$. The topological sum of $\mathbb{S}^1$ and $\mathbb{S}^2$ is an example of a countable dense homogeneous space which is not homogeneous. But for connected spaces it will turn out that countable dense homogeneity implies homogeneity.

Let X be a space and fix an arbitrary point $x \in X$. Let $\tau(x)$ denote the *type* of x, i.e.,

$$\{y \in X : (\exists\, h \in \mathcal{H}(X))(h(x) = y)\}.$$

Observe that $\tau(x)$ is *invariant* under $\mathcal{H}(X)$, that is, for all $y \in \tau(x)$ and all $h \in \mathcal{H}(X)$ the image $h(y)$ also belong to $\tau(x)$.

Theorem 1.4.9. *If X is countable dense homogeneous and $x \in X$ then $\tau(x)$ is clopen in X.*

Proof. The proof if in two steps.

Claim 1. $\tau(x)$ is closed.

Proof. To the contrary, assume that there exists $p \in \overline{\tau(x)} \setminus \tau(x)$. Let E be a countable dense subset of $\tau(x)$, and let F be a countable dense subset of $U = X \setminus \overline{\tau(x)}$. Then both

$$A = E \cup F, \quad B = A \cup \{p\}$$

are countable dense subsets of X and hence there exists by assumption an element $h \in \mathcal{H}(X)$ with $h[A] = B$. Since $p \notin h[E]$ because $h[\tau(x)] \subseteq \tau(x)$, there exists $b \in F$ with $h(b) = p$. But then $h[U]$ is a neighborhood of p and hence intersects $\tau(x)$. But this implies that for some $q \in U$ we have that $h(q) \in \tau(x)$, i.e., $q \in \tau(x)$. This is a contradiction. $\diamondsuit$

Claim 2. $\tau(x)$ is open.

Proof. It is easy to see that $\tau(x)$ is open if and only if $\operatorname{Int} \tau(x) \neq \emptyset$. Striving for a contradiction, suppose therefore that $\operatorname{Int} \tau(x) = \emptyset$. By Claim 1 it follows that $\tau(x)$ is a closed subset of X with empty interior, hence is nowhere dense. Let $E \subseteq U = X \setminus \tau(x)$ be countable and dense in X. There is an element $h \in \mathcal{H}(X)$ with $h[E] = E \cup \{x\}$. Pick $e \in E$ such that $h(e) = x$. But then $e \in \tau(x)$, which is a contradiction. $\diamondsuit$

So we are done. □

Corollary 1.4.10. *A connected countable dense homogeneous space is homogeneous.*

So for connected spaces, countable dense homogeneity is a strong form of homogeneity.

A space X is called *strongly locally homogeneous* provided that it has an open base $\mathcal{U}$ such that for all $U \in \mathcal{U}$ and points $x, y \in U$ there exists an element $h \in \mathcal{H}(X)$ such that $h(x) = y$ and $h(z) = z$ for $z \notin U$. Such a homeomorphism is said to be *supported* on U. It is geometrically obvious that the Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, are strongly locally homogeneous. Most of the well-known homogeneous continua are strongly locally homogeneous: the Hilbert cube, the universal Menger continua and manifolds without boundaries. The pseudo-arc is an example of a homogeneous continuum which is not SLH. Observe that a zero-dimensional homogeneous space is evidently SLH (the clopen sets do the job).

Theorem 1.4.11. *Let X be locally compact and strongly locally homogeneous. Then for all countable dense subsets A and B of X and for every closed $C \subseteq X$ which misses $A \cup B$, there is a homeomorphism $f\colon X \to X$ which restricts to the identity on C and which moreover satisfies $f[A] = B$.*

So by taking $C = \emptyset$ in this result, we obtain that every locally compact strongly locally homogeneous space is countable dense homogeneous.

Proof. Index A and B faithfully as $\{a_1, a_2, \dots\}$ and $\{b_1, b_2, \dots\}$, respectively. The hypothesis of strong local homogeneity implies that for each neighborhood U of a point $x \in X$, and for any dense $G \subseteq X$, there exists a homeomorphism of αX which is supported on U and takes x into G (use that $G \cap U \neq \emptyset$). Using Theorem 1.4.8, we construct a sequence $(h_n)_n$ of homeomorphisms of αX such that its infinite left product h is a homeomorphism and such that the following conditions (which ensure $h[A] = B$ and $h \restriction X \in \mathcal{H}(X)$) are satisfied:

(1) $h_n \circ \cdots \circ h_1(a_i) = h_{2i} \circ \cdots \circ h_1(a_i) \in B$ for each i and $n \geq 2i$,
(2) $(h_n \circ \cdots \circ h_1)^{-1}(b_i) = (h_{2i+1} \circ \cdots \circ h_1)^{-1}(b_i) \in A$ for each i and each $n \geq 2i+1$,
(3) for all n, h_n restricts to the identity on $C \cup \{\infty\}$.

Assume $h_1, \dots, h_{2i-1}$ have been defined for certain i.

If $h_{2i-1} \circ \cdots \circ h_1(a_i) \in B$, take $h_{2i} = 1_{\alpha X}$. Otherwise, choose a small neighborhood U_{2i} of $h_{2i-1} \circ \cdots \circ h_1(a_i)$ having compact closure in X which moreover is disjoint from the set

$$C \cup \{b_1, \dots, b_{i-1}\} \cup h_{2i-1} \circ \cdots \circ h_1[\{a_1, \dots, a_{i-1}\}].$$

Take f_{2i} to be a homeomorphism of X supported on U_{2i} and such that

$$f_{2i} \circ h_{2i-1} \circ \cdots \circ h_1(a_1) \in B.$$

Finally, let $h_{2i} = \alpha f_{2i}$.

If $\big(h_{2i} \circ \cdots \circ h_1(b_i)\big)^{-1} \in A$, take $h_{2i+1} = 1_{\alpha X}$. Otherwise, choose a small neighborhood U_{2i+1} of b_i having compact closure in X which moreover is disjoint from the set

$$C \cup \{b_1, \ldots, b_{i-1}\} \cup h_{2i} \circ \cdots \circ h_1[\{a_1, \ldots, a_{i-1}\}].$$

Take f_{2i+1} to be a homeomorphism of X supported on U_{2i+1} and such that

$$f_{2i+1}^{-1}(b_i) \in (h_{2i} \circ \cdots \circ h_1)[A].$$

Finally, let $h_{2i+1} = \alpha f_{2i+1}$.

If the neighborhoods U_{2i} and U_{2i+1} are chosen small enough, the conditions of the Inductive Convergence Criterion are satisfied. In addition, (3) easily implies that $h(\infty) = \infty$, i.e., $h \restriction X \in \mathcal{H}(X)$, and h restricts to the indentity on C. $\square$

Exercise 1.4.12. *Let X be homogeneous, locally compact and strongly locally homogeneous. If $A \subseteq X$ is a countable dense set, then for all $x, y \in X \setminus A$ there is a homeomorphism $f\colon X \to X$ such that*

(1) $f[A] = A$,
(2) $f(x) = y$.

Since the Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, are strongly locally homogeneous, we get:

Corollary 1.4.13. *The Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, are countable dense homogeneous.*

1.5. Examples

We will collect some interesting examples of homeomorphism groups here.

Exercise 1.5.1. $\mathcal{H}(\mathsf{C}) \approx \mathbb{P}$.

Exercise 1.5.2. *Prove that $\mathcal{H}(\mathbb{N})$ is a topological group, and $\mathcal{H}(\mathbb{N}) = \mathbb{S}_\infty \approx \mathbb{P}$.*

Let us now consider the group $\mathcal{H}(\mathbb{I})$. If $f \in \mathcal{H}(\mathbb{I})$ then clearly $f[\{0,1\}] = \{0,1\}$. A moments reflection shows that f is either (strictly) increasing or (strictly) decreasing. If f is increasing, hence $f(0) = 0$ and $f(1) = 1$ then

$$\{g \in \mathcal{H}(\mathbb{I}) : \hat{\varrho}(f, g) < {}^1\!/\!{}_2\}$$

consists entirely of increasing homeomorphisms. Hence the collection of all increasing homeomorphisms is open, and so is the collection of all decreasing homeomorphisms. We conclude that

$$\mathcal{H}_0(\mathbb{I}) = \{f \in \mathcal{H}(\mathbb{I}) : f \text{ is increasing}\}$$

is a clopen subset of $\mathcal{H}(\mathbb{I})$. Clearly, $\mathcal{H}_0(\mathbb{I}) \approx \mathcal{H}(\mathbb{I}) \setminus \mathcal{H}_0(\mathbb{I})$.

Example 1.5.3. $\mathcal{H}_0(\mathbb{I}) \approx \mathbb{R}^\infty$ and $\mathcal{H}(\mathbb{I}) \approx \{0,1\} \times \mathbb{R}^\infty$.

Proof. We will show that $\mathcal{H}_0(\mathbb{I}) \approx \prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i}$. Let

$$(x_{n,i}) \in \prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i}$$

be arbitrary. We will define an increasing homeomorphism h of $\mathbb{I}$ associated with $(x_{n,i})$. Suppose that n is gives and that we have defined sets

$$A_n = \{0 = \alpha_0^n < \alpha_1^n < \cdots < \alpha_{2^n}^n = 1\}$$

and

$$B_n = \{0 = \beta_0^n < \beta_1^n < \cdots < \beta_{2^n}^n = 1\}$$

and a 'partial' homeomorphism h such that $h(\alpha_i^n) = \beta_i^n$ for $i = 0, 1, \ldots, 2^n$. We extend h to a set $A_{n+1} \supseteq A_n$ onto $B_{n+1} \supseteq B_n$ with each of A_{n+1} and B_{n+1} having $2^{n+1} + 1$ points. If n is odd, then let z_i be the midpoint of the interval $[\alpha_{i-1}^n, \alpha_i^n]$ for $i = 1, 2, \ldots, 2^n$, and let

$$y_i = h(z_i) = x_{n,i}(\beta_i^n - \beta_{i-1}^n) + \beta_{i-1}^n.$$

If n is even then let y_i be the midpoint of the interval $[\beta_{i-1}^n, \beta_i^n]$ for $i = 1, 2, \ldots, 2^n$, and let

$$z_i = h^{-1}(y_i) = x_{n,i}(\alpha_i^n - \alpha_{i-1}^n) + \alpha_{i-1}^n.$$

Put

$$A_{n+1} = A_n \cup \{z_i : i = 1, \ldots, 2^n\}$$

and

$$B_{n+1} = B_n \cup \{y_i : i = 1, \ldots, 2^n\},$$

respectively.

Put $A = \bigcup_{n=1}^{\infty} A_n$, and $B = \bigcup_{n=1}^{\infty} B_n$. Then both A and B are dense in $\mathbb{I}$, and $h\colon A \to B$ is an order preserving bijection. Thus h has an order preserving extension $\bar{h}\colon \mathbb{I} \to \mathbb{I}$. It is not hard to prove that the assignment $(x_{n,i}) \to \bar{h}$ is a homeomorphism between $\prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i}$ and $\mathcal{H}_0(\mathbb{I})$. $\square$

Exercise 1.5.4. *Fill in all details in the proof of Example 1.5.3.*

The Anderson Theorem states that $\mathbb{R}^\infty$ and ℓ^2 are homeomorphic. See VAN MILL [**?**, Chapter 6] for details. So Example 1.5.3 implies that $\mathcal{H}_0(\mathbb{I}) \approx \ell^2$ and $\mathcal{H}(\mathbb{I}) \approx \{0,1\} \times \ell^2$.

