Fourier Series

$f \in C^0_{pw}([-\pi, \pi])$

- a_n, b_n all exist
- $\displaystyle\sum_{n=1}^{\infty}(a_n^2 + b_n^2) < \infty$
- $a_n, b_n \to 0, \quad n \to \infty$
- $\displaystyle\pi\Big[\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)\Big] \le \int_{-\pi}^{\pi} f^2(x)dx$

$f \in C^1_{pw}([-\pi, \pi])$

- $\forall\ x \in [-\pi, \pi]$

$$\frac{f(x+0) + f(x-0)}{2} = FS(f)(x)$$

- If the function is continuous at x then

$$f(x) = FS(f)(x)$$

$$f \in C^0(\mathbf{R}),\ f' \in C^0_{pw}([-\pi,\pi])$$

- a'_n, b'_n Fourier series of f'
- $a_n = -\frac{1}{n} b'_n, \quad b_n = \frac{1}{n} a'_n$
- $\sum_{n=1}^{\infty} (|a_n| + |b_n|) \leq C \Big(\sum_{n=1}^{\infty} (a'^2_n + b'^2_n) \Big)^{1/2} < \infty$
- $\sum_{n=N+1}^{\infty} (|a_n| + |b_n|) \to 0$
- $\max_{x \in [-\pi,\pi]} |FS_N(f)(x) - f(x)|$

$$\leq C \sum_{n=N+1}^{\infty} (|a_n| + |b_n|) \to 0$$

Conclusion: $f \in C^1_{pw}([-\pi, \pi]) \cap C^0(\mathbf{R})$

then the Fourier series $FS(f)$

converges to f uniformly in

$x \in [-\pi, \pi]$

Conversely, if $$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$$

then the Fourier series converges

uniformly to f and f is continuous on $\mathbf{R}$.

$f \in C^0_{pw}([-\pi, \pi])$

Theorem: $FS_n(f) \to f$ in L^2

if and only if

$$\pi\Big[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\Big] = \int_{-\pi}^{\pi} f^2(x)dx$$

This is called Parceval's inequality.

It shows the Fourier orthogonal system is complete with respect to piecewise continuous functions.

Integration:

$$f \in C^0_{pw}([-\pi, \pi])$$

$$\int_0^x f(x)dx =$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \frac{a_n \sin(nx) - b_n \cos(nx)}{n}$$

The integral is continuous.

Differentiation:

$$f \in C^0(\mathbf{R}),\ f' \in C^0_{pw}([-\pi, \pi])$$

$$FS(f') = \sum_{n=1}^{\infty} n\big(b_n \cos(nx) - a_n \sin(nx)\big)$$

In particular:

$$na_n, nb_n \to 0, \quad n \to \infty.$$

If $f \in C^{m-1}(\mathbf{R}),\ f^{(m)} \in C^0_{pw}([-\pi, \pi])$

then

$$n^m a_n, n^m b_n \to 0, \quad n \to \infty.$$

Conversely, if

$$|n^m a_n| \leq M, \ |n^m b_n| \leq M, \quad m \geq 2,$$

then the Fourier series defined by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

defines a function f in

$$C^{m-2}(\mathbf{R})$$