Linear, Branching Time and Joint Closure Semantics for Temporal Logic*

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Abstract. Temporal logic can be used to describe processes: their behaviour is characterized by a set of temporal models axiomatized by a temporal theory. Two types of models are most often used for this purpose: linear and branching time models. In this paper a third approach, based on socalled joint closure models, is studied using models which incorporate all possible behaviour in one model. Relations between this approach and the other two are studied. In order to define constructions needed to relate branching time models, appropriate algebraic notions are defined (in a category theoretical manner) and exploited. In particular, the notion of joint closure is used to construct one model subsuming a set of models. Using this universal algebraic construction we show that a set of linear models can be merged to a unique branching time model. Logical properties of the described algebraic constructions are studied. The proposed approach has been successfully aplied to obtain an appropriate semantics for nonmonotonic reasoning processes based on default logic. References are discussed that show the details of these applications.

Keywords: temporal logic, semantics, joint closure, linear time, branching time

1 Introduction

Temporal logic provides techniques to build formal models of dynamics: processes are described by temporal models that satisfy some set of temporal axioms. This approach may be used to describe the dynamics of (material) processes in the external world, as well as mental or computational processes. In our research we focus on formal models for the behaviour of compositional (knowledge- or agent-based) reasoning systems by means of temporal logic. States in a reasoning system are characterized by the (incomplete) information that has been obtained so far; usually they are called *information states*.

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A characteristic of dynamics is that there is often a number of possible alternative behavioural patterns. During the process in some way or another a choice between these alternatives is made. These (intended) behavioural patterns can be formalized by a set of possible (intended) linear time temporal models of the temporal theory involved. A different way of formalization of the variety of patterns is by branching time temporal models of a temporal theory, where each branch represents one of the patterns.

Formalization by a set of linear time models has the advantage of a very simple model structure. But the disadvantage is that the possible choices and the time points at which they should be made are not covered explicitly in the formalization itself. Branching time models represent these choices as points where the flow of time branches. However, a given branching time model may only describe a subset of the set of all possible behaviours. Different papers in the literature on temporal logic discuss the usefulness of linear time temporal logic versus branching time temporal logic depending on the type of applications; for example, [EH86], [Gl94].

A third formalization of temporal semantics, obtained as a result of the notions defined below is shown; this approach only looks at branching time models which incorporate all possible behaviour (joint closure models). This third approach has been successfully applied to describe the semantics of multiple nonmonotonic (default) reasoning processes for a given default theory in one (a kind of standard) joint closure model, which can be associated to the default theory (cf. [ET96]).

Given that these approaches are formalizations of (more or less) the same phenomenon, it is natural to study formal connections between them. We will define universal (algebraic) constructions on models which allow us to connect the approaches and to study logical properties of these constructions.

The world whose properties over time we are interested in, can be described using a language, called the *object-level* language. The states of this world are formalized by models of the object-level language together with a satisfaction relation which describes which formulae are true in a particular state (object-level model). We do not a priori pose any restrictions on the language or the models.

In Section 2, we introduce our temporalized logic (inspired by e.g., [FG92], [BPM83]). In Section 3, we define our notion of homomorphism between temporal models and establish properties of the notion. A class of formulae that are persistent under homomorphisms is identified. Furthermore, we show how a category can be defined, based on these homomorphisms. In Section 4, we present some category-theoretic results that have implications for our category. In particular, we define a universal construction to merge models that is used to show in Section 5 how to construct a branching time model in a canonical manner out of a set of linear time models. The logical properties of this universal construction are further worked out in Section 6. Finally, in Section 7 some conclusions are drawn.

2 Temporalized logic

In this section we introduce our temporalized logic (see also [ET93]). Our approach is in line with what in [FG92] is called *temporalizing* a given logic. We start defining the flows of time we use in Subsection 2.1, next we define temporalized models in Subsection 2.2 and finally we define temporal formulae and their interpretation in Subsection 2.3.

2.1 Flows of time

Definition 2.1 (flow of time)

A (discrete) *flow of time* **(T, <)** is a pair consisting of a nonempty set **T** of time points, and a binary relation < on **T x T**, called the *immediate successor relation* that is irreflexive, antisymmetric and antitransitive. Here for **s, t** in **T** the expression **s < t** denotes that **t** is an *immediate successor* of **s**, and that **s** is an *immediate predecessor* of **t**. We also introduce the transitive closure « of this binary relation: « = $<^{+}$. A flow of time is called *linear* if « is a total ordering.

Definition 2.2 (sub-ft and branch)

a) A flow of time **(T', <')** is called a *sub-ft (sub-flow of time)* of a flow of time **(T, <)** if **T'** ⊆ **T** and <' = < ∩ **T' x T'**. It is also called the sub-ft of **(T, <)** *defined by* **T'**, or the *restriction of* **(T, <)** to **T'**.

b) A sub-flow of time **T'** is *right* or *successor* (respectively *left* or *predecessor*) *complete* in **T** with respect to **t** in **T'** if for all **u** in **T** with **t « u** (respectively **u « t**) we have **u** in **T'**.

c) A *branch* in a flow of time **T** is a sub-ft **B** = **(T', <')** of **T** such that:

(i) «' = « ∩ **T' x T'** is a total ordering on **T' x T'**

(ii) Every **t** ∈ **T'** with a successor in **T** also has a successor in **T'**:
for all **s** ∈ **T'**, **t** ∈ **T** : **s < t** ⇒ there is a **t'** ∈ **T'** : **s < t'**

(iii) Every **t** in **T'** with a predecessor in **T** also has a predecessor in **T'**:
for all **s** ∈ **T**, **t** ∈ **T'** : **s < t** ⇒ there is an **s'** ∈ **T'** : **s' < t**.

(iv) Every element of **T** that is in between elements of **T'** is itself in **T'**:
for all **s** ∈ **T'**, **t** ∈ **T**, **u** ∈ **T'** : **s « t « u** ⇒ **t** ∈ **T'**

Branches will be viewed as linear temporal models. For example, Definition 2.2(i) guarantees linearity, whereas Definition 2.1 guarantees discreteness. In addition, we will impose the following definitions.

Definition 2.3 (minimal element, root, path)

a) An element $\mathbf{t}$ of $\mathbf{T}$ is called a *minimal element* if there exists no $\mathbf{s}$ with $\mathbf{s < t}$. We call $\mathbf{t}$ a *root* if for all $\mathbf{u}$ in $\mathbf{T}$ it holds $\mathbf{u = t}$ or $\mathbf{t \ll u}$.

b) We call $\mathbf{T}$ *well-founded* if for there do not exist infinite descending chains of elements $\mathbf{s_i < s_{i-1}}$.

c) A *(finite) path* is a finite sequence of successors: $\mathbf{s_0}$, ..., $\mathbf{s_n}$ such that: $\mathbf{s_i < s_{i+1}}$ for all $\mathbf{0 \leq i \leq n\text{-}1}$. We call $\mathbf{s_0}$ the starting point and $\mathbf{s_n}$ the endpoint of the path.