The question naturally arises what can be said about the homeomorphism groups $\mathcal{H}(\mathbb{I}^n)$ for $n \geq 2$. In fact, it is natural to think of the following closed subgroup of $\mathcal{H}(\mathbb{I}^n)$:

$$\mathcal{H}_\partial(\mathbb{I}^n) = \{h \in \mathcal{H}(\mathbb{I}^n) : h \restriction \partial\mathbb{I}^n = 1_{\partial\mathbb{I}^n}\};$$

here $\partial\mathbb{I}^n$ is the union of the endfaces of $\mathbb{I}^n$. It was shown by LUKE and MASON [**?**] that $\mathcal{H}_\delta(\mathbb{I}^2)$ is an AR, which implies that $\mathcal{H}_\delta(\mathbb{I}^2) \approx \ell^2$ (apply e.g., DOBROWOLSKI and TORUŃCZYK [**?**]). For $n \geq 3$ it is widely open wether $\mathcal{H}_\delta(\mathbb{I}^n)$ is an AR. This is one of the most interesting open problems in infinite-dimensional topology.

For $n = \infty$, the analogous problem has been solved (observe that Q has no boundary).

Example 1.5.5. $\mathcal{H}(Q) \approx \ell^2$.

The proof of this is difficult. See FERRY [**?**] and TORUŃCZYK [**?**] for details.

Let $\mathcal{P}$ be a decomposition of a space X into pairwise disjoint nonempty closed sets. In addition, let $q\colon X \to \mathcal{P}$ be the function sending $x \in X$ to the unique element of $\mathcal{P}$ containing x. Call a subset $\mathcal{U}$ of $\mathcal{P}$ *open* if and only if $q^{-1}[\mathcal{U}]$ is open in X. In this way we endow $\mathcal{P}$ with the *quotient topology* derived from X and q. With this topology we denote $\mathcal{P}$ by $X/\mathcal{P}$ and call it X *modulo* $\mathcal{P}$.

A function $f\colon X \to Y$ is called *quotient* provided that $U \subseteq X$ is open if and only if $f^{-1}[U]$ is open. Observe that a quotient map is continuous. It is clear that if $f\colon X \to Y$ is quotient and surjective then Y is canonically homeomorphic to the space $X/\mathcal{P}$, where $\mathcal{P}$ is the decomposition

$$\{f^{-1}(y) : y \in Y\}$$

of X.

If X is a space, and $A \subseteq X$ is closed, then $\mathcal{P} = \{A\} \cup \big\{\{x\} : x \in X \setminus A\big\}$ is a partition. It will be convenient to denote $X/\mathcal{P}$ by X/A in this case. So X/A is the quotient space obtained from X by collapsing A to a single point.

Lemma 1.5.6. *Let $\mathcal{P}$ be a decomposition of a space X into pairwise disjoint closed sets. The following statements are equivalent.*

(1) *The natural quotient mapping $q\colon X \to X/\mathcal{P}$ is closed.*
(2) *$\mathcal{P}$ is upper semi-continuous.*

Proof. For every closed set $A \subseteq X$ we have

$$q^{-1}\big[q[A]\big] = \bigcup\{P \in \mathcal{P} : P \cap A \neq \emptyset\}.$$

From this the implication (1) $\Rightarrow$ (2) is obvious. On the other hand, if (2) holds then for every closed set $A \subseteq X$ we have that $q^{-1}\big[q[A]\big]$ is closed from which it follows that $q[A]$ is closed since we are dealing with the quotient topology. $\square$

Due to our self-chosen convention to deal with separable metrizable spaces exclusively, we are in a very unpleasant situation now. It is rather trivial to prove that the quotient topology just defined is indeed a topology, but it is not necessarily separable and metrizable.

Exercise 1.5.7. *Let $\mathcal{P}$ be a partition of a space into nonempty closed sets. Prove that the quotient topology on $X/\mathcal{P}$ is indeed a topology. Prove that $\mathbb{R}/\mathbb{N}$ is not metrizable.*

We are primarily interested in decompositions of which the individual elements are compact. Fortunately, in that particular situation the decomposition spaces involved are separable and metrizable, as the following result shows.

Theorem 1.5.8. *Let X be a space and let $\mathcal{P}$ be an upper semi-continuous decomposition of X consisting of compact sets. Then $X/\mathcal{P}$ is separable and metrizable.*

Proof. First observe that $Y = X/\mathcal{P}$ is T_1 since the elements of $\mathcal{P}$ are closed sets. We next claim that Y is regular. Consider the natural quotient map

$$q\colon X \to Y.$$

Then q is closed by Lemma 1.5.6. Let $y \in Y$ and let U be an open neighborhood of it. Since $q^{-1}(y)$ is closed, and $q^{-1}[U]$ is open, there is a closed neighborhood V of $q^{-1}(y)$ such that $V \subseteq q^{-1}[U]$. Since q is closed there is a neighborhood W of y such that $q^{-1}[W] \subseteq V$. But then $y \in W \subseteq \overline{W} \subseteq U$ since $W \subseteq q[V] \subseteq U$ and $q[V]$ is closed.

So Y is regular and T_1 and it consequently suffices to prove by Urysohn's Metrization Theorem mentioned on Page 1 that it has a countable base. To this end, let $\mathcal{B}$ be a countable open base for X which is closed under finite unions. For every $B \in \mathcal{B}$ put

$$(*) \qquad U(B) = Y \setminus q[X \setminus B].$$

Since q is closed, $U(B)$ is open.

Now if U is an open neighborhood of y in Y then $q^{-1}[U]$ is an open neighborhood of the compact set $q^{-1}(y)$ in X. So there is a finite subcollection $\mathcal{E}$ of $\mathcal{B}$ such that $q^{-1}(y) \subseteq \bigcup \mathcal{E} \subseteq q^{-1}[U]$. Since $\mathcal{B}$ is closed under finite unions, clearly $\bigcup \mathcal{E} \in \mathcal{B}$. So there exists an element $B \in \mathcal{B}$ such that

$$q^{-1}(y) \subseteq B \subseteq q^{-1}[U].$$

Since $q^{-1}(y) \subseteq B$ it follows that $y \in U(B) \subseteq U$. We conclude that the collection $\{U(B) : B \in \mathcal{B}\}$ is a countable open base for Y. □

A *null-sequence* in a space X is a sequence $(A_n)_n$ of closed subsets of X such that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$.

Exercise 1.5.9. *Let X be a space, and let $(A_n)_n$ be a null-sequence consisting of pairwise disjoint nonempty compact subsets of X. In addition, for every n let $f_n\colon A_n \to B_n$ be a continuous surjection. Then the decomposition*

$$\mathcal{P} = \{f_n^{-1}(y) : y \in A_n, n \in \mathbb{N}\} \cup \Big\{\{x\} : x \notin \bigcup_{n=1}^{\infty} A_n\Big\}$$

is upper semi-continuous (and so $X/\mathcal{P}$ is separable and metrizable).

A space X is called *rigid* provided that $\mathcal{H}(X)$ consists of the identity 1_X only. So a rigid space has a very uninteresting homeomorphism group. A space containing one point only is clearly rigid. But there are more interesting examples of rigid compacta.

Example 1.5.10. There is an infinite rigid compact space.

Proof. Let K be a Cantor set, and let $(K_i)_i$ be a null sequence of nowhere dense Cantor subsets with dense union. For every i let Z_i be an i-pointed star (i.e., Z_i is the one-point compactification of $\bigcup_{j=1}^{i}[0,1)_i$), and let $\alpha_i\colon K_i \to Z_i$ be a continuous surjection. Consider the upper semi-continuous decomposition

$$\mathcal{P} = \{\alpha_i^{-1}(y) : y \in Z_i, i \in \mathbb{N}\} \cup \Big\{\{x\} : x \notin \bigcup_{i=1}^{\infty} K_i\Big\}$$

of K (Exercise 1.5.9). We claim that $X = K/\mathcal{P}$ is rigid. To this end, let $f\colon X \to X$ be any homeomorphism. Since f permutes the components of X, for every i there exists j such that $f[Z_i] = Z_j$. Since $Z_i \not\approx Z_j$ if $i \neq j$, this shows that $f[Z_i] = Z_i$ for every i. Now suppose that f is not the identity, i.e., there is an element $x \in X$ such that $f(x) \neq x$. Pick a neighborhood U of x in X such that $f[U] \cap U = \emptyset$. Since the sequence $(Z_i)_i$ is null in X and has dense union, there exists i such that $Z_i \subseteq U$. But then $f[Z_i] \cap Z_i = \emptyset$, which is impossible by the above observation. □

In §1.3 we proved that $C(\mathbb{I})$ is a 'universal' space for all Banach spaces in the same way as Q is 'universal' for spaces (see Page 2). Since $C(Q)$ contains an isomorphic copy of $C(\mathbb{I})$, $C(Q)$ is 'universal' as well. We will prove in §**??** that $\mathcal{H}(Q)$ is a universal topological group, i.e., a topological group containing an isomorphic copy of any topological group. So Q is 'universal' for spaces, $C(Q)$ is 'universal' for Banach spaces, and $\mathcal{H}(Q)$ is 'universal' for topological groups.

1.6. The compact-open topology is natural

In this section we will prove that the compact-open topology is a very natural topology on a set of homeomorphisms.

Let G be a group. There are generally many admissible topologies on G. The discrete topology is obviously admissible. MARKOV [**?**] asked whether every infinite group admits a non-discrete admissible topology. Here topology means T_0 topology to make life interesting. Note that T_0 is in general a very weak separation axiom. But not so for topological groups for it is known that a T_0 topological group is completely regular (HEWITT and ROSS [**?**, 8.4]). It is not difficult to see that for Abelian groups, Markov's question has a positive answer. SHELAH [**?**] constructed under the Continuum Hypothesis the first example of a group of cardinality ω_1 of which the only admissible topology is the discrete topology. This suggested the question whether every countably infinite group admits a non-discrete admissible topology. This interesting question was answered by OL′SHANSKIĬ in the negative.

We will argue here that the compact-open topology on a group of homeomorphisms is very natural. As support of this claim we will show that $\mathcal{H}(\mathbb{I})$ admits a unique minimum admissible T_0 topology (one contained in all admissible T_0 topologies), which is the compact-open topology.

If $f\colon X \to X$ is continuous then we define

$$\mathrm{Move}(f) = \{x \in X : f(x) \neq x\}, \quad \mathrm{Fix}(f) = \{x \in X : f(x) = x\}.$$

Exercise 1.6.1. *Let X be a space. If $f, g \in \mathcal{H}(X)$ then*

(1) $f[\mathrm{Move}(f)] = \mathrm{Move}(f)$.
(2) *if* $\mathrm{Move}(f) \cap \mathrm{Move}(g) = \emptyset$ *then* $f \circ g = g \circ f$.

