# Meta-level Selection Techniques for the Control of Default Reasoning[+]

Victor Allis[1]
Yao-Hua Tan[2]
Jan Treur[1]

[1]Vrije Universiteit Amsterdam, Department of Mathematics and Computer Science, Artificial Intelligence Group
De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
Emails: {victor,treur}@cs.vu.nl

[2]EURIDIS, Erasmus Universiteit Rotterdam,
P.O. Box 1738, 3000 DR Rotterdam. The Netherlands
Email: ytan@euridis.fbk.eur.nl

**Abstract**

The problem how to control default reasoning is addressed. In earlier work it was described how selection functions (on default conclusions) added to default logic can be exploited to describe control. In this paper specific properties of selection functions are described. Moreover, we investigate under which conditions selection functions can be expressed in the form of declarative control knowledge at the meta-level. Furthermore, we show that normal default logic with additional control by selection functions is (in some sense) equivalent to default logic in general. Finally, an alternative approach to control is introduced based on inhibition relations between default rules.

## 1 Introduction

In a defeasible reasoning process often some type of non-determinism plays a role: there exist more than one extension (in terms of default logic), and the reasoning should come up with a construction of one of them. This implies a serious problem concerning the control of the reasoning. Usually some kind of branching will play a role, i.e., many reasoning paths can be taken in the search space of information states where the reasoning takes place. In a successful defeasible reasoning process a choice will be made between the possible branches. The chosen branch should have internal consistency (i.e., it should not defeat itself later on), and should lead to a construction of the preferred extension among the extensions that are possible. In [TT92] it is described how constructive default logic can be exploited to describe the control issue as sketched. Here the notion of a selection function plays a central role.

In this paper we will establish that the use of selection functions allows one to restrict to normal default theories (Section 2). Furthermore, we study specific properties of selection functions (Section 3) and we investigate under what conditions selection functions can be expressed in the form of declarative control knowledge at the meta-level (Section 4). Finally we introduce another selection technique: inhibition relations (Section 5).

Constructive default logic is default logic in which the normal fixed-point extensions are replaced by so-called *constructive extensions* (see [TT92]). Constructive extensions are parameterized by a selection function. A selection function can be considered as a setting of a set of control parameters that may guide the reasoning. We assume familiarity with Reiter's default logic (see [Rei80]). Let **Th(S)** denote the deductive closure of a set **S** of **L**-formulas, i.e. $\mathbf{Th(S) = \{\varphi \mid S \vdash \varphi\}}$.

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**Definition 1.1**
A set of sentences $\mathbf{E}$ is an *extension* of the default theory $\Delta = < \mathbf{W}, \mathbf{D} >$, if $\mathbf{E} = \bigcup_{i=0}^{\infty} \mathbf{E_i}$, where each layer $\mathbf{E_i}$ is defined as follows:
for $\mathbf{i = 0}$
$\mathbf{E_0 = W}$
and for $\mathbf{i \geq 0}$
$\mathbf{E_{i+1} = Th(E_i) \cup \{ \omega \mid (\alpha : \beta_1,\ldots, \beta_n / \omega) \in D, \alpha \in E_i, \text{ and } \neg \beta_1,\ldots, \neg \beta_n \notin E\}}$.

Note that Definition 1.1 is a fixed-point definition, which in principle is not constructive. In the definition of the layer $\mathbf{E_{i+1}}$ it is required that the formulas $\neg\beta_1,\ldots, \neg\beta_n$ are not contained in $\mathbf{E}$. Hence, the definition of each layer $\mathbf{E_{i+1}}$ depends on the *final outcome* $\mathbf{E}$.

A selection function, which will be denoted by $\boldsymbol{\sigma}$, selects a subset of default conclusions from the set of all default conclusions that can be derived at a certain level. Indices $\mathbf{i}$ from an index set $\mathbf{I}$ are added to indicate that the selection is made at the $\mathbf{i}$-th reasoning step.

**Definition 1.2**
a) Let $\mathbf{L}$ be a logic and let $\mathbf{U}$ be a set of ground formulas of $\mathbf{L}$. Let $\mathbf{P(U)}$ denote the set of all subsets $\mathbf{V \subseteq U}$, and let $\mathbf{I}$ be an index set. The function

$$\boldsymbol{\sigma} : \mathbf{I \times P(U) \rightarrow P(U)}$$

is called a *selection function* if for every subset $\mathbf{V}$ of $\mathbf{U}$ and every index $\mathbf{i}$ : $\boldsymbol{\sigma}\mathbf{(i, V) \subseteq V}$.
b) Let $\mathbf{D}$ be a set of defaults. The *set of consequences* of $\mathbf{D}$, denoted as $\mathbf{Cons(D)}$, is defined by

$$\mathbf{Cons(D) = \{\omega \mid (\alpha : \beta_1,\ldots, \beta_n / \omega) \in D\}}.$$

Suppose $\Delta = < \mathbf{W, D} >$ is a default theory and $\boldsymbol{\sigma} : \mathbf{I \times P(U) \rightarrow P(U)}$ is a selection function. We call $\boldsymbol{\sigma}$ a *selection function related to* $\Delta$ if $\mathbf{U = Cons(D)}$. The set of selection functions related to $\Delta$ is denoted by $\mathbf{Sel(\Delta)}$.

In this paper the index set $\mathbf{I}$ will always be the set of natural numbers, with the usual ordering. Instead of $\boldsymbol{\sigma}\mathbf{(i, V)}$ we will also write simply $\boldsymbol{\sigma}_\mathbf{i}\mathbf{(V)}$. It is possible that $\mathbf{i < j}$, and $\boldsymbol{\sigma}_\mathbf{i}\mathbf{(V)} \neq \boldsymbol{\sigma}_\mathbf{j}\mathbf{(V)}$. Next we define the notion of a constructive extension. This is an extension of which the construction is controlled by a selection function $\boldsymbol{\sigma}$.

**Definition 1.3**
Let $\Delta = < \mathbf{W, D} >$ be a default theory and suppose $\boldsymbol{\sigma}$ is a selection function related to $\Delta$. A set of sentences $\mathbf{C^\sigma}$ is called the $\boldsymbol{\sigma}$*-constructive extension* of the default theory $\Delta$, if $\mathbf{C^\sigma} = \bigcup_{i=0}^{\infty} \mathbf{C^\sigma{}_i}$, where the $\mathbf{C^\sigma{}_i}$ are defined as follows:

for $\mathbf{i = 0}$
$\mathbf{C^\sigma{}_0 = W}$
for $\mathbf{i \geq 0}$
$\mathbf{C^\sigma{}_{i+1} = Th(C^\sigma{}_i) \cup \sigma_{i+1}(CD^\sigma{}_{i+1})}$
where $\mathbf{CD^\sigma{}_{i+1} = Cons(D^\sigma{}_{i+1})}$ with
$\mathbf{D^\sigma{}_{i+1} = \{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in D \mid \alpha \in C^\sigma{}_i, \text{ and } \neg \beta_1,\ldots,\neg \beta_n \notin Th(C^\sigma{}_i) \}}$

For a given default theory $\Delta = < \mathbf{W, D} >$ there is a collection of such $\boldsymbol{\sigma}$-constructive extensions, parameterized by selection functions $\boldsymbol{\sigma} \in \mathbf{Sel(\Delta)}$.

**Definition 1.4**
Let $\Delta = < \mathbf{W, D} >$ be a default theory and $\boldsymbol{\sigma}$ a selection function related to $\Delta$. A default $\mathbf{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in D}$ is called *applicable* at stage $\mathbf{i}$ if $\mathbf{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in D^\sigma{}_{i+1}}$. We say a default $\mathbf{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in D}$ *in principle is used* by $\boldsymbol{\sigma}$ at stage $\mathbf{i}$ if it is applicable and $\mathbf{\omega \in \sigma_{i+1}(CD^\sigma{}_{i+1})}$. In this case we also say that the formula $\boldsymbol{\omega}$ *is selected* by $\boldsymbol{\sigma}$ at stage $\mathbf{i}$. A default $\mathbf{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in D}$ that in principle is used at stage $\mathbf{i}$ by $\boldsymbol{\sigma}$ is called *defeated* by $\boldsymbol{\sigma}$ if there is some $\mathbf{j > i}$ such that $\mathbf{\neg \beta_k \in Th(C^\sigma{}_j)}$ for some $\mathbf{k}$. We call $\boldsymbol{\sigma}$ *self-defeating* if there is some $\mathbf{i}$ and some formula $\boldsymbol{\omega}$ that is selected by $\boldsymbol{\sigma}$ at stage $\mathbf{i}$ such that all defaults $\mathbf{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in D^\sigma{}_{i+1}}$ with consequence $\boldsymbol{\omega}$ are defeated by $\boldsymbol{\sigma}$.

Notice that since $\mathbf{C^\sigma}$ is the union of all $\mathbf{C^\sigma{}_j}$ it is deductively closed. Therefore the condition in Definition 1.4 that there is some $\mathbf{j > i}$ such that $\mathbf{\neg \beta_k \in Th(C^\sigma{}_j)}$ for some $\mathbf{k}$ is equivalent to $\mathbf{\neg \beta_k \in C^\sigma}$ for some $\mathbf{k}$. This observation can easily be used to prove the following Lemma.