We will make additional assumptions on the flow of time: that it describes a discrete tree structure where time branches in the direction of the future; see definitions below.

Definition 2.4 (tree and forest)

a) The following properties are defined:

(i) *Successor existence*

Every time point has at least one successor:

for all $\mathbf{s \in T}$ there exists a $\mathbf{t \in T}$ such that $\mathbf{s < t}$.

(ii) *Rooted*

A flow of time is rooted with root $\mathbf{r}$ if $\mathbf{r}$ is a (unique) smallest element:

for all $\mathbf{t}$ it holds $\mathbf{r = t}$ or $\mathbf{r \ll t}$.

(iii) *Left linear*

For all $\mathbf{t}$ the set of $\mathbf{s}$ with $\mathbf{s \ll t}$ is totally ordered by $\mathbf{\ll}$.

b) A flow of time is called a *tree* if it is rooted and left linear.

c) A flow of time is called a *forest* if it is well-founded and left linear.

Note that a forest is just a disjoint union of trees. From now on we will assume all flows of time to be forests satisfying successor existence.

Lemma 2.5

a) Suppose $\mathbf{T}$ is well-founded.

For every element $\mathbf{t}$ there is a minimal element $\mathbf{s}$ and a (finite) path from $\mathbf{s}$ to $\mathbf{t}$.

b) If there exists a root $\mathbf{r}$, then $\mathbf{T}$ is well-founded and for every $\mathbf{t}$ there exists a path from $\mathbf{r}$ to $\mathbf{t}$.

c) Suppose $\mathbf{T}$ is a forest.

Every non-minimal element has a unique predecessor. For every $\mathbf{t}$ in $\mathbf{T}$ there is a unique minimal element $\mathbf{m}$ with $\mathbf{m \ll t}$ and a unique path with $\mathbf{t}$ as end point and $\mathbf{m}$ as starting point; this path gives a finite ordered enumeration of $\mathbf{\{ s \mid s \ll t \} \cup \{t\}}$.

The proof of this lemma is straightforward and omitted. The number of elements in the path from t to its corresponding minimal element, minus one, is called the *depth* of **t**. Using this depth function the time points of a branch may be identified with $\mathbb{N}$.

Definition 2.6 (isolated and generated sub-ft)

A sub-ft **T'** of **T** is called an *isolated* sub-ft of **T** if there do not exist **t** in **T\T'** and **t'** in **T'** with $t < t'$ or $t' < t$. A minimal isolated sub-ft is called a *connectivity component*.

We call **T'** a sub-flow of time *generated by* a subset **B** of **T** if it is both left and right complete with respect to all elements of **B**.

The *smallest sub-ft generated by* **B** is the sub-ft generated by **B** given by

$$\mathbf{T} = \{s \mid \exists\ t \in \mathbf{B} \text{ with } s \ll t\} \cup \mathbf{B} \cup \{u \mid \exists\ t \in \mathbf{B} \text{ with } t \ll u\}$$

The sub-flow of time generated by one element **t** in **T'** is the tree consisting of the path from the minimal element under **t** to **t** and all **u** with $t \ll u$.

Proposition 2.7

a) **T'** is a sub-ft of **T** (self-)generated by **T'** iff it is an isolated sub-ft of **T**.

b) **T** is a forest iff all its connectivity components are trees.

Remark: we will sometimes, if no confusion can arise, use the same character $<$ to denote two different relations on different sets, for example as in $(\mathbf{T}, <)$ and $(\mathbf{T'}, <)$.

2.2 Temporalized models

As we want to be able to describe temporal changes in any domain, we will just assume we have an object-level language, $\mathcal{L}_0$, whose formulae describe the domain. The domain states based on this language will be supposed to form a class $\mathcal{M}_0$ of object models. An object-level satisfaction relation $\models_0 \subseteq \mathcal{M}_0 \times \mathcal{L}_0$ indicates which formulae are true in a model. Thus for $\mathbf{M} \in \mathcal{M}_0$ and $\varphi \in \mathcal{L}_0$, $\mathbf{M} \models_0 \varphi$ means that φ is true in **M**. We could take, for example, a propositional language with classical propositional models. We could also take the same language but with three-valued models under the Strong Kleene semantics. Or we could take a modal language with modal Kripke models. Thus the choice of language and models can be varied at will. From now on we will assume a fixed object-level language, model class and satisfaction relation.

Definition 2.8 (Temporal model)

Let $(\mathbf{T}, <)$ be a flow of time.

A *temporalized model* $\mathbb{M}$ based on flow of time $(\mathbf{T}, <)$ is a triple $(\mathbf{M}, \mathbf{T}, <)$, where **M** is a mapping

$$\mathbf{M : T} \;\rightarrow\; \mathcal{M}_{\mathbf{0}}$$

So at any point in time we have an object-level model describing what is true in the domain at that time. We will sometimes refer to $\mathbf{M}$ as a temporal model based on $(\mathbf{T}, <)$. If φ is an object-level formula, and $\mathbf{t}$ is a time point in $\mathbf{T}$, and $\mathbf{M_t} \models_{\mathbf{0}} \varphi$, then we say that in this model $\mathbf{M}$ *at time point* $\mathbf{t}$ *the formula* φ *is true*.

Definition 2.9

The temporal model $\mathbf{M'}$ is *sub-model* of $\mathbf{M}$ if $(\mathbf{T'}, <')$ is a sub-flow of time of $(\mathbf{T}, <)$ with $\mathbf{M(t) = M'(t)}$ for all $\mathbf{t}$ in $\mathbf{T}$. We also call $\mathbf{M'}$ the *restriction* of $\mathbf{M}$ to $\mathbf{T'}$, denoted by $\mathbf{M|T'}$. If $\mathbf{T'}$ is a branch of $\mathbf{T}$ then $\mathbf{M'}$ is called a *branch* of $\mathbf{M}$. For a temporal model $\mathbf{M}$, the set of its branches is denoted by $\mathbf{Br(M)}$.

Also the other notions defined in the above subsection for flows of time are inherited by models.