Theorem 1.6.2. *Let τ be an admissible T_0 topology on $\mathcal{H}(\mathbb{I})$. Then τ contains the compact-open topology.*

Proof. We will construct a family $\mathcal{U}$ of subsets of $\mathcal{H}(\mathbb{I})$ which must be open in any admissible topology on $\mathcal{H}(\mathbb{I})$, so that if V is a neighborhood of $1_{\mathbb{I}}$ in $\mathcal{H}(\mathbb{I})$ then there is an element $U \in \mathcal{U}$ such that $1_{\mathbb{I}} \subseteq U \subseteq V$.

For $0 \leq a < b \leq 1$ choose $p, q \in \mathcal{H}(\mathbb{I})$ (depending on a and b) so that $p \neq 1_{\mathbb{I}}$, $q \neq 1_{\mathbb{I}}$, $\mathrm{Move}(p) \subseteq (a, b)$, $\mathrm{Move}(q) \subseteq (a, b)$, and $p \circ q \neq q \circ p$. Finally, define

$$T(a, b) = \{g \in \mathcal{H}(\mathbb{I}) : g \circ p \circ g^{-1} \text{ does not commute with } q\}.$$

Let $\mathcal{U}$ be the collection of all finite intersection of $T(a, b)$'s. We will prove a series of claims which demonstrates that $\mathcal{U}$ is as required.

Claim 1. $1_{\mathbb{I}} \in T(a, b)$ for every $0 \leq a < b \leq 1$.

Proof. This is clear since p does not commute with q. ♢

Claim 2. If $0 \leq a < b \leq 1$ then $T(a,b)$ is open in any admissible T_0 topology on $\mathcal{H}(\mathbb{I})$, and if $g \in T(a,b)$ then $g(x) \in (a,b)$ for some $x \in (a,b)$.

Proof. Let τ be an arbitrary admissible T_0 topology on $\mathcal{H}(\mathbb{I})$. As observed above, τ is completely regular, hence T_1. As a consequence $W = \mathcal{H}(\mathbb{I})\setminus\{1_{\mathbb{I}}\} \in \tau$. Define $\xi\colon \mathcal{H}(\mathbb{I}) \to \mathcal{H}(\mathbb{I})$ by

$$\xi(g) = gpg^{-1}q(gpg^{-1})^{-1}q^{-1}.$$

Then ξ is evidently τ-continuous. But $T(a,b) = \xi^{-1}[W]$, i.e., $T(a,b) \in \tau$.

Let $I = (a,b)$. We will prove that for every $g \in T(a,b)$ we have $g[I] \cap I \neq \emptyset$. Striving for a contradiction, assume that some $g \in T(a,b)$ we have $g[I] \cap I = \emptyset$. Note that if $y \notin g[I]$, then $g^{-1}(y) \notin I$, so $pg^{-1}(y) = g^{-1}(y)$ since $\operatorname{Move}(p) \subseteq I$. As a consequence, $gpg^{-1}(y) = y$. Thus $y \notin \operatorname{Move}(gpg^{-1})$, i.e., $\operatorname{Move}(gpg^{-1}) \subseteq g[I]$. But $g[I] \cap I = \emptyset$ by assumption, hence,

$$\operatorname{Move}(gpg^{-1}) \cap \operatorname{Move}(q) = \emptyset.$$

Thus gpg^{-1} and q commute by Exercise 1.6.1, a contradiction. ♢

Claim 3. Every neighborhood V of $1_{\mathbb{I}}$ in $\mathcal{H}(\mathbb{I})$ contains a U in $\mathcal{U}$.

Proof. We may assume that V is of the form

$$B_\varepsilon = \{g \in \mathcal{H}(\mathbb{I}) : (\forall\, x \in \mathbb{I})(|g(x) - x| < \varepsilon)\}$$

for some $\varepsilon > 0$. Pick $n > 1$ such that $1/n < 1/2\varepsilon$, and put

$$U = T(0, 1/n) \cap T(1/n, 2/n) \cap \cdots \cap T((n-1)/n, 1).$$

We claim that U is as required. If not then for some $x \in \mathbb{I}$ and $h \in U$, $|h(x) - x| \geq \varepsilon > 2/n$. Let i be minimal such that $x \in [i/n, (i+1)/n]$.

Assume first that $0 < i < n-1$. By Claim 2 we may pick $x_{-1}, x_0, x_1 \in \mathbb{I}$ such that

$$\begin{aligned} x_{-1}, h(x_{-1}) &\in ((i-1)/n, i/n), \\ x_0, h(x_0) &\in (i/n, (i+1)/n), \\ x_1, h(x_1) &\in ((i+1)/n, (i+2)/n). \end{aligned}$$

Observe that h is either increasing or decreasing. Since $x_0 < x_1$ and $h(x_0) < h(x_1)$ it follows that h is increasing.

We distinguish between the following cases:

Case 1. $x_0 \leq x$. Since $x_0 \leq x < x_1$ we get $h(x_0) \leq h(x) < h(x_1)$. But then $|x - h(x)| < 2/n$, which is a contradiction.

Case 2. $x < x_0$. Since $x_{-1} < x < x_0$ we get $h(x_{-1}) \leq h(x) < h(x_0)$. But then $|x - h(x)| < 2/n$, which is again a contradiction.

Assume next that $i = 0$. By Claim 2 we may pick $x_0, x_1 \in \mathbb{I}$ such that

$$x_0, h(x_0) \in (0, {}^1\!/_n),$$
$$x_1, h(x_1) \in ({}^1\!/_n, {}^2\!/_n).$$

Observe that $h(x) > {}^2\!/_n$. Hence $x < x_1$ and $h(x_1) < h(x)$. This means that h is decreasing. But $x_0 < x_1$ and $h(x_0) < h(x_1)$, so this is a contradiction.

Finally, assume that $i = n - 1$. By Claim 2 we may pick $x_0, x_1 \in \mathbb{I}$ such that

$$x_0, h(x_0) \in ({}^{(n-1)}\!/_n, 1),$$
$$x_1, h(x_1) \in ({}^{(n-2)}\!/_n, {}^{(n-1)}\!/_n).$$

Observe that $h(x) < {}^{(n-2)}\!/_n$. Hence $x_1 < x$ and $h(x) < h(x_1)$. This means that h is decreasing. But $x_0 < x_1$ and $h(x_0) < h(x_1)$, so this again is a contradiction. $\diamondsuit$

This finishes the proof. $\square$

CHAPTER 2

Actions of topological groups and homogeneous spaces

In this chapter we first discuss some basic material on actions of topological groups on spaces. Then we apply these results to get conclusions on various relations between strong forms of homogeneity. Among other things we will prove that BLABLA.

2.1. Actions of topological groups

An *action* of a topological group G on a space X is a continuous function

$$(g, x) \mapsto gx \colon G \times X \to X$$

such that $ex = x$ for every $x \in X$ and $g(hx) = (gh)x$ for $g, h \in G$ and $x \in X$.

Exercise 2.1.1. *Prove that for each $g \in G$ the function $x \mapsto gx$ is a homeomorphism of X whose inverse is the function $x \mapsto g^{-1}x$.*

Observe that for the continuity of the action it suffices to check the following conditions:

(I) For every $g \in G$ we have that the function $x \mapsto gx$ is continuous on X.

(II) If $(g_n)_n$ is a sequence in G converging to e and $(x_n)_n$ is a sequence in X converging to $x \in X$ then the sequence $(g_n x)_n$ converges to x.

To check this, let $(g, x) \in G \times X$ and let $(g_n)_n$ be an arbitrary sequence in G and $(x_n)_n$ an arbitrary sequence in X such that $g_n \to g$ and $x_n \to x$. Since $g^{-1}g_n \to e$, by (II) we have

$$g^{-1}g_n x_n \longrightarrow x \quad (n \to \infty),$$

hence by (I),

$$g_n x_n = g(g^{-1}g_n x_n) \longrightarrow gx \quad (n \to \infty),$$

as required.

If $x \in X$ and $U \subseteq G$ then Ux abbreviates the set $\{gx : g \in U\}$. The action of G on X is *transitive* if $Gx = X$ for some (equivalently: every) $x \in X$.

The action $G \times X \to X$ is called *effective* if for all distinct $g, h \in G$ there is an element $x \in X$ such that $gx \neq hx$. In addition, it is called *free* if for all distinct $g, h \in G$ and *all* $x \in X$ we have $gx \neq hx$. It is easily seen that the action is free if and only if no homeomorphism $x \mapsto gx$ has a fixed-point, unless $g = e$.

For every $x \in X$ let $\gamma_x \colon G \to X$ be defined by $\gamma_x(h) = hx$. Then γ_x is a continuous and a surjection if G acts transitively.

In general it is difficult to decide which groups act on a given space. For example, it is still unknown whether an infinite compact zero-dimensional group can act effectively on a finite dimensional topological manifold (that such an action does not exist is the famous Hilbert-Smith conjecture).

We will now present some important examples of actions.

(1) A *continuous flow* is an action of $\mathbb{R}$ on a space X. If $x \in X$ then the set $\{tx : t \in \mathbb{R}\}$ is called the *orbit* of x in X. It is well-known that under suitable conditions, an autonomous differential equation in $\mathbb{R}^n$ defines a continuous flow such that the integral curves of the equation are just the orbits of the flow. See e.g., de Vries [**?**] for more information.

(2) A *discrete flow* on a space X is action of $\mathbb{Z}$ on X. So a discrete flow is nothing but a pair (X, f), where X is a space and $f \colon X \to X$ is a homeomorphism.

(3) Every topological group G acts on itself. Simply define

$$(g, h) \mapsto gh : G \times G \to G$$

This action is transitive since $Ge = G$ and free.

(4) Let G and H be topological groups with continuous surjective homomorphism $\alpha \colon G \to H$. Define an action of G on H by

$$(x, y) \mapsto \alpha(x)y : G \times H \to H.$$

This action is transitive since $Ge = H$. (Observe that the above action is a special case of this.) This action is not effective if α is not one-to-one. We call this action the *standard* action of G on H.

(5) Let (X, ϱ) be a compact space. The *natural action* of $\mathcal{H}(X)$ on X is defined by the formula

$$(h, x) \mapsto h(x) : \mathcal{H}(X) \times X \to X.$$

Observe that this action is transitive if and only if X is homogeneous. In addition, it is clearly effective (but not necessarily free).

The natural action of $\mathcal{H}(X)$ on X is continuous by Lemma 1.2.8.

Observe that $\mathcal{H}(X)$ is Polish by Corollary 1.4.5. So the natural action of $\mathcal{H}(X)$ on X is a transitive action of a Polish group on X.

(6) Let X be locally compact. As in the case of compact spaces, the *natural action* of $\mathcal{H}_\alpha(X)$ on X is defined by the formula

$$(h, x) \mapsto h(x) : \mathcal{H}_\alpha(X) \times X \to X.$$

This action is evidently continuous by (5). Observe that this action is transitive if and only if X is homogeneous. In addition, it is clearly effective (but not necessarily free). Observe that $\mathcal{H}_\alpha(X)$ is Polish since it is a closed subgroup of the Polish group $\mathcal{H}(\alpha X)$. So the natural action of $\mathcal{H}_\alpha(X)$ on X is a transitive action of a Polish group on X.