**Lemma 1.5**
Let $\Delta = < \mathbf{W}, \mathbf{D} >$ be a default theory and suppose $\boldsymbol{\sigma}$ is a selection function related to $\Delta$. Then the following are equivalent:
(i) $\boldsymbol{\sigma}$ is not self-defeating
(ii) For every $\mathbf{i}$ and every formula $\boldsymbol{\omega}$ that is selected by $\boldsymbol{\sigma}$ at stage $\mathbf{i}$,
there is a default $(\boldsymbol{\alpha} : \boldsymbol{\beta}_1,\ldots, \boldsymbol{\beta}_n / \boldsymbol{\omega}) \in \mathbf{D}$ such that $\boldsymbol{\alpha} \in \mathbf{C}^{\sigma}{}_{\mathbf{i}}$ and $\neg \boldsymbol{\beta}_1,\ldots, \neg \boldsymbol{\beta}_n \notin \mathbf{C}^{\sigma}$.
(iii) $\boldsymbol{\sigma}_{\mathbf{i}}(\mathbf{CD}^{\sigma}{}_{\mathbf{i}}) \subseteq \mathbf{CD}^{\sigma}{}_{\mathbf{j}}$ for $\mathbf{i} \leq \mathbf{j}$.

Being not self-defeating is a necessary but not sufficient condition for a selection function to define an extension in the sense of Definition 1.1. It is easy to give examples of selection functions that are not self-defeating, but that do not define an extension, simply because they are not exhaustive; e.g. some applicable defaults may be not selected at any stage. The notion of exhaustiveness is related to the concept of closedness under a set of defaults.

**Definition 1.6**
Let $\mathbf{D}$ be a set of defaults. The set of sentences $\mathbf{S}$ is *closed under* $\mathbf{D}$ if for any default $(\boldsymbol{\alpha} : \boldsymbol{\beta}_1,\ldots, \boldsymbol{\beta}_n / \boldsymbol{\omega}) \in \mathbf{D}$ with $\boldsymbol{\alpha} \in \mathbf{S}$ and $\neg \boldsymbol{\beta}_1,\ldots, \neg \boldsymbol{\beta}_n \notin \mathbf{S}$ it holds $\boldsymbol{\omega} \in \mathbf{S}$.
Let $\Delta = < \mathbf{W}, \mathbf{D} >$ be a default theory and suppose $\boldsymbol{\sigma}$ is a selection function related to $\Delta$. We call $\boldsymbol{\sigma}$ *exhaustive* for $\mathbf{D}$ if $\mathbf{C}^{\sigma}$ is closed under $\mathbf{D}$.

Now we are in a position to state the following result. The proof is given in the Appendix.

**Theorem 1.7**
Every extension of a default theory $\Delta = < \mathbf{W}, \mathbf{D} >$ can be obtained as a $\boldsymbol{\sigma}$-constructive extension for some selection function $\boldsymbol{\sigma}$ related to $\Delta$. More precisely, for any set of sentences $\mathbf{E}$ the following conditions are equivalent.
(i) $\mathbf{E}$ is a extension of $\Delta$
(ii) There exists a selection function $\boldsymbol{\sigma}$ related to $\Delta$ that is exhaustive and not self-defeating such that $\mathbf{E} = \mathbf{C}^{\sigma}$

Theorem 1.7 shows that, although the original definition of extensions as given by Reiter (see Definition 1.1) is a fixed point definition, the extensions of a default theory can be studied from a constructive point of view. This is a great advantage when it comes to implementing default logic. Of course, we do not claim that we have shown default logic to be more tractable than it actually is (for an analysis of the complexity problems of default logic see [KS91]). The selection functions of $\boldsymbol{\sigma}$-constructive extensions that generate extensions in the sense of Definition 1.1 might be very hard to specify, in the worst case it may even be impossible to define these selection functions in a constructive manner.

## 2 Default Logic = Normal Default Logic + Explicit Control

In this section the claim we want to defend is that all conflicts between defaults can be handled by explicit control expressed by selection functions. Reiter and Criscuolo were the first to suggest in [RC83] that semi-normal default rules were essential for adequate knowledge representation. A default rule is called *semi-normal* if its (single) justification inplies its conclusion, if its justification equals its conclusion it is called *normal*. Basically, their argument was that the justification in semi-normal default rules has to be used to block the application of the rule if it conflicts with a more specific default rule. This can be viewed as a way to code the control implicitly in the default rules themselves. Their famous example runs as follows. Consider the default theory $\Delta = < \mathbf{W}, \mathbf{D} >$ with two defaults $(\mathbf{b}: \mathbf{f}/\mathbf{f})$ and $(\mathbf{p}: \neg \mathbf{f}/ \neg \mathbf{f})$, where $\mathbf{b}$ stands for bird, $\mathbf{p}$ for penguin and $\mathbf{f}$ for can-fly. If we have the information that something is both a bird as well as a penguin, i.e. $\mathbf{W} = \{\mathbf{b}, \mathbf{p}\}$, then $\Delta$ has two extensions: $\mathbf{E}_1$ that contains the conclusion $\mathbf{f}$ and $\mathbf{E}_2$ that contains $\neg \mathbf{f}$. The first extension $\mathbf{E}_1$ is generated by the first default, and the other $\mathbf{E}_2$ by the second default. Most people are of the opinon that the first default should be overruled by the second one, because the latter one is more specific than the first one. This is called the *specificity principle*. Penguins are a subset of birds, hence the second default gives more adequate information than the first one. Reiter and Criscuolo proposed that this overruling can be obtained by adding the prerequisite of the more specific default rule to the justification of the more general rule. In this particular example, $(\mathbf{b}: \mathbf{f}/\mathbf{f})$ has to be replaced by $(\mathbf{b}: \mathbf{f} \wedge \neg \mathbf{p}/\mathbf{f})$. The first default is normal, and the latter one is semi-normal. It is clear that with this semi-normal default rule the extension $\mathbf{E}_1$ is no longer generated. The message of this example is that conflicts between default rules can be controlled by adding extra information to the justifications of default rules. The price one has to pay for this solution is that it drives us out of the computational paradise of normal default rules. Some years later Brewka argued in [Br92] that this implicit control can be replaced by adding proof paths as he did in his cumulative default logic. Brewka made a case that if the control about specificity is coded explicitly, as is done by adding the proof paths, then one does not need semi-normal defaults any more; normal default rules suffice. More recently, Brewka published several papers that further elaborate on this idea by introducing default logics that only use normal default rules, and in which the specificity conflicts between default are handled by explicit priority orderings that are added to the logic (see [Br94a] and [Br94b]). We want to generalize this perspective. Selection functions as introduced in [TT92] can be viewed as a generalization of Brewka's explicit priority orderings. The basic intuition is that instead of using justifications as techniques to control the specificity conflicts between defaults implicitly, we can as well code all the control explicitly in selection functions and restrict ourself to normal default rules only. The advantage of such a move is obvious, since computation for normal default rules is much simpler than for general default rules. Our claim could be expressed in the following equation.

**Default Logic = Normal Default Logic + Explicit Control**

In this equation explicit control has to be understood as selection function. Below we will present a proposition which supports this claim. The proposition states that if we have a default theory $\Delta = < \mathbf{W}, \mathbf{D} >$, and we replace this theory by the corresponding default theory $\Delta^* = < \mathbf{W}, \mathbf{D}^* >$ in which every default rule $(\alpha : \beta_1, \ldots, \beta_n / \omega)$ from $\mathbf{D}$ is replaced by its '*normalized*' version $(\alpha : \omega / \omega)$ in $\mathbf{D}^*$, then there is a set of selection functions that generate exactly the set of constructive extensions for $\Delta^*$ which is identical to the set of Reiter extensions of $\Delta$.

First we introduce some notation. If $\mathbf{D}$ is a set of default rules, then we say that $\mathbf{D}^*$ is the set $\mathbf{D}^* = \{ (\alpha : \omega / \omega) \mid (\alpha : \beta_1, \ldots, \beta_n / \omega) \in \mathbf{D}\}$ of *normalized* default rules. We will also say that $\mathbf{D}^*$ is the *corresponding normalized default set* of $\mathbf{D}$. If $< \mathbf{W}, \mathbf{D} >$ is a default theory, then we say that $< \mathbf{W}, \mathbf{D}^* >$ is the *corresponding normalized default theory*. Furthermore, we use the notation $\mathbf{CGD(E_i)}$ to denote the set of conclusions of generating defaults of level $\mathbf{E_i}$ of an extension $\mathbf{E}$ of a default theory $< \mathbf{W}, \mathbf{D} >$, i.e.

$$\mathbf{CGD(E_i, E)} = \{ \omega \mid (\alpha : \beta_1, \ldots, \beta_n / \omega) \in \mathbf{D}, \alpha \in \mathbf{E_{i-1}}, \text{ and } \neg \beta_1, \ldots, \neg \beta_n \notin \mathbf{E} \}.$$

If $\mathbf{E}$ is an extension, then we say that $\sigma_\mathbf{E}$ is the *characteristic* selection function for $\mathbf{E}$ if

$$\sigma_\mathbf{E}(\mathbf{i}, \mathbf{V}) = \mathbf{CGD(C_i, E)} \cap \mathbf{V}.$$

**Proposition 2.1**
If $\mathbf{E}$ is an extension of a default theory $< \mathbf{W}, \mathbf{D} >$, then the characteristic selection function $\sigma_\mathbf{E}$ generates the constructive extension $\mathbf{C}$ of the corresponding normalized default theory $<\mathbf{W}, \mathbf{D}^* >$ such that $\mathbf{E} = \mathbf{C}$.