2.3 Temporal formulae and their interpretation

We will now define the temporal language $\mathcal{L}_{\mathbf{T}}$ in terms of the object-level language using temporal operators to describe truth of object-level and temporal formulae over time. Because our temporal models based on forests have a more differentiated structure towards the future than towards the past, we will need more operators describing the future than the past. Also, we do not want any interaction between object-level formulae and temporal formulae. Therefore the object-level formulae are "shielded" by an operator $\mathbf{C}$:

Definition 2.10 (temporal language)

The temporal language $\mathcal{L}_{\mathbf{T}}$ is defined to be the least set such that:

(i) $\varphi \in \mathcal{L}_{\mathbf{0}} \Rightarrow \mathbf{C}\varphi \in \mathcal{L}_{\mathbf{T}}$

(ii) $\varphi, \psi \in \mathcal{L}_{\mathbf{T}} \Rightarrow \neg\varphi,\ \varphi \wedge \psi,\ \varphi \vee \psi,\ \varphi \rightarrow \psi \in \mathcal{L}_{\mathbf{T}}$

(iii) $\varphi \in \mathcal{L}_{\mathbf{T}} \Rightarrow \mathbf{O}\varphi \in \mathcal{L}_{\mathbf{T}}$ (where $\mathbf{O} \in \{\exists\mathbf{F}, \forall\mathbf{F}, \exists\mathbf{G}, \forall\mathbf{G}, \exists\mathbf{X}, \forall\mathbf{X}, \mathbf{P}, \mathbf{H}\}$)

The temporal language is similar to a modal propositional language where the atomic propositions consist of the $\mathbf{C}$ operator applied to an object-level formula. In these definitions, for a temporal model $\mathbf{M}$ based on $(\mathbf{T}, <)$, $\mathbf{t} \in \mathbf{T}$, and $\alpha \in \mathcal{L}_{\mathbf{T}}$, $(\mathbf{M}, \mathbf{t}) \models \alpha$ means that α is true in $\mathbf{M}$ at time point $\mathbf{t}$.

Definition 2.11 (Semantics)

Let a temporal model $\mathbf{M}$ based on $(\mathbf{T}, <)$, and a time point $\mathbf{t} \in \mathbf{T}$ be given, then inductively define:

(i) for $\alpha \in \mathcal{L}_{\mathbf{0}}$:

$(M, t) \models C\alpha \quad \Leftrightarrow \quad M_s \models_o \alpha$

(ii) for $\varphi, \psi \in \mathcal{L}_T$:

a) $(M, t) \models \neg \varphi \quad \Leftrightarrow \quad$ it is not the case that $(M, t) \models \varphi$

b) $(M, t) \models \varphi \wedge \psi \quad \Leftrightarrow \quad (M, t) \models \varphi$ and $(M, t) \models \psi$

(iii) for $\varphi \in \mathcal{L}_T$:

a) $(M, t) \models \exists F\varphi \quad \Leftrightarrow \quad \exists s \in T \; [\, t \ll s \;\&\; (M, s) \models \varphi \,]$

b) $(M, t) \models \exists G\varphi \quad \Leftrightarrow \quad$ there exists a branch including t such that

for all s in that branch

$[\, t \ll s \Rightarrow (M, s) \models \varphi \,]$

c) $(M, t) \models \exists X\varphi \quad \Leftrightarrow \quad \exists s \in T \; [\, t < s \;\&\; (M, s) \models \varphi \,]$

d) $(M, t) \models P\varphi \quad \Leftrightarrow \quad \exists s \in T \; [\, s \ll t \;\&\; (M, s) \models \varphi \,]$

Furthermore we introduce the following abbreviations:

$\varphi \vee \psi \equiv_{def} \neg(\neg \varphi \wedge \neg \psi)$,

$\varphi \rightarrow \psi \equiv_{def} \neg \varphi \vee \psi$,

$\top \equiv_{def} C\alpha \vee \neg C\alpha$ (for an $\alpha \in \mathcal{L}_o$),

$\bot \equiv_{def} \neg \top$,

$\forall F\varphi \equiv_{def} \neg \exists G(\neg \varphi)$,

$\forall G\varphi \equiv_{def} \neg \exists F(\neg \varphi)$,

$\forall X\varphi \equiv_{def} \neg \exists X(\neg \varphi)$ and

$H\varphi \equiv_{def} \neg P(\neg \varphi)$.

For a temporal model M, by $M \models \varphi$ we mean $(M, t) \models \varphi$ for all $t \in T$ and by $M \models K$ we mean $M \models \varphi$ for all $\varphi \in K$, where K is a set of temporal formulae.

The property of successor existence can be axiomatized by the formula $\exists F(\top)$. If in a model M the formula $P(\top)$ is true at time point t then t must have a predecessor.

3 Homomorphisms and persistency

As mentioned before, we assume the models to be forests satisfying successor existence. In this chapter M and M' denote temporal models based on the flows of time $(T, <)$ and $(T', <')$ respectively. As we are interested in linear and branching time models, we need a way of relating models and we will do this using a special class of functions between models, called homomorphisms. In the following definition, the symbol $\equiv$ denotes equality on the class of object models $\mathcal{M}_o$ (see first paragraph of Section 2.2).

Definition 3.1 (homomorphism)

A mapping $f : T \rightarrow T'$ is called a *homomorphism* of M to M' if

(i) $s < t \Rightarrow f(s) <' f(t)$

(ii) $M(s) \equiv M'(f(s))$

(iii) If s is a minimal element of T then $f(s)$ is minimal element of T'

A homomorphism preserves the temporal ordering $<$, object-level models (up to object-level equivalence), and minimal elements. Intuitively, a homomorphism can embed a model in a bigger model and it can identify points which have the same (up to isomorphism) path from their corresponding minimal elements. If a branching occurs at a certain point in time and there are equivalent object-level models at a number of next points, then we can defer the branching at this point by identifying these next points. If such a situation does not occur in a model (we shall later call such a model *closed*) then a homomorphism with this model as its domain can only be injective (in the branching time logic **CTL** (see [GK94]) a structure with this property is called *deterministic*).

Lemma 3.2

Let $f : M \rightarrow M'$ be a homomorphism.

a) The following conditions are satisfied:

(i) For all t in T and s' in T' with $s' < f(t)$ there exists an s in T with $s < t$ and $f(s) = s'$.

(ii) For every s' in T' with $s' \ll f(t)$ there exists an s in T with $s' = f(s)$ and $s \ll t$.

(iii) For all s, t in T it holds:
$f(s) < f(t)$ iff there exists a $u < t$ with $f(u) = f(s)$

(iv) For all s, t in T it holds:
$f(s) \ll f(t)$ iff there exists a $u \ll t$ with $f(u) = f(s)$

(v) If s' in T' is not in the image of f, then all t' with $s' \ll t'$ are not in the image either.

b) The following are equivalent:

(i) f is injective

(ii) for all s, t in T it holds $s < t$ if and only if $f(s) < f(t)$.

c) Let t in T be given with path P from a minimal element r to t.

Then $f(P)$ is the path from $f(r)$ to $f(t)$ and f is a bijection between P and $f(P)$.

d) f is a surjective homomorphism to the submodel $f(M) \equiv M'|f(T)$ of M'.

e) If B is a branch in M then f is injective on B, and $f(B)$ is a branch of M'; the restriction $f|B$ of f to B is an isomorphism from B onto $f(B)$.