Actions on compact spaces can be 'characterized' rather easily. To see this, let X be a compact space and let G be a topological group. For each $g \in G$ the function $x \mapsto gx$ is a homeomorphism of X. We denote this homeomorphism by $\varphi(g)$.

Exercise 2.1.2. *Prove that $\varphi\colon G \to \mathcal{H}(X)$ is a continuous homomorphism.*

Conversely, if $\varphi\colon G \to \mathcal{H}(X)$ is a continuous homomorphism then the composition

$$G \times X \xrightarrow{\varphi \times 1_X} \mathcal{H}(X) \times X \to X$$

is an action of G on X.

So on a compact space X there is basically only one action: the natural action $\mathcal{H}(X) \times X \to X$. All 'other' actions come from continuous homomorphisms from topological groups into the group $\mathcal{H}(X)$.

Observe that one-to-one homomorphisms $\varphi\colon G \to \mathcal{H}(X)$ precisely correspond to effective actions. So an effective action simply boils down to the natural action of a subgroup G of $\mathcal{H}(X)$ on X, where the topology of G is possibly weaker than the topology it inherits from $\mathcal{H}(X)$. The situation for actions on noncompact spaces is more complex. See §**??** for more details.

(A) Actions on second category spaces. Open mapping theorems are very important in topological groups as the classical Open Mapping Theorem in Functional Analysis demonstrates. We will prove here that a transitive action of a group on a second category space is 'almost open'.

Theorem 2.1.3. *Let G act transitively on X. If X is second category then for every neighborhood U of e in G and $x \in X$, $\overline{Ux}$ is a neighborhood of x.*

Proof. The neutral element e of G has an open neighborhood base $(U_n)_n$ having the following properties:

(†) U_n is symmetric and $U_1 = G$,
(‡) $U_{n+1} \subseteq U_{n+1}^2 \subseteq U_n$.

Observe that since U_n is symmetric, $x \in U_n y$ if and only if $y \in U_n x$. Also notice that $U_1 x = X$ for every $x \in X$.

Claim 1. For every $x \in X$, $n \in \mathbb{N}$, open $V \subseteq X$ and $z \in V \cap U_n x$ there exists $m \in \mathbb{N}$ such that $U_m z \subseteq V \cap U_n x$.

Proof. There is an element $h \in U_n$ such that $\gamma_x(h) = hx = z$. The set $E = \gamma_x^{-1}[V]$ is an open neighborhood of h, so $Eh^{-1} \cap U_n h^{-1}$ is a neighborhood of e. Pick m so large that $U_m \subseteq Eh^{-1} \cap U_n h^{-1}$. We claim that m is as required. To this end, pick an arbitrary element $p \in U_m z$. There is an element $g \in U_m$ such that $gz = p$. So $\gamma_x(gh) = (gh)x = p$ and $gh \in E \cap U_n$, which proves that $p \in V \cap U_n x$. ♢

Claim 2. If $x \in X$ and $n \in \mathbb{N}$ then $U_n x$ is not meager in X.

Proof. Since G is Lindelöf, there is a countable set $F \subseteq G$ such that $FU_n = G$. If $\varphi \in F$ then $(\varphi U_n)x$ is the image of $U_n x$ under the homeomorphism $p \mapsto \varphi p$ of X. So if $U_n x$ is meager then X is meager in itself since

$$X = \bigcup \{(\varphi U_n)x : \varphi \in F\},$$

the action of G on X being transitive. This contradicts that X is of the second category. ♢

Claim 3. For every $x \in X$ and $n \in \mathbb{N}$ the set $U_n x$ is nowhere meager in X.

Proof. This follows immediately from Claims 1 and 2. ♢

Claim 4. For every $x \in X$ and $n \in \mathbb{N}$ the set $\overline{U_n x}$ has dense interior. In addition, x belongs to the interior of $\overline{U_n x}$.

Proof. Let V be an open subset of X which intersects $\overline{U_n x}$. Then by Claim 3, $V \cap U_n x$ is not meager. Hence $\overline{V \cap U_n x}$ is not nowhere dense, i.e., has nonempty interior. This proves that the interior of $\overline{U_n x}$ is dense in $\overline{U_n x}$.

Let V be a nonempty open subset of X which is contained in $\overline{U_{n+1} x}$. Then V intersects $U_{n+1} x$, say $hx \in V$ for certain $h \in U_{n+1}$. So

$$x \in h^{-1}V \subseteq h^{-1}\overline{U_{n+1} x} = \overline{h^{-1}U_{n+1} x} = \overline{\{(h^{-1}\varphi)x : \varphi \in U_{n+1}\}} \subseteq \overline{U_n x}$$

by (†) and (‡) and the fact that the map $p \mapsto h^{-1}p$ is a homeomorphism. ♢

Now if U is an arbitrary neighborhood of e in X then pick n such that $U_n \subseteq U$. Hence we are done by Claim 4. □

Let X and Y be spaces. We say that a continuous function $f\colon X \to Y$ is *d-open* provided that for every open $U \subseteq X$ we have that $f[U]$ is contained in the interior of $\overline{f[U]}$.

Exercise 2.1.4. *Let $f\colon X \to Y$ be continuous. The following statements are equivalent:*

(1) *f is d-open,*
(2) *for every open $V \subseteq Y$, $\overline{f^{-1}[V]} = f^{-1}[\overline{V}]$.*

Corollary 2.1.5. *Let G act transitively on the second category space X. Then for every $x \in X$, the function $\gamma_x\colon G \to X$ is continuous, surjective and d-open.*

Proof. This is a direct consequence of Theorem 2.1.3. □

(B) Analytic spaces. A space X is called *analytic* provided that it is a continuous image of the space of all irrational numbers $\mathbb{P}$.

Exercise 2.1.6. *Prove that every Polish space is analytic.*

Hint. Let X be Polish. We assume that X is a subspace of the Hilbert cube Q. There is a continuous surjection $f\colon \mathsf{C} \to Q$. If $f^{-1}[X]$ would be homeomorphic to $\mathbb{P}$ then we would be done. But this is too much asked for. How do we fix this?

Let X be a space. An element of the smallest σ-algebra of subsets of X containing all the open subsets of X is called a *Borel* subset of X.

Exercise 2.1.7. *Let X be a Polish space with Borel subset B. Prove that B is analytic.*

Exercise 2.1.8. *Give an example of an analytic space that is not Polish.*

(C) The Effros Theorem. Now that we know that actions are sometime 'almost open', it is natural to ask when they are 'open'. Let G be a group acting on a space X. The action is *micro-transitive* if for every $x \in X$ and every neighborhood U of e in G the set Ux is a neighborhood of x in X.

Exercise 2.1.9. *Let G be a group acting transitively on a space X. Then the following statements are equivalent.*

(1) *The action of G on X is micro-transitive.*
(2) *For every $x \in X$ the function $\gamma_x\colon G \to X$ is open.*
(3) *For some $x_0 \in X$ the function $\gamma_{x_0}\colon G \to X$ is open.*

We will prove the following generalization of the Effros Theorem from [**?**], and to present some applications.

Theorem 2.1.10 (Open Mapping Principle)**.** *Suppose that an analytic group G acts transitively on a space X. If X is of the second category then G acts micro-transitively on X.*

Before we are in a position to present the proof of Theorem 2.1.10, we need to present some preliminary results.

We call a subset of X *fat* if it is both nowhere meager and dense in X. Observe that if A is fat then so is every larger set.

Lemma 2.1.11. *Suppose that S and A are subsets of X with S fat and A meager. Then $S \setminus A$ is fat.*

Proof. Let V be a nonempty open subset of X. If $V \cap (S \setminus A)$ is meager then so is the nonempty relatively open subset $V \cap S$ of S. Indeed, simply observe that $V \cap S$ is contained in the meager set $\big(V \cap (S \setminus A)\big) \cup A$. So $V \cap (S \setminus A)$ is not meager, which proves simultaneously that $S \setminus A$ is dense and nowhere meager. $\square$

The following result is our main tool in the proof of Theorem 2.1.10.

Proposition 2.1.12. *Any two fat analytic subspaces of a space intersect.*

Proof. Let S and T be fat in X. Let $\alpha\colon P \to S$ be a continuous surjection, where P is Polish. We denote by $\mathcal{U}$ the collection of all open subsets U of P such that $\alpha[U]$ is meager in X. We claim that $\alpha[\bigcup \mathcal{U}]$ is meager in X. To see this, let $\mathcal{V}$ be a countable subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ (use that X has a countable base). Since

$$\alpha\big[\bigcup \mathcal{U}\big] = \alpha\big[\bigcup \mathcal{V}\big] = \bigcup\{\alpha[V] : V \in \mathcal{V}\},$$

this establishes our claim. If $P' = P \setminus \bigcup \mathcal{U}$ then P' is Polish, being closed in P, hence $\alpha[P']$ is analytic. In addition, $\alpha[P']$ contains $S \setminus \alpha[\bigcup \mathcal{U}]$ and hence is fat by by Lemma 2.1.11. Finally, if $W' \subseteq P'$ is nonempty and (relatively) open then $\alpha[W']$ is not meager in X. For let W be an open subset of P such that $W \cap P' = W'$. Then $W \notin \mathcal{U}$, hence $\alpha[W]$ is not meager; but $\alpha[W] \subseteq \alpha[W'] \cup \alpha[\bigcup \mathcal{U}]$ and $\alpha[\bigcup \mathcal{U}]$ is meager, so $\alpha[W']$ is not meager.

So these considerations prove that we may assume without loss of generality that $\alpha\colon P \to S$ has the additional property that for every nonempty open subset V of P, $\alpha[V]$ is not meager in X; so $\overline{\alpha[V]}$ is not nowhere dense, i.e., has nonempty interior. We may assume that there are a Polish Q and $\beta\colon Q \to T$ with similar properties.

All our metrics are bounded by 2^{-1}, and on P respectively Q we use complete metrics. By induction on n we will construct a nonempty open subset $U_n \subseteq P$, a nonempty open subset $V_n \subseteq Q$, and a nonempty open subset W_n of X, having the following properties:

(1) $\operatorname{diam} U_n < 2^{-n}$, $\operatorname{diam} V_n < 2^{-n}$, and $\operatorname{diam} W_n < 2^{-n}$,
(2) $\overline{U}_{n+1} \subseteq U_n$, $\overline{V}_{n+1} \subseteq V_n$, $\overline{W}_{n+1} \subseteq W_n$,
(3) $\overline{W}_{n+1} \subseteq \overline{\alpha[U_n]} \subseteq \overline{\beta[V_n]} \subseteq W_n$.

