Midterm/Final exam Afdeling Wiskunde
Introduction Partial Differential Equations Faculteit der Exacte Wetenschappen
for students Mathematics and Physics Vrije Universiteit Amsterdam

Date: Monday December 20 2004, 9:30– 11:30/12:30 (2/3 hours)
Instructions: 3/5 problems; *motivate all answers.*
No calculators, no books, no formula sheets.
For midterm exam 2: problems 1 through 3, total time 2 hours.
For the final exam: problems 1 through 5, total time 3 hours.

1. (a) Calculate Fourier transform of the following functions

$$f(x) = \frac{1}{x^2 - 2x + 2}, \quad g(x) = (x^2 + x)e^{-x^2}.$$

   (b) Compute, using the definition, the Laplace transform of

$$g(x) = \sin(x).$$

   (c) Use the convolution formula for the Fourier transform to find

$$\mathcal{F}^{-1}\Big(\frac{\mathcal{F}(\alpha)}{\alpha^2 + 8\alpha + 20}\Big).$$

2. Consider the fourth order equation

$$u_t = -u_{xxxx}, \quad x \in [0, \pi], \quad t > 0$$

with initial condition $u(0, x) = f(x) \in C^0([0, \pi])$ and boundary conditions $u(t, 0) = u_{xx}(t, 0) = 0$, and $u(t, \pi) = u_{xx}(t, \pi) = 0$.

   (a) Give the general solution using separation of variables.
   (b) Derive the formulas for the Fourier coefficients in terms of the initial function f.
   (c) Let $f(x) = x$ on $[0, \pi]$. Compute the solution $u(t, x)$.
   (d) Prove that $\lim_{t\to\infty} u(t, x) = 0$ uniformly in $x \in [0, \pi]$.

3. Consider the elliptic equation on a domain $\Omega \subset \mathbb{R}^2$

$$Lu = \Delta u + \mathbf{x} \cdot \nabla u = f, \qquad \mathbf{x} \in \Omega,$$
$$Bu = \frac{\partial u}{\partial n} = g, \qquad \mathbf{x} \in \partial\Omega,$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{n}$ the outward unit normal on $\partial\Omega$, and $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$.

(a) Show that the adjoint differential operator L^* is given by

$$L^*\phi = \Delta\phi - \mathbf{x}\cdot\nabla\phi - 2\phi,$$

by establishing the identity

$$\int_\Omega [\phi L\psi - \psi L^*\phi] d_\Omega \mathbf{x} = 0,$$

for all $\phi, \psi \in C_0^\infty(\Omega)$. Recall that $C_0^\infty(\Omega)$ are infinitely smooth functions whose support is strictly contained in Ω; no boundary contributions. (Hint: use the fact that $\mathrm{div}(\phi\psi\mathbf{x}) = 2\phi\psi + \phi\mathbf{x}\cdot\nabla\psi + \psi\mathbf{x}\cdot\nabla\phi$.)

(b) Derive the Green's idenity

$$\int_\Omega [vLu - uL^*v] d_\Omega \mathbf{x} = ...$$

(c) Find the adjoint boundary operator B^*.

(d) Formulate the differential equation for the Green's function $G(\mathbf{x}, \mathbf{y})$.

(e) Assuming that the Green's function G exists, write down a representation formula for the solution $u(\mathbf{x})$.

4. (a) Determine the general solution of the following first order equation

$$yu_x + xu_y = 0.$$

(b) Find the solution that matches the the initial condition $u(0, y) = \frac{1}{1+y^4}$.

5. (a) Calculate the Fourier series of $|x|$ on $[-\pi, \pi]$.

(b) Compute the sine Fourier series of the function $f(x) = x$ on $[0, \pi]$.

(c) Explain the different rates of convergence of the Fourier series in (a) and (b).

The cosine Fourier series for f are $\sum_{n=0}^{\infty} a_n \cos(nx)$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx$, $n \geq 1$, and $a_0 = \frac{1}{\pi}\int_o^\pi f(x)dx$.

The sine Fourier series for f are $\sum_{n=1}^{\infty} a_n \sin(nx)$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx$, $n \geq 1$.

The Fourier series for f are $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)$.

The Fourier transform of f is given by $\mathcal{F}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx$, and the Laplace transform by $\hat{f}(s) = \int_0^\infty f(x)e^{-sx}dx$.

Good luck