Proof

a) (i). Suppose $s' < f(t)$, then $f(t)$ is not minimal, therefore t is not minimal in M and thus has a (unique) predecessor s and therefore $f(s) < f(t)$ and $f(s) = s'$.

a) (v) Suppose $s' < t'$ and $t' = f(t)$. From (i) it follows that $f(s) = s'$ for some s. Therefore all immediate successors of s' are not in the image, and by induction none of the t' with $s' \ll t'$ are in the image of f.

The other parts of the proof are similar.

b) For any homomorphism it holds $s < t \Rightarrow f(s) < f(t)$, so suppose also $f(s) < f(t) \Rightarrow s < t$, but f not injective. Then there exist s, t in T with $f(s) = f(t)$, which can be taken at minimal depth (distance from the minimal elements). If s and t are root of their components, then there are s', t' with $s < s'$ and $t < t'$, and thus $f(s) < f(s')$ and $f(t) < f(t')$, but as $f(s) = f(t)$ we also have $f(s) < f(t')$ from which it follows that $s < t'$ which is impossible since they are in different components. Let s and t now not be root. Then there are s', t' with $s' < s$ and $t' < t$ but $f(s') \neq f(t')$, since s and t were at minimal depth. But then $f(s')$ and $f(t')$ are both predecessors of $f(s)$, which is impossible in a tree. Now suppose f is injective and suppose we have s, t with $f(s) < f(t)$. Then t is not a root, so it has a predecessor t', and then $f(t') < f(t)$ so it must hold that $f(s) = f(t')$ but then $s = t'$ and therefore $s < t$.

We are interested in preservation of truth of formulae under these homomorphisms:

Definition 3.3

Let $f : M \rightarrow M'$ be a homomorphism.

a) The *forward persistency* property for a formula α (under f) is defined by

$$(M, t) \vDash \alpha \quad \Rightarrow \quad (M', f(t)) \vDash \alpha$$

for all time points t in T.

The *backward persistency* property for a formula α (under f) is defined by

$$(M, t) \vDash \alpha \quad \Leftarrow \quad (M', f(t)) \vDash \alpha$$

for all time points t in T.

If α is both forward and backward persistent, we call it *two-sided persistent*.

b) We say a logical connective X or temporal operator Y *preserves* forward (backward) persistency (under f) if for any forward (backward) persistent formula(s) α and β (under f) also the formulae $\alpha\ X\ \beta$, $X\ \alpha$, $Y(\alpha)$ are forward (backward) persistent (under f).

We say a logical connective X or temporal operator Y *reverses* forward (backward) persistency (under f) if for any forward (backward) persistent formula(s) α and β (under f) the formulae $\alpha\ X\ \beta$, $X\ \alpha$, $Y(\alpha)$ are backward (forward) persistent (under f).

The following theorem gives an overview of all preservation properties with respect to persistent formulae (see also Table 1).

g	H	P	∃F	∀F	∃G	∀G	∃X	∀X
preserves forward persistency	+	+	+	-	+	-	+	-
preserves backward persistency	+	+	-	+	-	+	-	+

Table 1. Preservation of persistency.

Theorem 3.4

Let $\mathbf{f : M \to M'}$ be a homomorphism.

a) Any temporal atom $\mathbf{C\alpha}$ is two-sided persistent under **f**.

b) The temporal operators **H** and **P** preserve forward and backward persistency under **f**.

c) The temporal operators **∃F**, **∃G** and **∃X** preserve forward persistency, but not necessarily backward persistency under **f**.

The temporal operators **∀F**, **∀G** and **∀X** preserve backward persistency, but not necessarily forward persistency under **f**.

d) The logical connectives $\wedge$ and $\vee$ on temporal formulae preserve both forward and backward persistency under **f**.

The logical connective $\neg$ on a temporal formula reverses forward and backward persistency under **f**.

If the temporal formula α is backward (forward) persistent and β forward (backward) persistent then $\alpha \to \beta$ is forward (backward) persistent (under **f**).

Proof

a) This is trivial, since $\mathbf{M'_{f(t)} \equiv M_t}$ for all **t** in **T**.

b) For the operator **P** we do the following. Suppose α is forward persistent and $\mathbf{(M, t) \models P(\alpha)}$. Then for some **s** with $\mathbf{s \ll t}$ it holds $\mathbf{(M, s) \models \alpha}$. By forward persistency of α we have $\mathbf{(M', f(s)) \models \alpha}$. From $\mathbf{s \ll t}$ it follows $\mathbf{f(s) \ll f(t)}$. So there exists an $\mathbf{s' \ll f(t)}$ with $\mathbf{(M', s') \models \alpha}$, or $\mathbf{(M', f(t)) \models P(\alpha)}$.

Next the case of α backward persistent: From $\mathbf{(M', f(t)) \models P(\alpha)}$ it follows that there exists an **s'** in **T'** with $\mathbf{s' \ll f(t)}$ such that $\mathbf{(M', s') \models \alpha}$.

From Lemma 3.2 it follows that there is an **s** in **T** with **s' = f(s)** and **s « t**. Now we can apply the backward persistency of **α** and conclude that **(M, s) ⊨ α** and so **(M, t) ⊨ P(α)**.

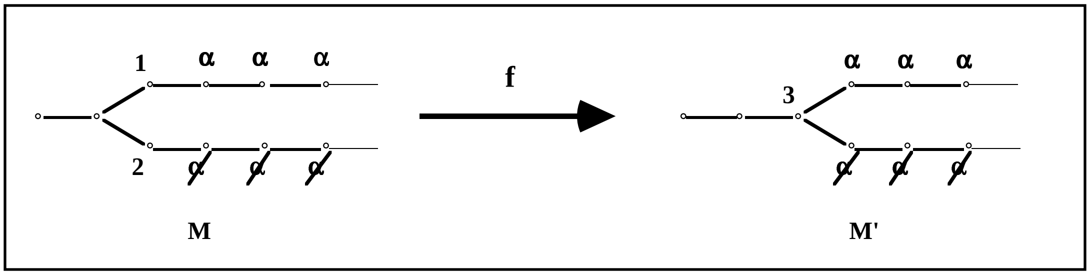
The proof for **H** is similar.

c) Suppose **α** is forward persistent and **(M, s) ⊨ ∃F(α)**. Then there is some **t** in **T** with **s « t** such that **(M, t) ⊨ α**. This implies **f(s) « f(t)** and **(M', f(t)) ⊨ α** and therefore **(M', f(s)) ⊨ ∃F(α)**.