Put $U_1 = P$, $V_1 = Q$, and $W_1 = X$. Suppose that U_n, V_n and W_n have been found. Pick a nonempty open subset W of X such that $\operatorname{diam} W < 2^{-(n+1)}$ and $W \subseteq \overline{\alpha[U_n]}$. Since $\beta^{-1}[W] \cap V_n$ is nonempty, we may pick a nonempty open subset F of Q such that $\overline{F} \subseteq V_n$, $\operatorname{diam} F < 2^{-(n+1)}$ and $\overline{\beta[F]} \subseteq W$. Let W' be a nonempty open subset of X which is contained in $\overline{\beta[F]}$. Since

$\alpha^{-1}[W'] \cap U_n$ is nonempty, we may pick a nonempty open subset E of P such that $\overline{E} \subseteq U_n$, $\operatorname{diam} E < 2^{-(n+1)}$ and $\overline{\alpha[E]} \subseteq W'$. So we conclude that $U_{n+1} = E$, $V_{n+1} = F$ and $W_{n+1} = W$ satisfy our inductive requirements.

Since our metrics on P respectively Q are complete, there are

$$p \in \bigcap_{n=1}^{\infty} U_n, \quad q \in \bigcap_{n=1}^{\infty} V_n.$$

Since $\operatorname{diam} W_n < 2^{-n}$ for every n, $\alpha(p) = \beta(q)$, i.e., $S \cap T \neq \emptyset$. □

A slightly more complicated argument shows that the intersection of countably many analytic fat subspaces is fat.

Proof of Theorem 2.1.10. Let G be a topological group acting transitively on a space X. The neutral element e of G has an open neighborhood base $(U_n)_n$ having the following properties:

(†) U_n is symmetric and $U_1 = G$,
(‡) $U_{n+1} \subseteq U_{n+1}^2 \subseteq U_n$.

Observe that since U_n is symmetric, $x \in U_n y$ if and only if $y \in U_n x$. Also notice that $U_1 x = X$ for every $x \in X$. By Theorem 2.1.3, if $x \in X$ and $n \in \mathbb{N}$ then $\overline{U_n x}$ is a neighborhood of x. This uses the fact that X is of the second category only. To finish the proof, we now additionally assume that G is analytic.

Since U_n is an open subspace of the analytic space G, it is analytic. As a consequence, $U_n x$ is analytic being a continuous image of U_n.

Claim 1. If $x \in X$ and $n \in \mathbb{N}$ then the interior of $\overline{U_{n+1} x}$ is contained in $U_n x$.

Proof. Let z be an arbitrary element of the interior V of $\overline{U_{n+1} x}$. Consider the set $U_{n+1} z$, and let W be the interior of $\overline{U_{n+1} z}$. Then $E = V \cap W$ is an open neighborhood of z. Since $E \subseteq \overline{U_{n+1} x}$, $U_{n+1} x \cap E$ is dense in E. Similarly, $U_{n+1} z \cap E$ is dense in E. Since $U_{n+1} x$ and $U_{n+1} z$ are analytic, so are $U_{n+1} x \cap E$ and $U_{n+1} z \cap E$. By Proposition 2.1.12, it follows that $U_{n+1} x \cap E$ and $U_{n+1} z \cap E$ intersect, say in the element y. Pick elements $g, h \in U_{n+1}$ such that $gx = y$ and $hz = y$, and put $\varphi = h^{-1} g$. Then clearly $\varphi x = z$ and $\varphi \in U_{n+1} U_{n+1} \subseteq U_n$ by (†) and (‡). We therefore conclude that $z \in U_n x$. ♢

So we get:

Claim 2. If $x \in X$ and $n \in \mathbb{N}$ then $U_n x$ is a neighborhood of x.

Observe that Claims 2 and 1 imply that $U_n x$ is open in X for every n. □

Corollary 2.1.13. *Suppose that a Polish group G acts transitively on a space X. Then the following statements are equivalent.*

(A) *G acts micro-transitively on X.*
(B) *X is Polish.*
(C) *X is of the second category.*

Proof. The implication (B) ⇒ (C) is simply the Baire Category Theorem for Polish spaces, and (A) ⇒ (B) is a consequence of **??**. □

We finish our discussion of the Effros Theorem by making some remarks.

(1) Corollary 2.1.13 was used extensively by continuum theorists in their study of homogeneous continua. See §**??** for more information.

(2) Let H be an analytic topological group of the first category which admits a continuous homomorphism α onto a topologically complete group G. Consider the standard action of H on G. Theorem 2.1.10 applies in this situation, in contrast to Corollary 2.1.13. Examples are easily found. For example, let
$$H = \{x \in \mathbb{R}^\infty : (\exists\, N \in \mathbb{N})(\forall\, n > N)(x_n = 0)\},$$
and $G = \mathbb{R}$, and let $\alpha\colon H \to G$ the projection $\alpha(x) = x_1$.

(3) It is natural to ask whether the assumptions in Theorem 2.1.10 actually imply that the space X is Polish. The answer to this question is in the negative, see §**??** for details.

(4) Theorem Theorem 2.1.10 clearly fails if X is not of the second category. Let $\mathbb{Q}_d$ be $\mathbb{Q}$ with the discrete topology. Then the identity $\mathbb{Q}_d \to \mathbb{Q}$ is not open.

(5) Nowhere in the proof of Theorem Theorem 2.1.10 did we use the full strength of the continuity of the action. In fact, we only used that the action is *separately continuous* (i.e., the maps $g \mapsto gx : G \to X$ for $x \in X$ and $x \mapsto gx : X \to X$ for $g \in G$ are all continuous). So we we actually proved a stronger result than stated.

(D) Application: The Open Mapping Theorem. The classical Open Mapping Theorem from Functional Analysis can be generalized to topological groups.

Exercise 2.1.14. *Let G and H be topological groups with G analytic and H Polish. If $\varphi\colon G \to H$ is a continuous surjective homomorphism then φ is open.*

Let G be a topological group. We let τG denote all topologies on G that make G into a topological group (so we vary the topology but leave the algebraic structure as it is) and satisfy some other unspecified condition $\mathcal{P}$. (The discrete topology on G is admissible but very uninteresting, so it is clear that we must demand something extra.) If H is a topological group for which there is a discontinuous surjective homomorphism $\varphi\colon G \to H$ then the topology of G is probably not maximal in τG. Simply observe that the graph $\{(x, \varphi(x)) : x \in G\}$ can be algebraically identified with G, while the subspace topology it inherits from $G \times H$ is stronger than the original topology on G (it has to satisfy $\mathcal{P}$ of course, but let us ignore this for the moment).

The following corollary to Exercise 2.1.14 shows that Polish group topologies are maximal among the analytic group topologies.

Corollary 2.1.15. *Let G and H be topological groups with G analytic and H Polish. If $\varphi\colon G \to H$ is a continuous one-to-one surjective homomorphism, then φ is an isomorphism.*

(E) Application: The Closed-Graph Theorem. Our next result is a generalization of the 'Closed Graph Theorem' in Functional Analysis.

Exercise 2.1.16. *Let G and H be topological groups with G Polish. If $\varphi\colon G \to H$ is a homomorphism such that its graph*

$$G(\varphi) = \big\{(g, \varphi(g)) : g \in G\big\} \subseteq G \times H$$

is analytic, then φ is continuous.

2.2. Coset spaces

Let G be a topological group with closed subgroup H. If $x, y \in G$ and $xH \cap yH \neq \emptyset$ then $xH = yH$. Hence the collection of all *left cosets* $G/H = \{xH : x \in G\}$ is a partition of G in closed sets. Let $\pi\colon G \to G/H$ be defined by $\pi(x) = xH$. We endow G/H by the quotient topology. A space X is a *coset space* provided that there is a closed subgroup H of a topological group G such that X and G/H are homeomorphic. In this section we will consider the following basic question: which spaces are coset spaces of topological groups?

Let G be a topological group with closed subgroup H. Then H is a subset of G and H is a point of G/H. This sometimes leads to confusion.

Lemma 2.2.1. *Let G be a topological group with closed subgroup H. Then*

1. *if $V \subseteq G/H$ is open then there is an open subset $U \subseteq G$ such that $V = \{xH : x \in U\}$,*
2. *if $U \subseteq G$ is open then $\pi[U] = \{xH : x \in U\}$ is open in G/H.*

Proof. For (1), let $U = \pi^{-1}[V]$. For (2), simply observe that

$$\pi[U] = \{xH : x \in U\} = \{xH : x \in UH\},$$

hence

$$\pi^{-1}\big[\pi[U]\big] = UH$$

is open. □

Let G be a topological group with closed subgroup H. We let G act transitively on G in the standard way by $(g, x) \mapsto gx$. We also let G act on G/H by

$$G \times G/H \to G/H : (g, xH) \mapsto gxH.$$

We call this the *natural* action of G on G/H. We will check that this action is continuous and transitive. Consider the diagram

$$(*) \qquad \begin{array}{ccc} G \times G & \xrightarrow{(g,x)\mapsto gx} & G \\ {\scriptstyle 1_G\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times G/H & \xrightarrow[(g,xH)\mapsto gxH]{} & G/H \end{array}$$

and observe that it clearly commutes.

Corollary 2.2.2. *Let G be a topological group with closed subgroup H. The natural action of G on G/H is continuous and transitive. As a consequence, for every $g \in G$ the function $xH \mapsto gxH$ is a homeomorphism of G/H, i.e., G/H is a homogeneous space.*

Proof. This is clear by the commutativity of $(*)$ and the fact that $1_G \times \pi$ is open by Lemma 2.2.1. □

By Corollary 2.2.2, if X is a coset space then X must be homogeneous. It is a natural question to ask whether the converse is true.

Corollary 2.2.3. *Let G be a topological group with closed subgroup H. Then G/H is separable and metrizable.*

Proof. First observe that G/H is second-countable since G is and π is open by Lemma 2.2.1(2). By Urysohn's Metrization Theorem, cf., Page 1, it consequently suffices to prove that G/H is regular and T_1. That every point in G/H is closed follows again from Lemma 2.2.1(2) since H and hence every coset xH is closed in G. By Corollary 2.2.2, we need only check that G/H is regular at $eH = H$. To this end, let U be a neighborhood of H in G/H. Let V be a symmetric neighborhood of e such that $V^2 \subseteq \pi^{-1}[U]$. Then $\pi[V]$ is an open neighborhood of H in G/H and is contained in U. Now if $\pi(x) = xH \in \overline{\pi[V]}$ then $\{vxH : v \in V\}$ is a neighborhood of $\pi(x)$ in G/H (Lemma 2.2.1(1)) and hence contains a point of $\pi[V]$. That is, there are points $v_0, v_1 \in V$ such that $v_0xH = v_1H$, i.e., $xH = v_0^{-1}v_1H \in \{wH : w \in V^{-1}V\} \subseteq \{uH : u \in \pi^{-1}[U]\} = U$. □

Corollary 2.2.4. *Let G be a Polish group with closed subgroup H. Then G/H is Polish.*

Proof. Since $\pi\colon G \to G/H$ is open, this follows from **??**. □

(A) Characterizing coset spaces. Let G be a topological group acting transitively on X. For every $x \in X$, put

$$G_x = \{g \in G : gx = x\}.$$

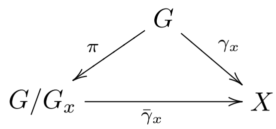
It is clear that G_x is a closed subgroup of G. It is called the the *stabilizer* of x. Observe that if $g \in G$ and $h \in G_x$ then

$$(gh)x = g(hx) = gx.$$

This means that the function $\bar{\gamma}_x\colon G/G_x \to X$ defined by

$$(*) \qquad \bar{\gamma}_x(gG_x) = \gamma_x(g) = gx$$

is well-defined. In addition, the diagram

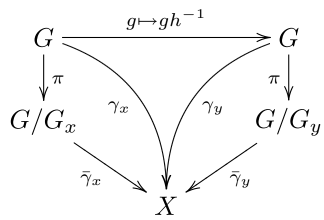


commutes. Since π is open (Lemma 2.2.1(2)) and γ_x is surjective, $\bar{\gamma}_x$ is a continuous surjection. We claim that $\bar{\gamma}_x$ is one-to-one. To this end, assume that $gG_x \neq g'G_x$ for certain $g, g' \in G$. Then $g^{-1}g' \notin G_x$, i.e., $g'x \neq gx$. So $\bar{\gamma}_x(gG_x) \neq \bar{\gamma}_x(g'G_x)$. This means that X is a coset space if $\bar{\gamma}_x$ is open.