The converse of this proposition, that every characteristic selection function generates a Reiter extension, holds only if every characteristic selection function is non-self-defeating and exhaustive. Fortunately, this is the case.

**Proposition 2.2**
Every characteristic selection function is non-self-defeating and exhaustive.

Now the following result follows.

**Theorem 2.3**
If $\Delta = < \mathbf{W}, \mathbf{D} >$ is a default theory, then the set of extensions of this default theory is identical to the set of constructive extensions that are generated by all the characteristic selection functions $\sigma_\mathbf{E}$ (for each Reiter extension $\mathbf{E}$ of $\Delta$ ) of the corresponding normalized default theories $<\mathbf{W}, \mathbf{D}^*>$.

## 3 Alternative types of selection functions

In this section we introduce some specific types of selection functions, called incremental and strict selection functions. Moreover, we discuss under which conditions the dependency of a selection function $\sigma(\mathbf{i+1}, \mathbf{V})$ on the index $\mathbf{i}$ (the stage of the construction) can be replaced by a dependency on what has been generated so far: $\mathbf{C}^\sigma_\mathbf{i}$.

**Definition 3.1**

Let $\boldsymbol{\sigma}$ be a selection function related to the default theory $\boldsymbol{\Delta} = < \mathbf{W}, \mathbf{D} >$.

We call $\boldsymbol{\sigma}$ *strict* if at each non-final stage it selects at least one default conclusion that is not already contained in $\mathbf{Th}(\mathbf{C}^{\sigma}{}_{\mathbf{i}})$, i.e. for all $\mathbf{i} \in \mathbf{I}$ with $\mathbf{C}^{\sigma}{}_{\mathbf{i+1}} \neq \mathbf{C}^{\sigma}$ it holds

$$\boldsymbol{\sigma}_{\mathbf{i+1}}(\mathbf{CD}^{\sigma}{}_{\mathbf{i+1}}) \not\subseteq \mathbf{Th}(\mathbf{C}^{\sigma}{}_{\mathbf{i}})$$

We call $\boldsymbol{\sigma}$ *incremental* if for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ with $\mathbf{i} \leq \mathbf{j}$ it holds

$$\boldsymbol{\sigma}_{\mathbf{i}}(\mathbf{CD}^{\sigma}{}_{\mathbf{i}}) \subseteq \boldsymbol{\sigma}_{\mathbf{j}}(\mathbf{CD}^{\sigma}{}_{\mathbf{j}})$$

The following Lemma gives some properties of strict and incremental selection functions.

**Lemma 3.2**

Let $\boldsymbol{\sigma}$ be a selection function related to the default theory $\boldsymbol{\Delta} = < \mathbf{W}, \mathbf{D} >$.

a) $\boldsymbol{\sigma}$ is strict if and only if for all $\mathbf{i}$

$$\boldsymbol{\sigma}_{\mathbf{i+1}}(\mathbf{CD}^{\sigma}{}_{\mathbf{i+1}}) \subseteq \mathbf{Th}(\mathbf{C}^{\sigma}{}_{\mathbf{i}}) \Rightarrow \mathbf{C}^{\sigma}{}_{\mathbf{i+1}} = \mathbf{C}^{\sigma}.$$

b) If $\boldsymbol{\sigma}$ is strict then the sequence of $\mathbf{C}^{\sigma}{}_{\mathbf{i}}$ is strictly increasing as long as $\mathbf{C}^{\sigma}$ has not been reached, i.e. in the case that $\mathbf{C}^{\sigma}{}_{\mathbf{i}} \neq \mathbf{C}^{\sigma}$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ it holds

$$\mathbf{i} < \mathbf{j} \Rightarrow \mathbf{C}^{\sigma}{}_{\mathbf{i}} \subset \mathbf{C}^{\sigma}{}_{\mathbf{j}} \text{ (proper subset)}$$

c) If $\boldsymbol{\sigma}$ is strict and incremental then for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ with $\mathbf{C}^{\sigma}{}_{\mathbf{j}} \neq \mathbf{C}^{\sigma}$ it holds

$$\mathbf{i} < \mathbf{j} \Rightarrow \boldsymbol{\sigma}_{\mathbf{i}}(\mathbf{CD}^{\sigma}{}_{\mathbf{i}}) \subset \boldsymbol{\sigma}_{\mathbf{j}}(\mathbf{CD}^{\sigma}{}_{\mathbf{j}}) \text{ (proper subset)}$$

It turns out that it is easy to replace any selection function by a strict and incremental selection function that essentially induces the same construction. Here we call a mapping $\boldsymbol{\pi}: \mathbf{I} \rightarrow \mathbf{I}$ *strictly monotonic* if $\mathbf{i} < \mathbf{j}$ implies $\boldsymbol{\pi}(\mathbf{i}) < \boldsymbol{\pi}(\mathbf{j})$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$.

**Proposition 3.3**

Suppose $\boldsymbol{\sigma} : \mathbf{I} \times \mathbf{P}(\mathbf{U}) \rightarrow \mathbf{P}(\mathbf{U})$ is a selection function related to the default theory given by $\boldsymbol{\Delta} = < \mathbf{W}, \mathbf{D} >$.

a) There exists a strictly monotonic mapping $\boldsymbol{\pi} : \mathbf{I} \rightarrow \mathbf{I}$ such that the mapping $\boldsymbol{\sigma}'$ defined by

$$\boldsymbol{\sigma}'_{\mathbf{i}}(\mathbf{V}) = \boldsymbol{\sigma}_{\boldsymbol{\pi}(\mathbf{i})}(\mathbf{V})$$

for all $\mathbf{i} \in \mathbf{I}$ is a strict selection function $\boldsymbol{\sigma}'$ related to $\boldsymbol{\Delta}$. Moreover, $\mathbf{C}^{\sigma'} = \mathbf{C}^{\sigma}$ and it holds $\mathbf{C}^{\sigma'}{}_{\mathbf{i}} = \mathbf{C}^{\sigma}{}_{\boldsymbol{\pi}(\mathbf{i})}$ and $\mathbf{CD}^{\sigma'}{}_{\mathbf{i}} = \mathbf{CD}^{\sigma}{}_{\boldsymbol{\pi}(\mathbf{i})}$ for all $\mathbf{i} \in \mathbf{I}$.

b) If $\boldsymbol{\sigma}$ is not self-defeating then the mapping $\boldsymbol{\sigma}^*$ defined by

$$\boldsymbol{\sigma}^*_{\mathbf{i}}(\mathbf{CD}^{\sigma}{}_{\mathbf{i}}) = \bigcup_{\mathbf{k} \leq \mathbf{i}} \boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{CD}^{\sigma}{}_{\mathbf{k}})$$

for all $\mathbf{i} \in \mathbf{I}$ is an incremental selection function related to $\boldsymbol{\Delta}$ such that $\mathbf{C}^{\sigma^*} = \mathbf{C}^{\sigma}$ and for all $\mathbf{i} \in \mathbf{I}$ it holds $\mathbf{C}^{\sigma^*}{}_{\mathbf{i}} = \mathbf{C}^{\sigma}{}_{\mathbf{i}}$ and $\mathbf{CD}^{\sigma^*}{}_{\mathbf{i}} = \mathbf{CD}^{\sigma}{}_{\mathbf{i}}$.

c) The following hold:

(i) If $\boldsymbol{\sigma}$ is not self-defeating then $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}^*$ are not self-defeating.

(ii) $\boldsymbol{\sigma}^*$ is exhaustive iff $\boldsymbol{\sigma}'$ is exhaustive iff $\boldsymbol{\sigma}$ is exhaustive

(iii) $\boldsymbol{\sigma}^*$ is strict iff $\boldsymbol{\sigma}$ is strict

(iv) $\boldsymbol{\sigma}'$ is incremental iff $\boldsymbol{\sigma}$ is incremental

d) Applying a) and b) to a selection function $\boldsymbol{\sigma}$ that is not self-defeating subsequently in any order results in a selection function $\boldsymbol{\sigma}'^* = \boldsymbol{\sigma}^{*\prime}$ related to $\boldsymbol{\Delta}$ that is both strict and incremental.

Now we can discuss the dependency of a selection function $\boldsymbol{\sigma}(\mathbf{i},\mathbf{V})$ of the index $\mathbf{i}$ (the stage of the construction). In practical terms one might think of maintaining a kind of counter that counts how many construction steps have been taken. However, this looks rather unnatural. If one thinks of human reasoners it seems more plausible that knowledge is used to evaluate the set accumulating what has been generated so far: $\mathbf{C}^{\sigma}{}_{\mathbf{i-1}}$. So, in fact we might be interested in selection functions the first argument of which actually could be interpreted as a set of formulae collected until then rather then as just an index $\mathbf{i}$. Lemma 3.2 shows that for strict selection functions this indeed is the case.

Let $\mathbf{S}$ be the set of all sentences of the language $\mathbf{L}(\mathbf{W})$ of $\mathbf{W}$ and $\mathbf{P}(\mathbf{S})$ the power set of $\mathbf{S}$. Let the subset $\mathbf{I}_0$ of $\mathbf{I}$ be defined by

$$\mathbf{I}_0 = \{ \mathbf{i} \in \mathbf{I} \mid \mathbf{C}^{\sigma}{}_{\mathbf{i}} \neq \mathbf{C}^{\sigma} \}$$

Notice that $\mathbf{I_0}$ either is the complete set $\mathbf{I}$ or is the set $\{ \mathbf{i} \in \mathbf{I} \mid \mathbf{i} \leq \mathbf{n} \}$ for some $\mathbf{n}$.