- Suppose **α** is forward persistent and **(M, s) ⊨ ∃G(α)**. So there is a branch **B** in **M** with **s** on **B** and for all **t ∈ B** with **s « t** it holds **(M, t) ⊨ α**. Then **B' := f[B]** is a branch with **f(s) ∈ B'**. Now take a point **t' ∈ B'** with **f(s) « t'**, say **t' = f(t)**, then **s « t** and therefore **(M, t) ⊨ α**. The forward persistency of **α** ensures that **(M', f(t)) ⊨ α**, so **(M', t') ⊨ α**. It follows that **(M', f(s)) ⊨ ∃G(α)**.

The operators **∀G** and **∀F** work similar (but reversed). Also the proofs for the operators **∃X** and **∀X** are similar.

The following homomorphism shows the negative results:



In this picture $\mathbf{M_1 \equiv M_2}$, **f(1) = f(2) = 3** and **α** is an object-level formula true in the upper models, not true in the lower ones. Now **Cα** is two-sided persistent, and **(M, 1) ⊨ ∀F(Cα), ∀G(Cα)** and **∀X(Cα)**, but **(M', f(1)) ⊭ ∀F(Cα), ∀G(Cα)** and **∀X(Cα)**, so these formulae are not forward persistent. Also, **(M', f(2)) ⊨ ∃F(Cα), ∃G(Cα)** and **∃X(Cα)** but **(M, 2) ⊭ ∃F(Cα), ∃G(Cα)** and **∃X(Cα)**, so these formulae are not backward persistent.

d) We show how the connective **¬** works. Suppose the temporal formula **α** is backward persistent, and assume **(M, t) ⊨ ¬α**, then **(M, t) ⊭ α** and because **α** is backward persistent we have **(M', f(t)) ⊭ α** whence **(M', f(t)) ⊨ ¬α**. So **¬α** is forward persistent. The proof for the other case is analogous.

Theorem 3.4 can be used to build up formula that are forward or backward persistent. For instance for an object-level formula **φ** the formula **∃F(¬ P(∀G(¬ ∃F(Cφ))))** is forward persistent, whereas **∃F(P(¬ ∀G(∃F(Cφ))))** in general is not. Another example: the formula **C φ → ∀G C φ**, expressing conservativity (things which are true remain true) is not forward persistent (for $\mathbf{t}$ separately). However, conservativity can be defined by the set of persistent formulae $\mathbf{P(C\,\varphi) \rightarrow C\,\varphi}$ for all objective φ.

Theorem 3.5

Let $\mathbf{f : M \rightarrow M'}$ be a homomorphism.

If α is backward persistent then

$\mathbf{M' \vDash \alpha \;\Rightarrow\; M \vDash \alpha}$

If $\mathbf{f}$ is surjective and α is forward persistent, then

$\mathbf{M \vDash \alpha \;\Rightarrow\; M' \vDash \alpha}$.

If $\mathbf{f}$ is surjective and α is two-sided persistent then

$\mathbf{M \vDash \alpha}$ **iff** $\mathbf{M' \vDash \alpha}$

So our notion of homomorphism (as we will see, strong enough to perform the algebraic constructions we have in mind) is too weak to ensure preservation of truth for all formulae. As the example in the proof of Theorem 3.4 shows, requiring only surjectivity is not enough. When identifying two points, there may be more branches through the image than through either of the two points, destroying truth of some formulae. So we need a stronger requirement:

Definition 3.6

A homomorphism $\mathbf{f : M \rightarrow M'}$ is called *branch-surjective* if for all $\mathbf{t \in T}$ and $\mathbf{B' \in Br(T')}$: if $\mathbf{f(t) \in B'}$ then there exists a branch $\mathbf{B \in Br(M)}$ such that $\mathbf{t \in B}$ and $\mathbf{f[B] = B'}$.

A homomorphism which is surjective and branch-surjective is called *strongly branch-surjective* .

As the definition suggests, branch-surjectivity does not imply surjectivity. If $\mathbf{M'}$ consists of only one component, then this is the case. Branch-surjective homomorphisms preserve truth:

Proposition 3.7

For a branch-surjective homomorphism $\mathbf{f : M \rightarrow M'}$, a temporal formula φ and $\mathbf{t \in T}$:

$$\mathbf{(M,\ t) \vDash \varphi \quad \Leftrightarrow \quad (M',\ f(t)) \vDash \varphi}$$

Proof

It is easy to show that the operators $\exists\mathbf{F}$, $\exists\mathbf{G}$ and $\exists\mathbf{X}$ preserve backward persistency under branch-surjective homomorphisms and that the operators $\forall\mathbf{F}$, $\forall\mathbf{G}$ and $\forall\mathbf{X}$ preserve forward persistency under branch-surjective homomorphisms. Since then any operator preserves two-sided persistency under branch-surjective homomorphisms, all formulae must be two-sided persistent under branch-surjective homomorphisms.

In the literature there are some different notions of homomorphism. In [Be83], a homomorphism is a surjective function which preserves $<$ (defined on flows of time, not on models). Thus a surjective homomorphism in our sense corresponds to a homomorphism which maps minimal elements to minimal elements in Van Benthem's sense. In [Be83] also the notion of a p-morphism is defined as a homomorphism which satisfies the additional "backward clause":

$$\forall t_1 \in T, t' \in T'(f(t_1) < t' \Rightarrow \exists t_2 \in T(t_1 < t_2 \wedge f(t_2) = t'))$$

$$\forall t_1 \in T, t' \in T'(t' < f(t_1) \Rightarrow \exists t_2 \in T(t_2 < t_1 \wedge f(t_2) = t'))$$

The second part of this clause is satisfied by our homomorphisms (see Lemma 3.2 a i), and implies that minimal elements are mapped to minimal elements. The first part is equivalent to branch-surjectivity. So our notion of branch-surjective homomorphism is equivalent to the notion of p-morphism (between forests) in [Be83].

Similar notions (between structures) can also be defined for **CTL*** (see for instance [GK94]). Loosely, a homomorphism from **M** to **M'** in our sense corresponds to a simulation relation from **M** to **M'** ([GK94]). They have a similar result as Theorem 3.5 for the **CTL*** fragment containing only $\forall$, respectively $\exists$.

We intend to use homomorphisms in a number of algebraic constructions on models, combining linear models into branching time models, and combining branching time models. As we suspect that similar constructions might be of interest in other (semantical) domains, we want to set up a more general framework. In order to do this we will use category theory. When reading the next section the reader can take temporal models and homomorphisms as an example (this will turn out to form a category).

4 Using category theoretical notions for model constructions

In this section we assume given any class of objects **MOD** (we will call them models) and a notion of morphisms (we will call them homomorphisms) between them, satisfying the basic rules of category theory (e.g., see [Pi91]).

Definition 4.1

a) A homomorphism **p** is called *monic* or a *monomorphism* if for any two homomorphisms **f** and **g**

$$pf = pg \quad \Rightarrow \quad f = g$$

A homomorphism **p** is called *epic* or an *epimorphism* if for any two homomorphisms **f** and **g**

$$fp = gp \quad \Rightarrow \quad f = g$$

b) The homomorphism $\mathbf{b : B \to C}$ is an *isomorphism* if there is a $\mathbf{c : C \to B}$ such that $\mathbf{bc = id}$ and $\mathbf{cb = id}$.