Proposition 2.2.5. *Let G be a topological group acting transitively on X. The following statements are equivalent:*

(1) *For some $x \in X$, $\bar{\gamma}_x\colon G/G_x \to X$ is open.*
(2) *For all $x \in X$, $\bar{\gamma}_x\colon G/G_x \to X$ is open.*
(3) *For some $x \in X$, $\gamma_x\colon G \to X$ is open.*
(4) *For all $x \in X$, $\gamma_x\colon G \to X$ is open.*
(5) *G acts micro-transitively.*

Proof. Take arbitrary $x, y \in X$, and pick $h \in G$ such that $hx = y$. The diagram



commutes. Now use that both functions π are open (Lemma 2.2.1(2)) and apply Exercise 2.1.9. □

This yields a characterization of coset spaces.

Theorem 2.2.6. *Let X be a space. The following statements are equivalent:*

(1) *X is a coset space.*
(2) *There is a topological group acting transitively on X such that for some (equivalently: for all) $x \in X$ the function $\gamma_x \colon G \to X$ is open.*
(3) *There is a topological group acting transitively and micro-transitively on X.*

Proof. Simply apply Corollary 2.2.2 and Proposition 2.2.5. □

We are now in a position to identify our firs important class of coset spaces.

Theorem 2.2.7. *Let X be a locally compact homogeneous space. Then X is a coset space.*

Proof. Consider the standard action of $\mathcal{H}_\alpha(X)$ on X. By Theorem 2.1.10, this action is micro-transitive. Hence we are done by Theorem 2.2.6. □

The question naturally arises whether all homogeneous spaces are coset spaces. The answer to this question is in the negative (see **??**).

2.3. Ungar's Theorem

Surprisingly, Theorem 2.1.10 implies an interesting statement about homogeneous compacta of which no direct proof is known.

Theorem 2.3.1. *Let X be a homogeneous compact space. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X$ with $\varrho(x, y) < \delta$ there is a homeomorphism $f \colon X \to X$ such that $f(x) = y$ while moreover f moves no point more than ε.*

(This result goes some way towards explaining the word micro-transitive.)

Proof. Consider the standard action of $\mathcal{H}(X)$ on X. By Theorem 2.1.10, this action is micro-transitive. Now let $\varepsilon > 0$, and let $V = \{f \in \mathcal{H}(X) : \hat{\varrho}(f, 1_X) < {}^1\!/_2\varepsilon\}$. By micro-transitivity, the set Vx is open in X for every $x \in X$. Let $\delta > 0$ be a Lebesgue number for the open cover

$$\{Vx : x \in X\}$$

of X. We claim that δ is as required. To prove this, let $a, b \in X$ be such that $\varrho(a, b) < \delta$. There is an $x \in X$ such that $a, b \in Vx$. Pick $f, g \in V$

such that $f(x) = a$ and $g(x) = b$. Then $h = g \circ f^{-1}$ sends a onto b and $\hat{\varrho}(h, 1_X) < \varepsilon$. □

So informally, one can say that in a compact homogeneous space 'close' points can be homeomorphed onto each other by a 'small' homeomorphism.

This elegant result seems to hold for compact spaces only. Let X be locally compact and homogeneous. Since $\mathcal{H}_\alpha(X)$ acts transitively on X, the ε–δ condition in Theorem 2.3.1 can be proved by the same method if the metric that we use on X is the restriction to X of an admissible metric on αX. But this is not a very useful result. Instead of metrics, one could try to measure closeness with open covers. This approach leads to an interesting open problem.

Let $\mathcal{U}$ be a cover of a space Y. We say that functions $f, g\colon X \to Y$ are *$\mathcal{U}$-close* provided that for every $x \in X$ there is an element $U \in \mathcal{U}$ such that $f(x), g(x) \in U$. We say that a space X has the *Effros property* provided that for every open cover $\mathcal{U}$ there is an open refinement $\mathcal{V}$ such that for every $V \in \mathcal{V}$ and $x, y \in V$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and f and 1_X are $\mathcal{U}$-close. So every compact homogeneous space has the Effros property by Theorem 2.3.1.

Problem 1. Characterize the spaces with the Effros property.

Proposition 2.3.2. *Let X be an SLH-space. Then X has the Effros property.*

Proof. Let $\mathcal{U}$ be an open cover of X. This cover can clearly be refined by an open cover $\mathcal{V}$ having the property that for all $V \in \mathcal{V}$ and $x, y \in V$ there is a homeomorphism $f \in \mathcal{H}(X)$ which is supported on V and moreover maps x onto y. This cover is clearly as required. □

So we see that for every n the euclidean space $\mathbb{R}^n$ has the Effros property. Not all locally compact homogeneous spaces have the Effros property, as the following simple example shows.

Example 2.3.3. The product $X = \mathsf{C} \times \mathbb{R}$ does not have the Effros property.

Proof. Let $(\mathcal{C}_n)_n$ be a sequence of clopen partitions of C such that mesh $\mathcal{C}_n < {}^1\!/_n$ and $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$. Consider the open cover

$$\mathcal{U} = \{C \times (-n, n) : C \in \mathcal{C}_n, n \in \mathbb{N}\}$$

of X, and let $\mathcal{V}$ be an arbitrary open refinement of $\mathcal{U}$. Let x and y be distinct elements of C for which there exists an element $V \in \mathcal{V}$ containing both $(x, 0)$ and $(y, 0)$. Let $f\colon X \to X$ be a homeomorphism such that $f\big((x, 0)\big) = (y, 0)$. Then evidently $f[\{x\} \times \mathbb{R}] = \{y\} \times \mathbb{R}$. Pick $n \in \mathbb{N}$ so large that ${}^1\!/_n \leq \varrho(x, y)$. Consider the points $z = (x, n)$ and $f(z) = (y, t)$. We claim that there does not exist an element $U \in \mathcal{U}$ containing both z and $f(z)$. Striving for a contradiction, assume that for m and $C \in \mathcal{C}_m$ we have

$$(x, n), (y, t) \in C \times (-m, m).$$

Since $x, y \in C$ and $1/n \leq \varrho(x, y)$ it follows that $m \leq n$. On the other hand, $n \in (-m, m)$, i.e., $n < m$. This is a contradiction. □

Problem 2. Let X be locally compact, locally connected and homogeneous. Does X have the Effros property?

The local compactness is essential in this question, as will be shown at the end of this section.

(A) Unique homogeneity. A space X is called *uniquely homogeneous* provided that for all $x, y \in X$ there is a *unique* homeomorphism moving x to y. So a uniquely homogeneous space is a space which is 'barely' homogeneous.

Theorem 2.3.4. *If X is compact and uniquely homogeneous then $|X| \leq 2$.*

Proof. Fix a point $x \in X$, and define $\gamma_x \colon \mathcal{H}(X) \to X$ by $\gamma_x(f) = f(x)$. By Theorem 2.1.10, γ_x is an open surjection. By unique homogeneity, γ_x is one-to-one. Hence γ_x is a homeomorphism. Hence $\mathcal{H}(X)$ is a compact topological group. So X is a uniquely homogeneous topological group.

For every $x \in X$ consider the inner isomorphism $y \mapsto x^{-1}yx$ of X. It sends the neutral element of X onto itself, hence must be the identity on X by unique homogeneity. We conclude that X is abelian.

Consider the isomorphism $x \mapsto x^{-1}$ of X. This homeomorphism also sends the neutral element of X onto itself, so is again the identity by unique homogeneity. This proves that $x = x^{-1}$ for every x, i.e., X is Boolean.

So by the remarks we made on Page **??**, X is finite or the Cantor set. If X is finite, then clearly $|X| \leq 2$ by unique homogeneity. The Cantor set is evidently not uniquely homogeneous, so we are done. □

Problem 3. Does there exist a Polish, uniquely homogeneous space?

We will present below an example of a uniquely homogeneous Baire space which admits the structure of a Boolean group. This explains why we ask in Problem 3 for a Polish space. The example will also be an example of a homogeneous space not having the $\mathcal{U}$–$\mathcal{V}$ condition in Problem 1.

2.4. Strong forms of homogeneity

In this section we will present some interesting applications of Theorem 2.1.10.

(A) Strong forms of homogeneity. A space X is *n-homogeneous* if for all subsets $F, G \subseteq X$ of size n there is a homeomorphism $f \colon X \to X$ such that $f[F] = G$. We will prove that if X is n-homogeneous for some $n > 1$, then X is $(n-1)$-homogeneous. As we will show, this has in fact nothing to do with topology, it is a combinatorial result.

If X is a set and $n \in \mathbb{N}$ then $[X]^n = \{A \subseteq X : |A| = n\}$. The symbol

$$\omega \to (\omega)^n_r$$

means that if a countably infinite set S and $[S]^n = \bigcup_{i=1}^r A_i$ then there are an infinite $T \subseteq S$ and $i \le r$ such that $[T]^n \subseteq A_i$. We say that T is a *homogeneous set* for the cover $\{A_1, \ldots, A_r\}$.

Theorem 2.4.1. $\omega \to (\omega)^n_r$.

Proof. Assume that S is a countably infinite set and let $\{A_1, \ldots, A_r\}$ be a partition of $[S]^n$. We prove the theorem by induction on n. If $n = 1$ there is clearly an A_i which is infinite. Assume the theorem is true for $n = k$ and assume $n = k + 1$. Choose an arbitrary $s_1 \in S$. Since the theorem is true for $n = k$, there is an infinite $S_1 \subseteq S \setminus \{s_1\}$ and an $i_1 \le r$ such that if $X \in [S_1]^k$ then $\{s_1\} \cup X \in A_{i_1}$. Choose $s_2 \in S_1 \setminus \{s_1\}$. Since the theorem is true for $n = k$, there is an infinite $S_2 \subseteq S_1 \setminus \{s_2\}$ and an $i_2 \le r$ such that if $X \in [S_2]^k$ then $\{s_2\} \cup X \in A_{i_2}$. Choose $s_3 \in S_2 \setminus \{s_2\}$. Etc. Since r is finite, there is an $i \le r$ and an infinite subset J of $\mathbb{N}$ such that $i_j = i$ for all $j \in J$. Clearly, if $T = \{s_j : j \in J\}$ then $[T]^{k+1} \subseteq A_i$. □

Let X be a set with infinite subset Y. If $\mathcal{G}$ is a group of bijections of X then $\mathcal{G}$ is *n-transitive* over Y if for all $A, B \in [Y]^n$ there is an element $g \in \mathcal{G}$ such that $g[A] = B$ (it is not required that $g[Y] = Y$).

