Assignment III Afdeling Wiskunde
Introduction Partial Differential Equations Faculteit der Exacte Wetenschappen
for students Mathematics and Physics Vrije Universiteit Amsterdam

Due: Wednesday December 13, 2006.
Instructions: include input and output; *motivate all answers.*
Individual assignment, written in English.

Given the eigenvalue problem

$$\begin{aligned} \varphi'' &= -\lambda\varphi, \quad x \in (0,\pi), \\ \varphi(0) &= \varphi(\pi) = 0. \end{aligned}$$

The eigenfunctions are given by $\varphi_n(x) = \sin(nx)$, and the associated eigenvalues are $\lambda_n = n^2$, $n = 1, 2, ...$ Given a function $v \in L^2(0,\pi)$, then $v(x) = \sum_{n=1}^{\infty} v_n \sin(nx)$, with $v_n = \frac{2}{\pi}\int_0^{\pi} v(x)\sin(nx)dx$, and $\int_0^{\pi} v^2(x)dx = \frac{\pi}{2}\sum_{n=1}^{\infty} v_n^2 < \infty$.

Consider the equation

$$\begin{aligned} -u'' + u &= f(x), \quad x \in (0,\pi), \\ u(0) &= u(\pi) = 0. \end{aligned}$$

From general theory we know that there exists a Green's function $G(x,y) = G(y,x)$ on $[0,\pi] \times [0,\pi]$, such that

$$u(x) = \int_0^{\pi} G(x,y)f(y)dy.$$

1. Use the above described eigenfunction expansion and use the eigenfunction expansion method to derive a Fourier series for the Green's function G.

2. Use MAPLE or MATHEMATICA with $N = 100$ modes to plot the Green's function G in a 3d graph.

3. Use the above to compute and plot the solution of $-u'' + u = 1$. Compare the graphs of the approximate solution with the exact solution

$$u(x) = \frac{\cosh(\pi) - 1}{\sinh(\pi)}\sinh(x) - \cosh(x) + 1.$$

Good luck