In this case **B** and **C** are called *isomorphic*.

c) If $\mathbf{a_1 : A \to B}$ and $\mathbf{a_2 : A \to C}$ then $\mathbf{b : B \to C}$ is an *isomorphism over* $\mathbf{(A, a_1, a_2)}$ if $\mathbf{ba_1 = a_2}$ and there is a $\mathbf{c : C \to B}$ such that $\mathbf{bc = id}$ and $\mathbf{cb = id}$.

In this case **B** and **C** are called isomorphic over $\mathbf{(A, a_1, a_2)}$.

In many categories monic morphisms are injective functions and epic morphisms are surjective functions.

Lemma 4.2

a) **p**, **q** epic $\Rightarrow$ **pq** epic

b) **p**, **q** monic $\Rightarrow$ **pq** monic

c) **pq** epic $\Rightarrow$ **p** epic

d) **pq** monic $\Rightarrow$ **q** monic

e) $\mathbf{pq} = \mathbf{id}$ $\Rightarrow$ **p** epic & **q** monic

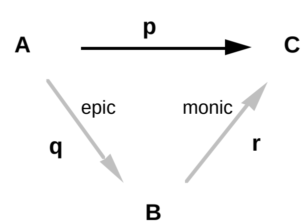
f) $\mathbf{pq} = \mathbf{id}$ & $\mathbf{qp} = \mathbf{id}$ $\Rightarrow$ **p** and **q** both epic and monic

g) $\mathbf{pq} = \mathbf{id}$ & $\mathbf{qr} = \mathbf{id}$ $\Rightarrow$ $\mathbf{p} = \mathbf{r}$ & $\mathbf{qp} = \mathbf{id}$

We will assume a number of constraints on our category. This is the first.

Constraint 1

For any $\mathbf{p : A \to C}$ there exists a model **B** and homomorphisms $\mathbf{q : A \to B}$ epic and $\mathbf{r : B \to C}$ monic such that $\mathbf{p = rq}$.

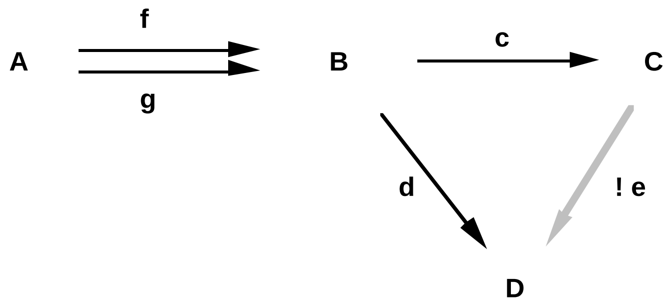


Definition 4.3

Suppose $\mathbf{f, g : A \to B}$ are given. We call $\mathbf{c : B \to C}$ a *coequalizer* of **f** and **g** if

(i) $\mathbf{cf = cg}$

(ii) For every $\mathbf{d : B \to D}$ with $\mathbf{df = dg}$ there exists a unique homomorphism $\mathbf{e : C \to D}$ such that $\mathbf{ec = d}$.



Loosely speaking a coequalizer identifies the two copies of **A** in **B** (by the two homomorphisms) without doing anything else. We will require the existence of coequalizers.

Constraint 2

For every $\mathbf{f, g : A \to B}$ there exists a coequalizer.

The word "unique" is usually meant to be up to isomorphism.

The notion of epic closure defined below is inspired by the notion of a λ-closed model as defined and exploited in [Tr76]. The terminology is inherited from the literature on model theory which aims at generalising the notion of algebraic closure from algebraic field theory, for example, [HW75], [Ho93], Ch. 8.

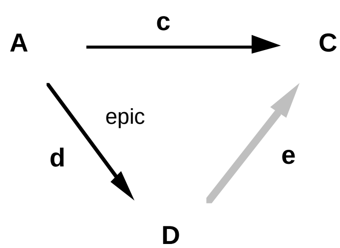
Definition 4.4

We call a model **C** *closed* if for any **D** any homomorphism $\mathbf{d : C \to D}$ is monic.

We call $\mathbf{c : A \to C}$ an *epic closure* of **A** if

(i) **c** is epic

(ii) For every **D** and epic $\mathbf{d : A \to D}$ there exists a homomorphism $\mathbf{e : D \to C}$ with $\mathbf{ed = c}$.



Intuitively, a closed model is one in which nothing can be identified further; sometimes such a model is also called a deterministic model. However, to preserve our relation to the source of literature in model theory we will use the terms ‘closed’ and ‘closure’. The epic closure of **A** maps **A** into a closed model.

Proposition 4.5

a) $\mathbf{C}$ is closed if and only if every epic $\mathbf{d : C \to D}$ is monic.

b) If $\mathbf{C}$ is closed then for any $\mathbf{A}$ there is at most one $\mathbf{a : A \to C}$.

c) Any epic closure is closed.

Proof

a) If $\mathbf{d : C \to D}$, then by Constraint 1 there are epic $\mathbf{b : C \to D'}$ and monic $\mathbf{c : D' \to D}$ with $\mathbf{cb = d}$. Then $\mathbf{b}$ is monic and by Lemma 4.2 also $\mathbf{d = cb}$ is monic.

b) Suppose $\mathbf{a, b : A \to C}$. Take a coequalizer $\mathbf{c : C \to D}$, i.e., $\mathbf{ca = cb}$. Because $\mathbf{C}$ is closed $\mathbf{c}$ is monic. Therefore $\mathbf{a = b}$.

c) Let $\mathbf{a : A \to C}$ be an epic closure and $\mathbf{f : C \to D}$ epic. Then $\mathbf{fa}$ is epic. Therefore there exists a homomorphism $\mathbf{g : D \to C}$ with $\mathbf{g\,fa = a}$. This implies $\mathbf{gf = id}$; therefore $\mathbf{f}$ is monic.

Lemma 4.6

Suppose homomorphisms $\mathbf{b : A \to B}$ and $\mathbf{c : A \to C}$ are given with $\mathbf{b}$ epic.

a) Then there exists at most one $\mathbf{d : B \to C}$ with $\mathbf{db = c}$.

If $\mathbf{c}$ is epic, then also such a $\mathbf{d}$ is epic.

b) The homomorphism $\mathbf{d}$ in Definition 4.4 (ii) is always unique and epic.

Proof

a) Suppose $\mathbf{d, e : B \to C}$ with $\mathbf{db = eb = c}$. From $\mathbf{b}$ epic it follows $\mathbf{d = e}$.