Theorem 2.4.2. *Let X be a set with infinite subset Y. If $\mathcal{G}$ is n-transitive over Y and $n > 1$ then $\mathcal{G}$ is $(n-1)$-transitive over Y.*

Proof. Let $A, B \in [Y]^{n-1}$ be arbitrary, and pick $S \in [Y]^\omega$ such that $B \subseteq S$. Let $\mathcal{A} = \{T \in [S]^{n-1} : (\exists\, g \in \mathcal{G})(g[A] = T)\}$ and $\mathcal{B} = [S]^{n-1} \setminus \mathcal{A}$.

Case 1. There is an infinite $E \subseteq S$ such that $[E]^{n-1} \subseteq \mathcal{A}$.

We may assume that E is maximal with the property that $[E]^{n-1} \subseteq \mathcal{A}$. We claim that $E = S$. Striving for a contradiction, assume that there is an element $x \in S \setminus E$. Let $C \in [E \cup \{x\}]^{n-1}$ be arbitrary such that $x \in C$. There are distinct $\alpha, \beta \in E \setminus C$. There is by assumption an element $g \in \mathcal{G}$ such that

$$g[C \cup \{\alpha\}] = (C \setminus \{x\}) \cup \{\alpha, \beta\}.$$

But then $g[C] \subseteq E$, hence there is an element $h \in \mathcal{G}$ such that $h[A] = g[C]$. So $g^{-1} \circ h[A] = C$. This clearly implies that $[E \cup \{x\}]^{n-1} \subseteq \mathcal{A}$, hence $E = S$. But then $B \in \mathcal{A}$, i.e., there is $f \in \mathcal{G}$ with $f[A] = B$.

Case 2. There is an infinite $F \subseteq S$ such that $[F]^{n-1} \subseteq \mathcal{B}$.

Pick $\alpha \in F$ and $B' \in [F \setminus \{\alpha\}]^n$. There is by assumption an element $g \in \mathcal{G}$ such that $g[A \cup \{\alpha\}] = B'$. But then $g[A] \subseteq F$, hence $g[A] \notin \mathcal{B}$ which is a contradiction.

So we are done by Theorem 2.4.1. □

The following natural strengthening of n-homogeneity does not seem to be combinatorial. A space X is *strongly n-homogeneous* if for all subsets $F, G \subseteq X$ of size n, every bijection $f: F \to G$ can be extended to a homeomorphism $\bar{f}: X \to X$. It is clear that every strongly n-homogeneous space is n-homogenous and strongly $(n-1)$-homogeneous if $n > 1$.

Let X be a space. For every m, put

$$F_m(X) = \{x \in X^m : (\forall\, i < j \leq m)(x_i \neq x_j)\}.$$

Exercise 2.4.3. *Prove by induction on m that $F_m(X)$ is connected if if no set of $m-1$ points of X separates X. How about the converse?*

For $n = 1$, a space X is n-homogeneous if and only if it is strongly n-homogeneous. Burgess [**?**] asked whether there is a continuum different from a simple closed curve which, for some n, is n-homogeneous but not strongly n-homogeneous.

Theorem 2.4.4. *Let X be an n-homogeneous compact space such that no set of $n-1$ points separates X. Then X is strongly n-homogeneous.*

Proof. It is clear that $F_m(X)$ is open in X^m, and hence is locally compact. Let G be the symmetric group on $\{1, 2 \ldots, n\}$ with the discrete topology. Define an action of $\mathcal{H}(X) \times G$ on $F_n(X)$, as follows:

$$\big((h, \pi), (x_1, \ldots, x_n)\big) \mapsto \big(h(x_{\pi(1)}), \ldots, h(x_{\pi(n)})\big).$$

It is clear that this action is continuous, and moreover transitive since X is n-homogeneous. Fix an arbitrary element $(x_1, \ldots, x_n) \in F_n(X)$. By Theorem 2.1.10 it follows that the function $\mathcal{H}(X) \times G \to F_n(X)$ defined by

$$T: (h, \pi) \mapsto \big(h(x_{\pi(1)}), \ldots, h(x_{\pi(n)})\big)$$

is open. So for every $\pi \in G$ we have that the set $A_\pi = T[\mathcal{H}(X) \times \{\pi\}]$ is open in $F_n(X)$. If $\pi \neq \pi'$ then clearly $A_\pi = A_{\pi'}$ or $A_\pi \cap A_{\pi'} = \emptyset$. So by connectivity (Exercise 2.4.3), $A_\pi = F_n(X)$ for every $\pi \in G$. This evidently implies that X is strongly n-homogeneous. □

Corollary 2.4.5. *If X is an n-homogeneous continuum, then X is strongly n-homogeneous or X is the circle.*

Proof. For $n = 1$ there is nothing to prove, so assume that $n > 1$. By Theorem 2.4.4 it follows that we may assume without loss of generality that there is a set E in X of size $n-1$ which separates X. Since X is n-homogeneous,

X is m-homogeneous for every $m \leq n$ by Theorem 2.4.2. As a consequence, X is homogeneous. But a homogeneous continuum which is separated by a countable set, is the circle by BURGESS [**?**, Theorem 13]. □

(B) 2-homogeneity and local connectivity.

We will now present two different proofs of the interesting fact that 2-homogeneous continua are locally connected. The first one is based on Theorem 2.1.10 and is a good illustration of how that theorem can be applied. But there is also a simpler direct argument that avoids Theorem 2.1.10. We will also present that argument.

Theorem 2.4.6. *Every 2-homogeneous continuum is locally connected.*

Proof. Let X be a 2-homogeneous continuum. Suppose first that X is not strongly 2-homogeneous. By Theorem 2.4.4 it consequently follows that X contains a cutpoint, say c. But X has also two distinct noncut points (VAN MILL [**?**, Exercise A10.6]), say a and b. By 2-homogeneity, there is a homeomorphism $f\colon X \to X$ such that $f[\{a,c\}] = \{a,b\}$. Since $f(c)$ is a cutpoint of X, this is a contradiction.

So we may assume without loss of generality that X is strongly 2-homogeneous.

Claim 1. Given a point $x \in X$ and an open neighborhood U of x there exists $\delta > 0$ such that if $a, b \in X \setminus U$ and $\varrho(a,b) < \delta$ then there there exists a homeomorphism h of X such that $h(x) = x$, $h(a) = b$ and $\hat{\varrho}(h, 1_X) < \varepsilon$.

Proof. The proof is similar to the proof of Theorem 2.3.1. Consider the closed subgroup

$$\mathcal{G} = \{h \in \mathcal{H}(X) : h(x) = x\}$$

of $\mathcal{H}(X)$. Since X is strongly 2-homogeneous, $\mathcal{G}$ acts transitively on $X \setminus \{x\}$. Since both $\mathcal{G}$ and $X \setminus \{x\}$ are Polish, we are in a position to apply Theorem 2.1.10. Now let $\varepsilon > 0$, and let $V = \{f \in \mathcal{G} : \hat{\varrho}(f, 1_X) < {}^1\!/\!_2\varepsilon\}$. By micro-transitivity, the set Vz is open in $X \setminus \{x\}$ for every $z \in X \setminus \{x\}$. Let $\delta > 0$ be a Lebesgue number for the open cover

$$\{Vz : z \in X \setminus U\}$$

of the compact space $X \setminus U$. We claim that δ is as required. To prove this, let $a, b \in X \setminus B(x, \varepsilon)$ be such that $\varrho(a,b) < \delta$. There is a $z \in X$ such that $a, b \in Vz$. Pick $f, g \in V$ such that $f(x) = a$ and $g(x) = b$. Then $h = g \circ f^{-1} \in \mathcal{G}$, sends a onto b and $\hat{\varrho}(h, 1_X) < \varepsilon$. ♢

Let V be a nonempty open subset of X with component C, and let $y \in C$. There exists $\varepsilon > 0$ such that $B(y, 4\varepsilon) \subseteq V$. We assume that ε is so small that $B(y, \varepsilon) \neq X$. Let D be the component of $D(y, {}^1\!/\!_2\varepsilon)$ that contains y. Then D must meet the boundary of $D(y, {}^1\!/\!_2\varepsilon)$ (VAN MILL [**?**, A.10.5]), hence D contains a point x different from y. Let U be a closed neighborhood of x

which does not contain y. Let $\delta > 0$ be as in the claim for x, the interior of U, and ε. We may assume that δ is so small that $B(y, \delta) \cap U = \emptyset$. We claim that $B(y, \delta) \subseteq C$. To prove this, take an arbitrary element $z \in B(y, \delta)$, and let $h\colon X \to X$ be a homeomorphism such that $h(y) = z$, $h(x) = x$ and $\hat{\varrho}(h, 1_X) < \varepsilon$. Since $\operatorname{diam}(D) \leq 2\varepsilon$, it follows that the diameter of $h[D]$ is smaller than 3ε. Observe that $h(x) = x$ so $x \in h[D]$. It follows that $D \cup h[D]$ is connected, has diameter smaller than 4ε, and contains both y and z. Since $D(y, 4\varepsilon) \subseteq V$, we conclude that y and z both belong to the same component of V, i.e., $z \in C$. □

We now come to the second more direct proof of Theorem 2.4.6 which is of independent interest.

Lemma 2.4.7. *Let X be a space such that for all $x, y \in X$, there is a continuum C in X containing both x and y, and an open neighborhood U of C such that the component of U containing C is nowhere dense in X. Then X is first category.*

Proof. Fix and arbitrary point $p \in X$, and let $\mathcal{U}$ be a countable open base for X which is closed under finite unions. Put

$$\mathcal{V} = \{U \in \mathcal{U} : p \in U\},$$

and for every $U \in \mathcal{V}$, let C_U be the component of U that contains p. Put

$$\mathcal{W} = \{U \in \mathcal{V} : C_U \text{ is nowhere dense}\}.$$

We claim that $\bigcup_{U \in \mathcal{W}} C_U = X$, which evidently does the job since $\mathcal{U}$ is countable. To prove this, let $x \in X$. There is a continuum C in X containing both p and x, and an open neighborhood U of C such that the component D of U containing C is nowhere dense in X. There is a finite subcollection $\mathcal{F}$ of $\mathcal{U}$ such that

$$C \subseteq \bigcup \mathcal{F} \subseteq U.$$

Hence $F = \bigcup \mathcal{F}$ belong to $\mathcal{U}$ and has the property that C_F contains C and is nowhere dense, being a subset of D. So $F \in \mathcal{W}$, and $x \in C \subseteq C_F$, as required. □

Lemma 2.4.8. *Let X be a homogeneous Polish space. If X is not locally connected, then every $x \in X$ has an open neighborhood U such that every component C of U is nowhere dense in X.*

Proof. By homogeneity, it suffices to prove that there is a nonempty open subset V of X, every component of which is nowhere dense in X.