From Lemma 2.2 we know that for a strict selection function $\sigma$ the mapping $\mu: \mathbf{I_0} \rightarrow \mathbf{P(S)}$ , given by $\mathbf{i} \mapsto \mathbf{C}^{\sigma}{}_{\mathbf{i}}$ is *strictly monotonic* : for all $\mathbf{i}, \ \mathbf{j} \in \mathbf{I_0}$ with $\mathbf{i} < \mathbf{j}$ it holds $\mathbf{C}^{\sigma}{}_{\mathbf{i}} \subset \mathbf{C}^{\sigma}{}_{\mathbf{j}}$ (proper subset). In particular this implies that this mapping is injective. Therefore in this case the dependency of the selection function on **i** (its first argument) can be viewed as a dependency on the set already constructed at that stage: $\mathbf{C}^{\sigma}{}_{\mathbf{i}}$ . This is stated more precisely in the following (second) main result.

**Theorem 3.4**

Suppose $\sigma : \mathbf{I} \times \mathbf{P(U)} \rightarrow \mathbf{P(U)}$ is a selection function related to the default theory given by $\Delta = < \mathbf{W}, \ \mathbf{D} >$. Let $\mathbf{S}$ be the set of sentences of the language $\mathbf{L(W)}$ of $\mathbf{W}$ and $\mathbf{P(S)}$ the power set of $\mathbf{S}$, and suppose $\mathbf{I_0} = \{ \mathbf{i} \in \mathbf{I} \mid \mathbf{C}^{\sigma}{}_{\mathbf{i}} \neq \mathbf{C}^{\sigma} \}$

a) If $\sigma$ is strict then $\sigma$ can be described by a function $\tau$ with $\mathbf{C}^{\sigma}{}_{\mathbf{i}}$ as its first argument instead of $\mathbf{i}$. More precisely, a mapping $\tau : \mathbf{P(S)} \times \mathbf{P(U)} \rightarrow \mathbf{P(U)}$ can be defined such that for all $\mathbf{C}$ and $\mathbf{V}$ and all $\mathbf{i} \in \mathbf{I_0}$ it holds

$$\tau(\mathbf{C}, \ \mathbf{V}) \subseteq \mathbf{V}$$
$$\tau(\mathbf{C}^{\sigma}{}_{\mathbf{i}}, \ \mathbf{CD}^{\sigma}{}_{\mathbf{i+1}}) = \sigma(\mathbf{i+1}, \ \mathbf{CD}^{\sigma}{}_{\mathbf{i+1}})$$

b) If $\sigma$ is both strict and incremental then the mapping $\tau$ described in a) may be defined in such a manner that additionally it satisfies the following monotonicity property:

$$\text{if } \mathbf{C} \subseteq \mathbf{C'} \text{ and } \mathbf{V} \subseteq \mathbf{V'} \text{ then } \tau(\mathbf{C}, \ \mathbf{V}) \subseteq \tau(\mathbf{C'}, \ \mathbf{V'})$$

c) If all default conclusions in $\mathbf{D}$ are literals then in the above statements $\mathbf{C}^{\sigma}{}_{\mathbf{i}}$ can be replaced by $\mathbf{Lit(C^{\sigma}{}_{i})}$, and $\mathbf{S}$ by $\mathbf{Lit(S)}$.

According to Proposition 3.3 and Theorem 3.4 we loose no constructions and constructive extensions if we restrict ourselves to selection functions that are strict. If we restrict ourselves to this type of selection functions then the dependency of a selection function on the index **i** can be replaced by a dependency on the set of formulae (or literals) constructed until that stage. Therefore we can use an alternative type of selection function that depend on sets of formulae and not on the index **i**. Notice that any type of control that restricts the default conclusions that are used can be described by a strict selection function $\sigma$, and hence by a selection function $\tau$ as defined in Theorem 2.4.

## 4 Declarative specification of control knowledge

Selection functions express some kind of control knowledge on a defeasible reasoning process. In general it is not always easy to specify control knowledge in a transparent manner; often one finds rather technical algorithmic descriptions of control knowledge. In this section we will discuss the question how the control knowledge that defines a selection function can be specified. We will do this by a proposal to specify the control knowledge in a declarative manner as meta-level knowledge.

Suppose $\Delta = < \mathbf{W}, \ \mathbf{D} >$ is a default theory. In this section we will discuss the question whether the functionality of a selection function $\sigma$ related to $\Delta$ can be expressed by means of a selection knowledge base $\mathbf{KB^s}$ containing control (meta-)knowledge in a declarative form. Reasoning with this knowledge makes use of input information on which default conclusions are possible and on the current (information) state of the process, defined by the literals generated so far. In the practical tasks and domains we studied (e.g., [BPT92], [GK92]) this always turned out to be possible, so the question arose whether in principle examples of selection functions can be defined that cannot be described in this manner. We will prove as a third main result that if we restrict ourselves to a $\mathbf{W}$ expressed in a finite propositional logic and only permit default conclusions in $\mathbf{D}$ that are literals then *any* strict selection function can be described in a declarative form by a knowledge base $\mathbf{KB^s}$. The idea for the theorem and its proof is adopted from earlier work, as described in [Tr91], where it has been investigated under which conditions in general functionalities can be expressed by means of declarative (rule-based) knowledge bases.

Before we can give a precise formulation of the following theorem we need some basic concepts. We define the propositional signature $\Sigma(\mathbf{AS})$ (i.e. set of symbols) of a meta-level language based on the atoms $\mathbf{T(c)}$, $\mathbf{PA(c)}$, $\mathbf{BA(c)}$ for all meta-level constants $\mathbf{c}$ (referring to object-level literals, i.e. literals from $\mathbf{L(W)}$ ). The meaning of these meta-predicates is as follows.

| | |
|---|---|
| **T(a)** | The object-level literal **a** of $\mathbf{L(W)}$ is contained in the current set $\mathbf{Lit(C^{\sigma}{}_{i})}$, i.e., is *true* in the current information state |
| **PA(a)** | The object-level literal **a** of $\mathbf{L(W)}$ is contained in the current set $\mathbf{CD}^{\sigma}{}_{\mathbf{i+1}}$ , i.e., is a *possible assumption* to be made. |
| **BA(a)** | The object-level literal **a** of $\mathbf{L(W)}$ is contained in the current set |

$\sigma_{i+1}(CD^{\sigma}{}_{i+1})$, i.e., is a *best assumption* to be made (is selected by $\sigma$).

Notice that a naming relation is needed where constants of the meta-level language refer to literals of the object-level language (that for reasons of presentation is assumed propositional). To this end for each object-level atom **a** a constant symbol **a'** at the meta-level is used, and we name a literal **¬ a** by the constant symbol **~a'**. If no confusion is expected, for convenience we sometimes leave out the primes used in the naming relation.

Let $\mathbf{Lit}(\mathcal{L}(\mathbf{W}))$ be the set of literals of $\mathcal{L}(\mathbf{W})$ and suppose $\mathbf{C} \subseteq \mathbf{Lit}(\mathcal{L}(\mathbf{W}))$ is a consistent set of object-level literals. This determines a set **C*** of meta-level atoms of the form $C^* = \{ T(c') \mid c \in C \}$. Furthermore, any subset $V \subseteq U$ of the set of all default conclusions **U** of **D** determines a set $V^* = \{ PA(c') \mid c \in V\}$. Moreover, we use the following notations for the related sets of literals:

$$
\begin{aligned}
C^{ml} &= C^* \cup \{ \neg\, \alpha \mid \alpha \notin C^* \} \\
&= \{ T(c') \mid c \in C \} \cup \{ \neg\, T(c') \mid c \in \mathrm{Lit}(\mathcal{L}(W))\backslash C \}
\end{aligned}
$$

$$
\begin{aligned}
V^{ml} &= V^* \cup \{ \neg\, \alpha \mid \alpha \notin V^* \} \\
&= \{ PA(c') \mid c \in V\} \cup \{ \neg\, PA(c') \mid c \in U\backslash V\}
\end{aligned}
$$

**Theorem 4.1**

Suppose $\sigma : I \times \mathcal{P}(U) \to \mathcal{P}(U)$ is a strict selection function related to the default theory given by $\Delta = \langle W, D \rangle$. Assume that the language $\mathcal{L}(\mathbf{W})$ of **W** is a finite propositional language, and that the set **U** of all default conclusions of **D** only contains literals, i.e. $U = \mathbf{Cons}(D) \subseteq L$ where $L = \mathbf{Lit}(\mathcal{L}(W))$. Let $\mathcal{P}(L)$ be the set of subsets of **L** and take $I_0 = \{ i \in I \mid C^{\sigma}{}_i \neq C^{\sigma}\}$.

Then the selection function $\sigma$ can be expressed by a knowledge base in rule format.