If $\mathbf{c}$ is epic, then from Lemma 4.2c) it follows that such a $\mathbf{d}$ (if it exists) is epic too.

b) This immediately follows from a).

The idea is that from a given model $\mathbf{A}$ by epimorphisms we form homomorphic images until nothing new can be obtained by epimorphisms. The following proposition shows that such a process of closure can lead to at most one closed model (up to isomorphism).

Proposition 4.7

Suppose $\mathbf{c : A \to C}$ is an epic closure.

a) If $\mathbf{d : C \to D}$ is epic, then there exists a $\mathbf{d' : D \to C}$ such that $\mathbf{dd' = id}$ and $\mathbf{d'd = id}$, i.e., $\mathbf{C}$ and $\mathbf{D}$ are isomorphic.

b) If $\mathbf{c : A \to C}$ and $\mathbf{d : A \to D}$ are epic closures then $\mathbf{C}$ and $\mathbf{D}$ are isomorphic over $\mathbf{(A, c, d)}$.

Proof

a) From Lemma 4.2a) it follows that $\mathbf{dc}$ is epic. Because $\mathbf{c : A \to C}$ is an epic closure there exists a $\mathbf{d' : D \to C}$ with $\mathbf{d'dc = c}$. Since $\mathbf{c}$ is epic it follows that $\mathbf{d'd = id}$. Now $\mathbf{(dd')\, dc = d\, (d'd)\, c = dc}$. Since $\mathbf{dc}$ is epic it follows that $\mathbf{dd' = id}$.

b) From the fact that $\mathbf{c : A \to C}$ is an epic closure it follows that there exists a homomorphism $\mathbf{a : D \to C}$ such that $\mathbf{c = ad}$. Similarly there exists a homomorphism $\mathbf{b : C \to D}$ such that $\mathbf{d = bc}$. It follows that $\mathbf{abc = ad = c}$, and since $\mathbf{c}$ is epic $\mathbf{ab = id}$. Similarly $\mathbf{ba = id}$.

Closed models behave quite convenient; this is reason enough to claim that there should exist enough closed models. We will see that in the interesting cases indeed these exist.

Constraint 3

Every model has an epic closure (unique up to isomorphism).

Epic closures are unique if they exist and the epimorphism is also unique. Moreover, in specific instances of categories often some canonical construction can be given for them; we will sometimes use the notation $\mathbf{cl(M)}$ to denote (a canonical construction for) an epic closure of $\mathbf{M}$.

Now we have the possibility to map any model onto its epic closure, a next step is to require that a number of these closures can be embedded in one (closed) model, or to require that we can form a joint closure of any set of models. For this purpose we will generalize the notion of epic closure.

Definition 4.8

Let $\mathbf{I}$ be any index set and let $\mathbf{(a_i)_{i \in I}}$ be a collection of homomorphisms $\mathbf{a_i : A_i \to B}$.

We call $\mathbf{(a_i)_{i \in I}}$ *jointly epic* if for all $\mathbf{f, g : B \to C}$ with $\mathbf{fa_i = ga_i}$ for every $\mathbf{i}$ it holds $\mathbf{f = g}$.

As a generalisation of Lemma 4.2 we have the following lemma.

Lemma 4.9

Suppose $(b_i)_{i \in I}$ and c are given and $a_i = c\, b_i$ for all i.

a) If $(a_i)_{i \in I}$ are jointly epic then c is epic.

b) If $(b_i)_{i \in I}$ are jointly epic and c is epic then $(a_i)_{i \in I}$ are jointly epic.

Proof

a) Suppose $fc = gc$, then $fa_i = fc\, b_i = gc\, b_i = ga_i$ for all i; therefore $f = g$.

b) Similar

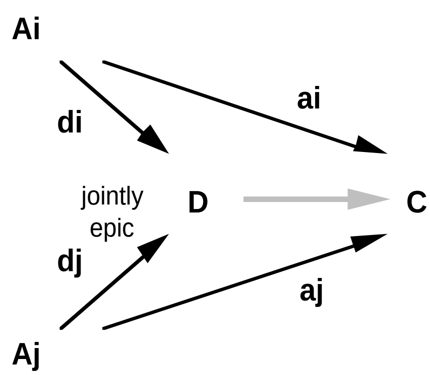
Definition 4.10 (joint closure)

Let I be any index set and let $(A_i)_{i \in I}$ be a collection of models.

We call $(C, (a_i)_{i \in I})$ with $a_i : A_i \to C$ a homomorphism for each i in I, a *joint (epic) closure* for $(A_i)_{i \in I}$ if

(i) $(a_i)_{i \in I}$ are jointly epic.

(ii) For every D with jointly epic homomorphisms $d_i : A_i \to D$ there exists a homomorphism $c : D \to C$ such that $ca_i = d_i$ for all i in I.



The notion joint closure is not present in the literature on category theory, as far as we know. It is a natural notion following the model-theoretic literature about generalisations of the notion of algebraic closure mentioned after Definition 4.4. We will give some of the details of how this notion relates to, for example, the wellknown notion of co-product here.

Lemma 4.11

Any joint closure is closed.

Proof

Let $(C, (a_i)_{i \in I})$ with $a_i : A_i \to C$ be a joint closure and $f : C \to D$ epic. Then (fa_i) are jointly epic. Therefore there exists a homomorphism $g : D \to C$ with $g\, fa_i = a_i$ for all i. This implies $gf = id$; therefore f is monic.

Proposition 4.12

Suppose $(A_i)_{i \in I}$ is a collection of models.

If a joint closure for $(A_i)_{i \in I}$ exists, it is unique up to isomorphism.

More precisely, if $(C, (c_i)_{i \in I})$ and $(D, (d_i)_{i \in I})$ are joint closures of $(A_i)_{i \in I}$ then there exists an isomorphism $f : C \to D$ such that for all i it holds $f c_i = d_i$.

Proof

Suppose C and D are joint closures for $(A_i)_{i \in I}$ with homomorphisms $d_i : A_i \to D$ and $c_i : A_i \to C$ respectively. Then there exist (unique) homomorphisms $f : D \to C$ and $g : C \to D$ such that for every i in I it holds

$fd_i = c_i$ and $gc_i = d_i$

Therefore

$fg\ c_i = fd_i = c_i$

Since also $id\ c_i = c_i$ and there is only one unique homomorphism with this property $fg = id$. Similarly $gf = id$. Therefore C and D are isomorphic.

Notice that for a joint closure $(C, (c_i)_{i \in I})$ the homomorphisms $A_i \to C$ are unique, so C already determines in a unique manner the joint closure. If in a certain category there exists a canonical construction for a joint closure, we will denote (the model C of) one by $jcl((A_i)_{i \in I})$.

Lemma 4.13

Suppose homomorphisms $b_i : A_i \to B$ and $c_i : A_i \to C$ are given with $(b_i)_{i \in I}$ jointly epic.

a) Then there exists at most one $d : B \to C$ with $db_i = c_i$ for all i.