So assume the contrary that every nonempty open subset of X has a component with nonempty interior. We will derive a contradiction. Let ϱ be an admissible complete metric on X. Let U_1 be a nonempty open set in X with diameter less than 2^{-1}, and pick a component C_1 of U_1 the closure of which has nonempty interior. Pick a nonempty open subset U_2 of X such that $\overline{U_2} \subseteq \overline{C_1} \cap U_1$ and $\operatorname{diam} U_2 < 2^{-2}$. Let C_2 be a component of U_2 with nonempty interior. Continuing in this way yields a sequence $(U_n)_n$ of open sets such that $\bigcap_{n=1}^{\infty} U_n$ contains a single point, say x (use that ϱ is complete). Clearly, X is locally connected at x. But then by homogeneity, X is locally connected at all points, which is a contradiction. □

Theorem 2.4.9. *Let X be a homogeneous compact space such that for all $x, y \in X$ and $\varepsilon > 0$ there are a homeomorphism $h\colon X \to X$ and a continuum C in X of diameter less than ε such that $h(x), h(y) \in C$. Then X is locally connected.*

Proof. Striving for a contradiction, assume that X is not locally connected. Pick arbitrary $x, y \in X$, and for every n a homeomorphism $h_n\colon X \to X$ and a continuum C_n of diameter less than $1/n$ such that $h_n(x), h_n(y) \in C_n$. We may assume without loss of generality that the sequence $(C_n)_n$ converges to a point $p \in X$ (with respect to the Hausdorff metric, cf., van Mill [**?**, 1.11.14]). By Lemma 2.4.8 there is a neighborhood U of p every component of which is nowhere dense in X. There is n such that $C_n \subseteq U$. Hence the component of C_n in U is nowhere dense in X. So $x, y \in C' = h_n^{-1}[C_n]$ and the component of C' in $h_n^{-1}[U]$ is nowhere dense in X. So X is first category by Lemma 2.4.7, which is a contradiction. □

Corollary 2.4.10. *Let X be a 2-homogeneous continuum. Then X is locally connected.*

Proof. Let $x \in X$ be arbitrary and $\varepsilon > 0$. We assume without loss of generality that $D(x, 1/2\varepsilon) \neq X$. Let D be the component of $D(x, 1/2\varepsilon)$ that contains x. Then D meets the boundary of $D(x, 1/2\varepsilon)$ (van Mill [**?**, A.10.5]), hence D contains a point y different from x. Since the diameter of D is at most ε, we are done by 2-homogeneity and Theorem 2.4.9. □

Question 1. Let X be connected, Polish and 2-homogeneous. Is X locally connected?

2.5. Homogeneous compacta are products

The aim of this section is to prove the surprising result that every homogeneous compactum is the product of a compact zero-dimensional (homogeneous) compactum and a (homogeneous) continuum.

Let X be a space. Call a collection $\mathcal{A}$ of subsets of X *invariant* if for all $h \in \mathcal{H}(X)$ and $A \in \mathcal{A}$ we have $h[A] \in \mathcal{A}$. Observe that both the collection of all clopen subsets of X and the collection of components of X are invariant.

If $\mathcal{A}$ is any family of subsets of X, and $U \subseteq X$ then

$$R(U, \mathcal{A}) = \bigcup\{A \in \mathcal{A} : A \cap U \neq \emptyset\}.$$

Lemma 2.5.1. *Let G act micro-transitively on X. If $\mathcal{A}$ is invariant and U is open then $R(U, \mathcal{A})$ is open.*

Proof. Let $A \in \mathcal{A}$ be such that $A \cap U \neq \emptyset$, say $x \in A \cap U$. We have to find a neighborhood of A which is contained in $R(U, \mathcal{A})$. Put $V = \gamma_x^{-1}[U]$. Then V is an open neighborhood of e in G. Let

$$W = \bigcup\{Vz : z \in A\}.$$

Since G acts micro-transitively, W is an open neighborhood of A. We claim that $W \subseteq R(U, \mathcal{A})$. To this end, pick an arbitrary element $y \in W$, say $y \in Vz$ for certain $z \in A$. There is by assumption an element $g \in V$ such that $gz = y$. Since $\{ga : a \in A\} \in \mathcal{A}$, $gx \in U$ and $gz = y$, this proves that $y \in R(U, \mathcal{A})$, which is as required. □

If X is a space and $x \in X$ then the *quasi-component* C_x of x is the intersection of all clopen subsets of X that contain x. Obviously, quasi-components are closed, and the collection

$$\mathcal{C} = \{C_x : x \in X\}$$

is an invariant partition of X. Let $\pi\colon X \to X/\mathcal{C}$ be the function sending each $x \in X$ onto C_x. We endow $X/\mathcal{C}$ with the quotient topology. We first claim that $X/\mathcal{C}$ is Hausdorff. This is easy. Simply observe that if $C_x \neq C_y$ then there is a clopen set E that contains x but misses y. It consequently suffices to remark that every clopen subset of X is the union of quasi-components.

Lemma 2.5.2. *Let G act micro-transitively on X. Then $\pi\colon X \to X/\mathcal{C}$ is open.*

Proof. Since we endowed $X/\mathcal{C}$ with the quotient topology, this follows directly from Lemma 2.5.1. □

Corollary 2.5.3. *Let G act micro-transitively on the locally compact space X. Then $X/\mathcal{C}$ is separable metrizable, locally compact and zero-dimensional.*

Proof. We will first prove that $X/\mathcal{C}$ is regular. To this end, Let $C_x \in X/\mathcal{C}$, and let $\mathcal{B} \subseteq X/\mathcal{C}$ be closed such that $C_x \notin \mathcal{B}$. Observe that $C_x \cap B = \emptyset$, where $B = \bigcup \mathcal{B}$ and that B is closed. There is a neighborhood U of x such that $\overline{U}$ is compact and misses B. By Lemma 2.5.2, $\pi[\overline{U}]$ is a compact neighborhood of C_x in $X/\mathcal{C}$ which misses $\mathcal{B}$. Since we already observed that $X/\mathcal{C}$ is Hausdorff, this proves that $\pi[\overline{U}]$ is closed in $X/\mathcal{C}$. Hence $X/\mathcal{C}$ is locally compact and regular.

Since $\pi\colon X \to X/\mathcal{C}$ is open, if $\mathcal{U}$ is a countable open base for X then $\pi[U] : U \in \mathcal{U}\}$ is a countable open base for $X/\mathcal{C}$. Hence $X/\mathcal{C}$ is second countable, and hence a separable metrizable space by Urysohn's Metrization Theorem (see Page 1).

We finally check that $X/\mathcal{C}$ is zero-dimensional. If $C_x \neq C_y$ then there is a clopen set E which contains C_x but misses C_y. Since clopen sets are unions of quasi-components and $\pi\colon X \to X/\mathcal{C}$ is open, it follows easily that $\pi[E]$ and $\pi[X \setminus E]$ are disjoint clopen sets in $X/\mathcal{C}$. As a consequence, $X/\mathcal{C}$ is totally disconnected, hence zero-dimensional by VAN MILL [**?**, Exercise 1.5.12]. □

We now come to the main result in this section.

Theorem 2.5.4. *Every locally compact homogeneous space is the product of a connected space and a zero-dimensional space.*

Proof. Let X be locally compact and homogeneous. First observe that there is a Polish group G that acts transitively on X (see Page 37). This action is micro-transitive by Theorem 2.1.10 since locally compact spaces are Baire spaces (VAN MILL [**?**, Page 482]).

Let Y be a fixed quasi-component of X, and let $Z = X/\mathcal{C}$. We claim that $X \approx Y \times Z$. Let $q \in Y$, and consider the continuous open surjection $\gamma_q\colon \mathcal{H}(X) \to X$.

We already know that Z is zero-dimensional (Corollary 2.5.3). The function $\pi\colon X \to Z$ is continuous and open by Lemma 2.5.2. The composition $\varphi = \pi \circ \gamma_q\colon \mathcal{H}(X) \to Z$ is consequently a continuous open surjection as well. So by **??**, there is a continuous function $s\colon Z \to \mathcal{H}(X)$ such that $\varphi \circ s = 1_Z$. Define $\psi\colon Y \times Z \to X$ by the formula

$$\psi(y, z) = s(z)(y).$$

It is clear that ψ is continuous. To prove that ψ is injective, assume $\psi(y_1, z_1) = \psi(y_2, z_2)$ for some $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$. Observe that since quasi-components are invariant for $i = 1, 2$ we have

$$\pi\big(s(z_i)(q)\big) = \pi\big(s(z_i)(y_i)\big).$$

Hence

$$z_i = \varphi\big(s(z_i)\big) = \pi\big(\gamma_q\big(s(z_i)\big)\big) = \pi\big(s(z_i)(q)\big) = \pi\big(s(z_i)(y_i)\big) = \pi\big(\psi(y_i, z_i)\big).$$

Hence $z_1 = z_2$. As $s(z_1) \in \mathcal{H}(X)$ we also get $y_1 = y_2$.

We next claim that ψ is surjective. That is easy since

$$(*) \qquad x = \psi\big(s\big(\pi(x)\big)^{-1}(x), \pi(x)\big)$$

for every $x \in X$.

We finally claim that ψ^{-1} is continuous. To this end, let $x_n \to x$ in X, and write $x = \psi(y, z)$ and $x_n = \psi(y_n, z_n)$ for all n. Then

$$\pi(x_n) = z_n \to z = \pi(x)$$

in Z $\big($use $(*)\big)$, hence $s(z_n) \to s(z)$ in $\mathcal{H}(X)$. Since $\mathcal{H}(X)$ is a topological group, $s(z_n)^{-1} \to s(z)^{-1}$ in $\mathcal{H}(X)$, hence

$$y_n = s(z_n)^{-1}(x_n) \to s(z)^{-1}(x) = y$$

in Y.

It remains to show that Y is connected. Striving for a contradiction, assume that we can write $Y = U \cup V$, where U and V are non-empty, relatively open and disjoint. Then $U \times Z$ and $V \times Z$ is a clopen partition of X. This contradicts the fact that Y is a quasi-component of X. $\square$

It is well-known that components and quasi-components agree in compact spaces (van Mill [**?**, A.10.1]). Simple examples show that this is not true in general. Surprisingly, Theorem 2.5.4 shows that for locally compact homogeneous spaces there are no problems.

Corollary 2.5.5. *Let X be a homogeneous locally compact space. Then for every $x \in X$ the component of x coincides with the quasi-component of x.*

Problem 4. Let X be a homogeneous Polish space. Is X the product of a connected space and a totally disconnected (zero-dimensional) space? NO!

Later we will present examples of σ-compact homogeneous spaces that do not admit the structure of a product. (Nul rij van Z-set Hilbert kubussen in de Hilbert kubus. Misschien via een resolutie doen?)