More precisely, there exists a meta-level knowledge base **KB'** of signature $\Sigma(AS)$ in rule format such that for each $i \in I_0$ and $\omega \in U$ it holds

$$\omega \in \sigma_{i+1}(CD^{\sigma}{}_{i+1}) \quad \text{iff} \quad \mathrm{Lit}(C^{\sigma}{}_i)^{ml} \cup (CD^{\sigma}{}_{i+1})^{ml} \cup KB' \vdash BA(\omega)$$

If, moreover, $\sigma$ is incremental, then there exists a meta-level knowledge base **KB*** of signature $\Sigma(AS)$ in rule format such that for each $i \in I$ and $\omega \in U$ it holds

$$\omega \in \sigma_{i+1}(CD^{\sigma}{}_{i+1}) \ \text{iff} \quad \mathrm{Lit}(C^{\sigma}{}_i)^* \cup (CD^{\sigma}{}_{i+1})^* \cup KB^* \vdash BA(\omega)$$

Notice that this theorem does not say anything about the complexity of the knowledge base that can be created. In the worst case this complexity may be high. In general the (theoretical) construction as given in the proof (see Appendix) will result in a knowledge base with a large number of rather complex rules, so for practical purposes this construction is of no use. On the other hand, as discussed earlier, in concrete application domains often this selection knowledge can be acquired (not seldom in a domain-specific form) from experts performing the task that is concerned.

In the above theorem (and its proof as given in the appendix), if $\sigma$ is not incremental it is essential that it can be expressed at the meta-level that a given literal is not in $\mathrm{Lit}(C^{\sigma}{}_i)$ (this is not the same as the opposite literal being in $\mathrm{Lit}(C^{\sigma}{}_i)$). This cannot be expressed at the object-level. Moreover, another essential point is that it is not only expressed explicitly what are the possible default conclusions but also what are *no* possible default conclusions: for the knowledge base **KB'** standard monotonic deduction is used, and the selection from a larger set of possible default conclusions as made by a selection function is not necessarily a larger set. However, in the case of an incremental selection function Theorems 4.1 shows that these issues are avoided. In that case only meta-level *atoms* are needed as inputs that express that an object literal is in $\mathrm{Lit}(C^{\sigma}{}_i)$ or is in a set of possible default conclusions $CD^{\sigma}{}_{i+1}$ (and no explicit information on the opposite is needed). This makes that in this case the reasoning to make a selection is quite standard.

In this incremental case the selection knowledge could even be encoded in the form of an object level theory and added to **W**. This can be done by extending the language $\mathcal{L}(\mathbf{W})$ to a language $\mathcal{L}'$ by adding new object-level atom symbols $PA\omega$ for each $\omega \in U$. Let **D'** be the set of defaults obtained from **D** by replacing each default conclusion $\omega$ by the atom $PA\omega$; the conclusions $BA(\omega)$ of the rules from **KB*** are encoded as $\omega$. The conditions of the rules are encoded either just as the object-level literals or as atoms of the form $PA\omega$. If $\sigma$ is exhaustive and not self-defeating then there is a theory $W' \supseteq W$ in the extended language $\mathcal{L}'$ such that the default theory $\Delta' = \langle W', D' \rangle$ has an extension **E'** subsuming $C^{\sigma}$ with $C^{\sigma} = L \cap E'$. A construction can be obtained based on $\Delta'$ using the trivial non-selecting selection function $\sigma' : I \times \mathcal{P}(U') \to \mathcal{P}(U')$ with $\sigma'(i, V) = V$ for all **i** and **V**. This construction satisfies $C^{\sigma}{}_i = L \cap C^{\sigma'}{}_i$ for all **i**.

## 5 Control through Inhibition Relations

In this section we introduce a new concept of control in normal default theories: inhibition relations. We will show that inhibited normal default theories are exactly as expressive as semi-normal default theories: for each semi-normal default theory, an inhibited normal default theory exists which has the same set of extensions, and vice versa. The main advantage of inhibition relations is that control of default theories is separated from the defaults themselves, which is particularly useful when incrementally modelling a real-world application with a default theory.

### 5.1 Inhibition Relations

In this section we define inhibition relations for default theories. Furthermore, we adapt the definition of Reiter extensions to include inhibition relations.

An *inhibition relation* for a default theory $\Delta_1 = \langle \mathbf{W},\mathbf{D} \rangle$, is a relation $\mathbf{I}: \mathcal{P}(\mathbf{D}) \rightarrow \mathbf{D}$. We will present a definition of extensions for inhibited default theories, which implies that for all extensions **E** of the inhibited default theory and for all inhibition relations **I(s, d)** of the default theory, **d** cannot be an element of the set of generating defaults of **E**, if **s** is a subset of **GD(E)**. In other words, **s** inhibits all extensions **E**, for which **d** is contained in **GD(E)**.

**Definition 5.1**

An *inhibition relation* for a default theory $\Delta = \langle \mathbf{W}, \mathbf{D} \rangle$, is a relation $\mathbf{I}: \mathcal{P}(\mathbf{D}) \rightarrow \mathbf{D}$, where $\mathcal{P}(\mathbf{D})$ is the power set of **D**. We say that a set of defaults **s** *inhibits* the default **d**, if **I(s, d}** and that $\langle \mathbf{W}, \mathbf{D}, \mathbf{I} \rangle$ is an *inhibited default theory*.

Formally, this leads to the following definition of extensions for an inhibited default theory $\Delta_2 = \langle \mathbf{W}, \mathbf{D}, \mathbf{I} \rangle$.

**Definition 5.2**

A set of sentences **E** is an *inhibited extension* of the inhibited default theory $\Delta = \langle \mathbf{W}, \mathbf{D}, \mathbf{I} \rangle$, if $\mathbf{E} = \bigcup_{i \in N} \mathbf{E_i}$ and $\mathbf{GD(E)} = \bigcup_{i \in N} \mathbf{GD(E)_i}$, where:

1. $\mathbf{E_0} = \mathbf{W}$.
2. $\mathbf{E_{i+1}} = \mathbf{Th(E_i)} \cup \{ \mathbf{Cons(d)} \mid \mathbf{d} \in \mathbf{D}$ applicable in $\langle E_i, \mathbf{GD(E)_i} \rangle \}$, for $i \geq 0$.
3. $\mathbf{GD(E)_0} = \emptyset$.
4. $\mathbf{GD(E)_{i+1}} = \mathbf{GD(E)_i} \cup \{ \mathbf{d} \mid \mathbf{d} \in \mathbf{D}$ applicable in $\langle \mathbf{E_i}, \mathbf{GD(E)_i} \rangle \}$, for $i \geq 0\}$
5. $\mathbf{d} \in \mathbf{D}$ is applicable in $\langle \mathbf{E_i}, \mathbf{GD(E)_i} \rangle$ if the following two conditions hold:
   (a) $\mathbf{d} = \alpha : \beta / \omega$, $\mathbf{E_i} \vdash \alpha$, $\mathbf{E} \nvdash \neg \beta$.
   (b) $\forall_{s \in \mathbf{GD(E)}} \neg \mathbf{I(s, d)}$.

In the definition, **GD(E)** is the set of generating defaults of extension **E**, while $\mathbf{GD(E)_i}$ is the set of defaults which has generated the set of sentences $\mathbf{E_i}$. Note that, if condition 5b would have been omitted, we get the standard definition of an uninhibited Reiter extension.

### 5.2 Semi-normal default logic = inhibited normal default logic

In this section we will show that for each semi-normal default theory $\Delta_1$ an inhibited normal default theory $\Delta_2$ exists, such that $\Delta_1$ and $\Delta_2$ have the same set of extensions. More specifically, we will show that such a $\Delta_2$ can be created from $\Delta_1$ in a straightforward manner.

**Definition 5.3**

Let $\Delta_1 = \langle \mathbf{W}, \mathbf{D_1} \rangle$ be a semi-normal default theory. We define $\Delta_2 = \langle \mathbf{W}, \mathbf{D_2}, \mathbf{I} \rangle$ , the *associated inhibited normal default theory associated to* $\Delta_1$ as follows.

(1) $\mathbf{D_2} = \{ (\alpha : \gamma) / \gamma \mid (\alpha : \beta) / \gamma \in \mathbf{D_1}$ **for some** $\beta \}$

(2) **I(s, d)** if and only if it holds that $\mathbf{W} \cup \{ \gamma \mid (\alpha : \gamma) / \gamma \in \mathbf{s} \} \vdash \neg \beta_i$ for some $\beta_i \in \mathbf{jus(d)}$

Note that in this definition the inhibition relation depends on **W**. This is not surprising, because whether two defaults are conflicting may depend on a particular **W**. For example, if we have the default set $\mathbf{D} = \{ (\alpha_1 : \beta_1) / \gamma$

and $(\alpha_2 : \beta_2) / \neg \gamma\}$, then these defaults will conflict when **W** implies both $\alpha_1$ and $\alpha_2$. If not, for example when $W = \{\alpha_1\}$ and $\alpha_1$ does not imply $\alpha_2$, then there will not be a conflict between these two defaults.

As for selection functions, different inhibition relations can be used for one default theory. In a system design this implies a form of modularity where the defaults form one module that can be used together with alternative meta-level modules representing inhibition (meta-)knowledge.

**Proposition 5.4**
Let $\Delta_1 = \langle W, D_1 \rangle$ be a semi-normal default theory. Then the set of extensions of $\Delta_1$ equals the set of inhibited extensions of its associated inhibited normal default theory $\Delta_2$ .

We remark that inhibition relations can be used to further restrict the set of extensions by adding additional inhibitions. When applied in such a context, inhibition relations are related to the Minimal Conflicting Sets described in Delgrande and Schaub (1994). While for both Minimal Conflicting Sets and inhibition relations we have the condition that they cannot be subset of the set of generating defaults of an extension, there are some differences. First, minimal conflicting sets are defined (and restricted to) sets of defaults which produce an inconsistency. Any set of defaults can be specified as inhibition relation, whether they produce inconsistencies or not. Second, Delgrande and Schaub (1994) apply minimal conflicting sets to normal default theories, while we have applied inhibition relations to general default theories.

## 5.3 Examples

In this section we present a set of examples of semi-normal defaults from the literature and show how these situations can be modelled using inhibited normal defaults.

**Example 1. (Italian communist)**

The following example is a shortened example from Lukaszewicz (1990, p. 162).

$$W = \{ \text{Italian} \wedge \text{Communist} \}$$
$$D = \{ (\text{Italian} : \text{Christian}) / \text{Christian}, (\text{Communist} : \neg \text{Christian}) / \neg \text{Christian} \}$$

In the above example, the justification of the first default can be extended with the conjunct $\neg$ **Communist** to give explicit preference to application of the second default if both are applicable, since the fact that someone is a Communist seems a stronger indication that he is not a Christian, then the fact that he is an Italian indicates that he is. Without the additional conjunct there are two extensions, one with **Christian** and one with $\neg$ **Christian**.

In inhibited normal default theories, it suffices to define

$$I(\{ \emptyset , (\text{Italian} : \text{Christian}) / \text{Christian})\},$$

to block the deduction of the extension with Christian. In this case **I** is used to eliminate the first default, because due to W it is inapplicable.

**Example 2.** Suppose we have the following two defaults:

$$d_1 = ( : \text{Happy})/\text{Happy}$$
$$d_2 = ( : \text{Busy})/\text{Busy}$$

Now let us assume that we feel that being **Happy** and **Busy** at the same time is counterintuitive, then we may want to extend the justification of $d_1$ with '**not Busy**', to indicate that **Happy** can be deduced unless we are also **Busy**. It can easily be checked that the only Reiter extension we get when extending $d_1$, is **{Busy}**.

The alternative using inhibition relations is to have default $d_2$ inhibit default $d_1$. Thus $I(\{d_2\}, d_1)$. Let us analyze the extensions we then get. In a first step, it seems that we can apply either $d_1$ or $d_2$. Let us assume applicability of $d_1$. Then we must determine whether $d_2$ is applicable. If we assume that $d_2$ is applicable, **GD(E)** will consist of at least $\mathbf{d_1}$ and $\mathbf{d_2}$. In that case, our assumption that $\mathbf{d_1}$ was applicable was incorrect. If we assume that $\mathbf{d_2}$ is not applicable, **GD(E)** will consist only of $\mathbf{d_1}$. However, in that case $\mathbf{d_2}$ is applicable. So, in both cases the fixed-point definition fails to provide us with an extension. Let us now assume applicability of $\mathbf{d_2}$. In that case, $\mathbf{d_1}$ is not applicable (condition 5b), and there we get as an extension **{Busy}**. We conclude that using an inhibition relation, we get exactly the same set of extensions we would have obtained when introducing a semi-normal default.[1]

## 6 Conclusions

In this paper the control problem for default reasoning was addressed. In earlier work it was described how selection functions (on default conclusions) added to default logic can be exploited to describe control. In this paper we investigated under what conditions selection functions can be expressed in the form of declarative control knowledge at the meta-level. This enables to determine selection in a dynamic manner by reasoning. Furthermore, we show that normal default logic with additional control by selection functions is (in some sense) equivalent to default logic in general. Finally, an alternative approach to control was introduced based on inhibition (meta-)relations between default rules. Also in this case one can restrict to normal defaults.

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[1] This example is based on an example by Gerd Brewka, when pointing at a flaw in a definition of inhibition relations in an earlier version of this paper.

## Appendix

**Proof of Theorem 1.7**

(ii) ⇒ (i) Let $\sigma$ be a selection function related to $\Delta$ that is exhaustive and not self-defeating and suppose $\mathbf{E} = \mathbf{C}^\sigma$. The sets $\mathbf{E_i}$ are defined as in Definition 1.0 with $\mathbf{E} = \mathbf{C}^\sigma$, $\mathbf{E_0} = \mathbf{W}$ and $\mathbf{E_{i+1}} = \mathbf{Th(E_i)} \cup \mathbf{Cons(D_{i+1})}$, where $\mathbf{D_{i+1}} = \{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in \mathbf{D} \mid \alpha \in \mathbf{E_i}, \text{ and } \neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{E}\}$. We will prove by induction that for every $\mathbf{i}$ it holds $\mathbf{C}^\sigma{}_\mathbf{i} \subseteq \mathbf{E_i} \subseteq \mathbf{E}$. For $\mathbf{i = 0}$ this is trivial. Induction step $\mathbf{i} \Rightarrow \mathbf{i+1}$. We first prove $\mathbf{E_{i+1}} \subseteq \mathbf{E}$. Since $\mathbf{E}$ is deductively closed, the induction hypothesis $\mathbf{E_i} \subseteq \mathbf{E}$ implies that (1) $\mathbf{Th(E_i)} \subseteq \mathbf{Th(E)} = \mathbf{E}$. From the fact that $\sigma$ is exhaustive it follows that $\mathbf{E}$ is closed under $\mathbf{D}$. Therefore, again using the induction hypothesis $\mathbf{E_i} \subseteq \mathbf{E}$ it follows that (2) $\mathbf{Cons(D_{i+1})} \subseteq \mathbf{E}$. From (1) and (2) it follows that $\mathbf{E_{i+1}} \subseteq \mathbf{E}$. Next we prove $\mathbf{C}^\sigma{}_\mathbf{i+1} \subseteq \mathbf{E_{i+1}}$. Recall that $\mathbf{C}^\sigma{}_\mathbf{i+1} = \mathbf{Th(C^\sigma{}_i)} \cup \sigma_\mathbf{i+1}(\mathbf{Cons(D^\sigma{}_{i+1})})$, where $\mathbf{D}^\sigma{}_\mathbf{i+1} = \{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in \mathbf{D} \mid \alpha \in \mathbf{C}^\sigma{}_\mathbf{i}, \text{ and } \neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{Th(C^\sigma{}_i)}\}$. From $\mathbf{C}^\sigma{}_\mathbf{i} \subseteq \mathbf{E_i}$ it immediately follows that (3) $\mathbf{Th(C^\sigma{}_i)} \subseteq \mathbf{Th(E_i)}$. Since $\sigma$ is not self-defeating, $\mathbf{C}^\sigma = \mathbf{E}$, and $\mathbf{C}^\sigma{}_\mathbf{i} \subseteq \mathbf{E_i}$ we have

$$\begin{aligned}&\sigma_{i+1}(\mathbf{Cons(D^\sigma{}_{i+1})}) \subseteq\\ (4)\quad &\mathbf{Cons}(\{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in \mathbf{D} \mid \alpha \in \mathbf{C}^\sigma{}_\mathbf{i}, \text{ and } \neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{Th(E)}\}) \subseteq\\ &\mathbf{Cons(D_{i+1})}\end{aligned}$$

From (3) and (4) it follows that $\mathbf{C}^\sigma{}_\mathbf{i+1} \subseteq \mathbf{E_{i+1}}$. Thus we have proved $\mathbf{C}^\sigma{}_\mathbf{i} \subseteq \mathbf{E_i} \subseteq \mathbf{E}$ for all $\mathbf{i}$. Hence, it follows that $\mathbf{C}^\sigma \subseteq \bigcup_{i=0}^{\infty} \mathbf{E_i} \subseteq \mathbf{E} = \mathbf{C}^\sigma$, therefore $\mathbf{E}$ is an extension.

(i) ⇒ (ii) Suppose $\mathbf{E}$ is an extension. We will construct a selection function $\sigma$ related to $\Delta$ such that $\mathbf{E_i} = \mathbf{C}^\sigma{}_\mathbf{i}$ for all i and prove that it is exhaustive and not self-defeating. The construction of sets $\mathbf{C_i}$ and proof are by induction. We take $\mathbf{C_0} = \mathbf{W}$. Assume $\mathbf{C_{i-1}}$ and $\sigma_\mathbf{i-1}$ have been defined such that $\mathbf{C_{i-1}} = \mathbf{E_{i-1}}$. Let $\mathbf{D_i}$ be the set of defaults in principle used in defining $\mathbf{E_i}$, i.e. $\mathbf{E_i} = \mathbf{Th(E_{i-1})} \cup \mathbf{Cons(D_i)}$, where $\mathbf{D_i} = \{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in \mathbf{D} \mid \alpha \in \mathbf{E_{i-1}} \text{ and } \neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{E}\}$. Let $\mathbf{D'_i} = \{(\alpha : \beta_1,\ldots, \beta_n / \omega) \in \mathbf{D} \mid \alpha \in \mathbf{C_{i-1}} \text{ and } \neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{Th(C_{i-1})}\}$, and $\mathbf{CD_i} = \mathbf{Cons(D'_i)}$. The basic idea of the proof is simple. First we show that $\mathbf{Cons(D_i)} \subseteq \mathbf{CD}^\sigma{}_\mathbf{i}$ and then we define the selection function $\sigma$ such that $\sigma_\mathbf{i}(\mathbf{CD}^\sigma{}_\mathbf{i}) = \mathbf{Cons(D_i)}$. Due to the induction hypothesis we have $\mathbf{Th(C_{i-1})} = \mathbf{Th(E_{i-1})}$, and hence it immediately follows that $\mathbf{C_i} = \mathbf{Th(C_{i-1})} \cup \sigma_\mathbf{i}(\mathbf{CD_i}) = \mathbf{Th(E_{i-1})} \cup \mathbf{Cons(D_i)} = \mathbf{E_i}$. To prove that $\mathbf{Cons(D_i)} \subseteq \mathbf{CD}^\sigma{}_\mathbf{i}$, it suffices to prove that $\mathbf{D_i} \subseteq \mathbf{D'_i}$. Suppose the default rule $\partial = (\alpha : \beta_1,\ldots, \beta_n / \omega)$ is in $\mathbf{D_i}$. This implies that $\alpha \in \mathbf{E_{i-1}}$ and $\neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{E}$. From the induction hypothesis it follows that (5) $\alpha \in \mathbf{C_{i-1}}$. From $\neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{E}$ it follows that $\neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{E_i}$, and hence also $\neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{Th(E_{i-1})}$. Due to the induction hypothesis this implies that (6) $\neg\beta_1,\ldots,\neg\beta_n \notin \mathbf{Th(C_{i-1})}$. From (5) and (6) it follows that $\partial \in \mathbf{D'_i}$, and consequently $\mathbf{D_i} \subseteq \mathbf{D'_i}$. Thus we have constructed a selection function $\sigma$ related to $\Delta$ such that $\mathbf{E_i} = \mathbf{C_i} = \mathbf{C}^\sigma{}_\mathbf{i}$ for all $\mathbf{i}$ . It is easy to verify that $\sigma$ is exhaustive and not self-defeating. ■

**Proof of Proposition 2.1.**
In the proof we use the following selection function $\sigma_\mathbf{E}(\mathbf{i}, \mathbf{V}) = \mathbf{CGD(C_i)} \cap \mathbf{V}$. We prove by induction that $\mathbf{E_i} = \mathbf{C_i}$ for each level $\mathbf{i}$. , $\mathbf{i = 0}$. Then it is trivial that $\mathbf{E_0} = \mathbf{Th(W)} = \mathbf{C_0}$.

$\mathbf{i = n}$.

To show that $\mathbf{E_n} \subseteq \mathbf{C_n}$ consider an arbitrary $\omega \in \mathbf{E_n} - \mathbf{E_{n-1}}$. (If $\omega \in \mathbf{E_{n-1}}$, then $\omega \in \mathbf{C_n}$, because of the induction hypothesis.) For this conclusion $\omega$ there is a default $(\alpha : \beta_1,\ldots, \beta_n / \omega) \in \mathbf{D}$ such that $\alpha \in \mathbf{E_{n-1}}$ and $\neg\beta_1,\ldots, \neg\beta_n \notin \mathbf{E}$. Hence, by the induction hypothesis, we have $\alpha \in \mathbf{C_{n-1}}$. We also have $(\alpha : \omega / \omega) \in \mathbf{D^*}$. And $\neg\omega \notin \mathbf{C_{n-1}}$, because otherwise $\neg\omega \in \mathbf{E_{n-1}}$, and then $\mathbf{E}$ would be inconsistent, which contradicts the fact that $\neg\beta_1,\ldots, \neg\beta_n \notin \mathbf{E}$. Hence, $\omega \in \mathbf{C_n}$.

To show that $\mathbf{C_n} \subseteq \mathbf{E_n}$ consider an arbitrary $\omega \in \mathbf{C_n} - \mathbf{C_{n-1}}$. (If $\omega \in \mathbf{C_{n-1}}$, then $\omega \in \mathbf{E_n}$, because of the induction hypothesis.) For this conclusion $\omega$ there is a default $(\alpha : \omega / \omega) \in \mathbf{D^*}$ such that $\alpha \in \mathbf{C_{n-1}}$ and $\neg\omega \notin \mathbf{C_{n-1}}$, and the $\omega$ is selected by $\sigma_E$ at level $\mathbf{n}$, which implies that $\omega \in \mathbf{CGD(E_n)}$, and hence $\omega \in \mathbf{E_n}$. ■

**Proof of Proposition 2.2**.
Let $\sigma$ be the characteristic selection function for the extension $\mathbf{E}$ of the default theory $< \mathbf{W}, \mathbf{D} >$. Suppose that $\sigma$ is self-defeating, then there is a stage $\mathbf{i}$ in the construction of the constructive extension $\mathbf{C}$ and a formula $\omega$ that is

selected by $\sigma$ at $i$ such that for all defaults $(\alpha : \beta_1,\ldots, \beta_n / \omega) \in D^{\sigma}_i$ with $\alpha \in C^{\sigma}_{i-1}$, and $\neg \beta_1, \ldots, \neg \beta_n \notin Th(C^{\sigma}_{i-1})$, there is some $j \geq i$ such that $\neg \beta_k \in Th(C^{\sigma}_j)$ for some $k$ with $1 \leq k \leq n$. But this would imply that each one of these justification violations $\neg \beta_k \in Th(C^{\sigma}_j)$ is in $CGD(Ej)$ for some $j \geq i$, and then it is impossible that $\omega$ is in $CGD(E_i)$, which contradicts the assumption that $\omega$ is selected by $\sigma$ at $i$.

Suppose $\sigma$ is not exhaustive, then there is a default $(\alpha : \beta_1,\ldots, \beta_n / \omega) \in D$ such that $\alpha \in C$ and $\neg \beta_1, \ldots, \neg \beta_n \notin C$ and $\omega \notin C$, where $C$ is the constructive extension generated by $\sigma$. Let $C^{\sigma}_i$ be the first level such that $\alpha \in C^{\sigma}_{i-1}$ and $\neg \beta_1,\ldots, \neg \beta_n \notin C^{\sigma}_j$ for all levels $j$ and $\omega \notin C^{\sigma}_i$. The only reason that $\omega$ is not in $C^{\sigma}_i$ can be that $\omega \notin CGD(E_i)$. But this implies that either $\omega \notin E_{i-1}$, or $\neg \beta_k \in E_j$ for some level $j$ and some $k$ with $1 \leq k \leq n$. Both cases contradict the property that $E_i = C_i$ for each level $i$, that we proved in the proof of Proposition 2.1. ■

**Proof of Proposition 2.3**.

Follows immediately from Propositions 2.1, 2.2 and Theorem 1.7 that says that for a default theory its set of constructive extensions generated by non-self-defeating and exhaustive selection functions is identical to its set of Reiter extensions. ■

**Proof of Lemma 3.2**

a), c) These are easy to verify.

b) If $\sigma$ is strict and incremental then for each $i$ such that $C^{\sigma}_{i+1} \neq C^{\sigma}$ it holds

$$\sigma_i(CD^{\sigma}_i) \subseteq \sigma_{i+1}(CD^{\sigma}_{i+1}) \not\subseteq Th(C^{\sigma}_i)$$

Since $\sigma_i(CD^{\sigma}_i) \subseteq C^{\sigma}_i$ this implies that it concerns a proper subset relation:

$$\sigma_i(CD^{\sigma}_i) \subset \sigma_{i+1}(CD^{\sigma}_{i+1})$$ ■

**Proof of Proposition 3.3**

a) Define the mapping $\pi : I \to I$ inductively as follows.

For $i = 0$, take $\pi(0) = 0$.

For $i \geq 0$, define $\pi(i+1)$ as follows:

$\pi(i+1) = \pi(i) + 1$ in case that $\sigma_j(C^{\sigma}_j) \subseteq Th(C^{\sigma}_{\pi(i)})$ for all $j > \pi(i)$

$\pi(i+1) = \min \{j \in I \mid j > \pi(i) \text{ and } \sigma_j(C^{\sigma}_j) \not\subseteq Th(C^{\sigma}_{\pi(i)}) \}$ otherwise

Notice that in the first case we have $C^{\sigma}_{\pi(i+1)} = C^{\sigma}$.

It is easy to verify that we have defined a strictly monotonic mapping $\pi : I \to I$ such that the selection function $\sigma'$ related to $\Delta$, defined by $\sigma'_i(V) = \sigma_{\pi(i)}(V)$ for all $V$ is strict, and constructs the sets as stated.

b) Since $\sigma$ is not self-defeating, from Lemma 2.6 it follows that

$$\cup_{k \leq i} \sigma_k(CD^{\sigma}_k) \subseteq CD^{\sigma}_i$$

for all $i$. Therefore $\sigma^*$ is a selection function related to $\Delta$. It is easy to verify that it is incremental, and constructs the same sets as $\sigma$.

c) (i) This is easy to verify.

(ii) and (iii) This immediately follows from a) and b).

(iv) Since $\pi$ is strictly monotonic for all $i, j \in I$ we have $i \leq j$ iff $\pi(i) \leq \pi(j)$.

Therefore the statement

$$\pi(i) \leq \pi(j) \Rightarrow \sigma_{\pi(i)}(CD^{\sigma}_{\pi(i)}) \subseteq \sigma_{\pi(j)}(CD^{\sigma}_{\pi(j)})$$

is equivalent to

$$i \leq j \Rightarrow \sigma_{\pi(i)}(CD^{\sigma}_{\pi(i)}) \subseteq \sigma_{\pi(j)}(CD^{\sigma}_{\pi(j)})$$

From this observation (iv) easily follows.

d) Using c) this can be verified in a straightforward manner. ■

**Proof of Theorem 3.4**

a) Since $\sigma$ is strict the mapping $\mu: I_0 \to P(S)$ defined by $i \mapsto C^{\sigma}_i$ is a strictly monotonic mapping. Therefore it is injective, and hence one can consider the selection function a function of the arguments $C^{\sigma}_i$ and $V$ instead of $i$ and $V$: in that case the index $i$ is uniquely determined, given a set $C$ that is equal to some $C^{\sigma}_i$. Formally, this is equivalent to the fact that we can define a mapping $\tau : P(S) \times P(U) \to P(U)$ such that for all $i \in I_0$ and $V$ it holds $\tau(\mu(i), V) = \sigma(i+1, V)$ ; it is easy to see that we can define $\tau$ in such a manner that it satisfies $\tau(C, V) \subseteq V$ for all $C$ and $V$.

b) In this case, we can define $\tau : P(S) \times P(U) \rightarrow P(U)$ by taking the intersection of the second argument $V \in P(U)$ with the union of all selected default conclusion sets for all stages $j+1$ with $j \in I_0$ and $C^\sigma{}_j \subseteq C$, where $C \in P(S)$ is the first argument; i.e.

$$\tau(C, V) = V \cap \bigcup_{j \in I_0, C^\sigma{}_j \subseteq C} \sigma_{j+1}(CD^\sigma{}_{j+1})$$

By this definition it is clear that

$$\tau(C, V) \subseteq V$$

We verify that this $\tau$ describes $\sigma$ (the second property stated in a)). Let $i \in I_0$ be given. Indeed from the fact that $\sigma$ is strict and incremental it follows that

$$\begin{aligned}\tau(C^\sigma{}_i, CD^\sigma{}_{i+1}) &= CD^\sigma{}_{i+1} \cap \bigcup_{j \in I_0, C^\sigma{}_j \subseteq C^\sigma{}_i} \sigma_{j+1}(CD^\sigma{}_{j+1}) \\ &= CD^\sigma{}_{i+1} \cap \bigcup_{j \in I_0, j \leq i} \sigma_{j+1}(CD^\sigma{}_{j+1}) \\ &= CD^\sigma{}_{i+1} \cap \sigma_{i+1}(CD^\sigma{}_{i+1}) \\ &= \sigma_{i+1}(CD^\sigma{}_{i+1})\end{aligned}$$

The monotonicity property is proved as follows. Suppose $C \subseteq C'$ and $V \subseteq V'$, then the union $\bigcup_{j \in I_0, C^\sigma{}_j \subseteq C'} \sigma_j(CD^\sigma{}_j)$ is taken over more sets then the union $\bigcup_{j \in I_0, C^\sigma{}_j \subseteq C} \sigma_j(CD^\sigma{}_j)$, so:

$$\bigcup_{j \in I_0, C^\sigma{}_j \subseteq C} \sigma_j(CD^\sigma{}_j) \subseteq \bigcup_{j \in I_0, C^\sigma{}_j \subseteq C'} \sigma_j(CD^\sigma{}_j)$$

Since $V \subseteq V'$ it follows

$$V \cap \bigcup_{j \in I_0, C^\sigma{}_j \subseteq C} \sigma_j(CD^\sigma{}_j) \subseteq V' \cap \bigcup_{j \in I_0, C^\sigma{}_j \subseteq C'} \sigma_j(CD^\sigma{}_j)$$

This proves the monotonicity property.

c) This is straightforward to verify. ■

**Proof of Theorem 4.1**

Suppose $\sigma : I \times P(U) \rightarrow P(U)$ is a strict selection function. By Theorem 2.11a),c) a mapping $\tau: P(L) \times P(U) \rightarrow P(U)$ can be defined such that for all $i \in I_0$, $C$ and $V$ it holds $\tau(Lit(C^\sigma{}_i), CD^\sigma{}_{i+1}) = \sigma(i+1, CD^\sigma{}_{i+1})$ and $\tau(C, V) \subseteq V$. We will show how in the case of a finite propositional set $Lit(C^\sigma{}_i)$ this $\tau$ can be expressed in first order logic by means of meta-rules. The finiteness condition enables us to apply the following construction (used earlier in [Tr91] to express functionalities of reasoning modules in terms of (rule-based) knowledge bases). We define the knowledge base $KB'$ of signature $\Sigma(AS)$ by:

$$KB' = \{ Con(C^{ml} \cup V^{ml}) \rightarrow BA(c) \mid c \in \tau(C, V), C \in P(L), V \in P(U) \}$$

Here $Con(L)$ means taking the conjunction of a set of literals $L$. It is straightforward to verify that by the definition of this knowledge base we have that

$$Lit(C^\sigma{}_i)^{ml} \cup (CD^\sigma{}_{i+1})^{ml} \cup KB' \vdash BA(\omega)$$

is equivalent to the fact that there is one rule of the knowledge base with conclusion $BA(\omega)$ and conditions that form the set $Lit(C^\sigma{}_i)^{ml} \cup (CD^\sigma{}_{i+1})^{ml}$, and this is equivalent to $\omega \in \tau(Lit(C^\sigma{}_i), CD^\sigma{}_{i+1})$.

Now assume that $\sigma$ is both strict and incremental. In this case we may assume that $\tau$ verifies the monotonicity condition stated in Theorem 3.4.

This time the knowledge base expressing $\tau$ can be given by:

$$KB^* = \{ Con(C^* \cup V^*) \rightarrow BA(c) \mid c \in \tau(C, V), C \in P(L), V \in P(U) \}$$

Now the statement

$$Lit(C^\sigma{}_i)^* \cup (CD^\sigma{}_{i+1})^* \cup KB^* \vdash BA(\omega)$$

is equivalent to the fact that there is at least one rule of the knowledge base

$$Con(C^* \cup V^*) \rightarrow BA(\omega)$$

such that $\omega \in \tau(C, V)$ and $C \subseteq Lit(C^\sigma{}_i)$ and $V \subseteq (CD^\sigma{}_{i+1})$. From the monotonicity condition it follows that this is equivalent to $\omega \in \tau(Lit(C^\sigma{}_i), CD^\sigma{}_{i+1})$. ■

**Proof of Proposition 5.4**

We will show through induction that for all $i$, $E_i(\Delta_1) = E_i(\Delta_2)$. As $E_0(\Delta_1) = E_0(\Delta_2) = W$, the basis for the induction is proven. Let us assume that $E_i(\Delta_1) = E_i(\Delta_2)$ for all $i$ less or equal to some natural $k$. Then, $E_{k+1}(\Delta_1) = Th(E_k(\Delta_1)) \cup \{ cons(d_1) \mid d_1 = \alpha : \beta \wedge \omega / \omega$ in $D$, $E_k(\Delta_1) \vdash \alpha$, $E \nvdash \neg \omega$, $E \nvdash \neg \beta\}$, and, $E_{k+1}(\Delta_2) = Th(E_k(\Delta_2)) \cup \{ cons(d_2) \mid d_2 = \alpha : \omega / \omega$ in $D$, $E_k(\Delta_2) \vdash \alpha$, $E \nvdash \neg \omega$, $GD(E) \notin I \}$

The definitions of $E_{k+1}(\Delta_1)$ and $E_{k+1}(\Delta_2)$ are based on the applicability of defaults in $E_k$. We will show that if and only if a default $d_1 = \alpha: \beta \wedge \omega / \omega$ is applicable in $E_k(\Delta_1)$, then default $d_2 = \alpha : \omega / \omega$ is applicable in $E_k(\Delta_2)$. In other words, we must show that the following two conditions are equivalent:

(i) $E_k(\Delta_1) \vdash \alpha$, $E \nvdash \neg \omega$, $E \nvdash \neg \beta$,

(ii) $\mathbf{E_k(\Delta_2) \vdash \alpha, E \nvdash \neg \omega, GD(E) \notin I}$.

It follows from the induction assumption that $\mathbf{E_k(\Delta_1) = E_k(\Delta_2)}$. What remains to be shown is that $\mathbf{E} \nvdash \neg \beta$ if and only if $\mathbf{GD(E) \notin I}$. Substituting $\mathbf{GD(E)}$ for $\mathbf{s}$ in condition 2 of the proposition yields:

$\mathbf{I(GD(E))}$ if and only if for some $\mathbf{d_{j_2}}$ in $\mathbf{GD(E)}$, $\mathbf{W \cup \{ \omega_i \mid d_{i_2} \in GD(E) \} \vdash \neg \beta_j}$.

Since $\mathbf{Th( W \cup \{ \omega_i \mid d_{i_2} \in GD(E) \} )}$ equals $\mathbf{E}$, we get:

$\mathbf{I(GD(E))}$ if and only if for some $\mathbf{d_{j_2}}$ in $\mathbf{GD(E)}$, $\mathbf{E \vdash \neg b_j}$.

It follows that $\mathbf{GD(E) \notin I}$, if and only if $\mathbf{E \nvdash \neg b}$. Thus, $\mathbf{E_{k+1}(\Delta_1) = E_{k+1}(\Delta_2)}$, and by induction $\mathbf{E_i(\Delta_1) = E_i(\Delta_2)}$ for all $\mathbf{i}$, and thus $\mathbf{E(\Delta_1) = E(\Delta_2)}$.