If the $(c_i)_{i \in I}$ are jointly epic, then also such a d is epic.

b) The homomorphism c in Definition 4.10 is always unique and epic.

The following constraint would guarantee the existence of closures of sets of models: for every set of models there exists a closure (unique up to isomorphism). However, the existence will follow from other, more general constraints. If we require the existence of coproducts we can follow an alternative path of construction.

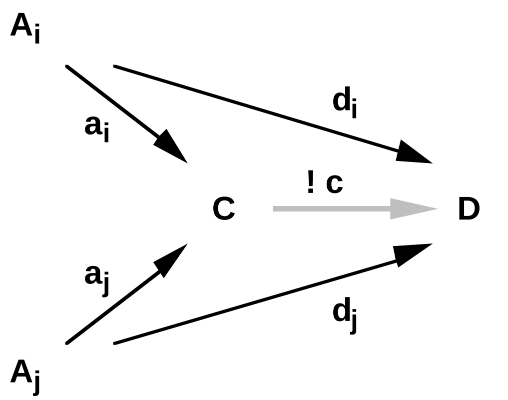
Definition 4.14

Let I be any index set and let $(A_i)_{i \in I}$ be a collection of models

We call $(C, (a_i)_{i \in I})$ a *coproduct* for $(A_i)_{i \in I}$ if

(i) $a_i : A_i \to C$ is a homomorphism for each i in I.

(ii) For every D with homomorphisms $d_i : A_i \to D$ there exists a unique homomorphism $c : C \to D$ such that $ca_i = d_i$ for all i in I.



A coproduct allows us to combine models into one model, without doing anything else.

Lemma 4.15

If $(C, (a_i)_{i \in I})$ is a coproduct for $(A_i)_{i \in I}$ then the $(a_i)_{i \in I}$ are jointly epic.

The following is well known from the literature. The proof is in fact similar to that of Proposition 4.13.

Proposition 4.16

If a coproduct exists, it is unique up to isomorphism.

Constraint 4

Every set of models has a coproduct (unique up to isomorphism).

Above we first introduced the notion of closed model and epic closure of a model. Now we can use the following construction. Given a collection of models, first take its coproduct, and next take the epic closure of the coproduct. The resulting model is the joint closure of the collection.

Theorem 4.17

The epic closure of the coproduct of a collection of models $(A_i)_{i \in I}$ is a joint closure of $(A_i)_{i \in I}$.

Proof

Let C be the coproduct of $(A_i)_{i \in I}$ (with homomorphisms $c_i : A_i \rightarrow C$) and D its epic closure (with homomorphism $d : C \rightarrow D$).

We will show that $(\mathbf{D}, (\mathbf{d\,c_i})_{i \in I})$ is a joint closure of $(\mathbf{A_i})_{i \in I}$. Suppose $\mathbf{E}$ and jointly epic homomorphisms $\mathbf{e_i : A_i \to E}$ are given. We will show how to map $\mathbf{E}$ to $\mathbf{D}$. First, because $\mathbf{C}$ is the coproduct of $(\mathbf{A_i})_{i \in I}$, there exists a homomorphism $\mathbf{f : C \to E}$ such that for all $\mathbf{i}$ in $\mathbf{I}$ it holds $\mathbf{f\,c_i = e_i}$.
From Lemma 4.9 it follows that $\mathbf{f}$ is epic. Because $\mathbf{D}$ with $\mathbf{d : C \to D}$ is the epic closure of $\mathbf{C}$ there exists a homomorphism $\mathbf{g : E \to D}$ with $\mathbf{gf = d}$. Therefore we have found a $\mathbf{g}$ with $\mathbf{g\,e_i = gf\,c_i = dc_i}$ for all $\mathbf{i}$.

Proposition 4.18

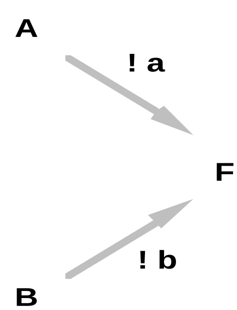
Let $(\mathbf{A_i})_{i \in I}$ and $\mathbf{C}$ be given with jointly epic homomorphisms $\mathbf{c_i : A_i \to C}$. Then $\mathbf{jcl((A_i)}_{i \in I})$ is isomorphic to $\mathbf{cl(C)}$.

Proof

Let $\mathbf{J}$ be the joint closure of $(\mathbf{A_i})_{i \in I}$ with canonical homomorphisms $\mathbf{a_i : A_i \to J}$. There is a (unique) homomorphism $\mathbf{c : C \to J}$ such that $\mathbf{c\,c_i = a_i}$ for all $\mathbf{i}$. By lemma 4.9 this is epic. We will prove that $\mathbf{J}$ is the epic closure of $\mathbf{C}$. Suppose an epic $\mathbf{d : C \to D}$ is given. Then by lemma 4.9 the homomorphisms $\mathbf{dc_i}$ are jointly epic. Therefore (joint closure property) there is a homomorphism $\mathbf{e : D \to J}$ with $\mathbf{ed = c}$. This proves that $\mathbf{J}$ is the epic closure of $\mathbf{C}$.

Definition 4.19 (final model)

Let $\mathbf{MOD'}$ be a sub-class of $\mathbf{MOD}$ and $\mathbf{F}$ any model in $\mathbf{MOD'}$.
The model $\mathbf{F}$ is called *final* in $\mathbf{MOD'}$ if for each model $\mathbf{M}$ in $\mathbf{MOD'}$ there is a unique homomorphism $\mathbf{f : M \to F}$.



Theorem 4.20

Let $\mathbf{MOD'}$ be a sub-set of $\mathbf{MOD}$ and $\mathbf{F}$ any model in $\mathbf{MOD'}$.
Then the following are equivalent .
(i) $\mathbf{F}$ is a joint closure of all models of $\mathbf{MOD'}$
(ii) $\mathbf{F}$ is a final model in $\mathbf{MOD'}$

Proof

(i) => (ii) Let $(F, (f_A)_{A \in MOD'})$ be a joint closure of all models in **MOD'**, then any **A** in **MOD'** is mapped to **F** by f_A. By Lemma 4.11 the model **F** is closed, and by Lemma 4.5 there exists at most one homomorphism $A \to F$; therefore f_A is unique. This proves that **F** is a final model in **MOD'**.

(ii) => (i) Suppose **F** is a final model in **MOD'**. Then for every **A** in **MOD'** there is a unique homomorphism $f_A : A \to F$. We will show that $(F, (f_A)_{A \in MOD'})$ is a joint closure for **MOD'**.

First, the $(f_A)_{A \in MOD'}$ are jointly epic, since if $g\, f_A = h\, f_A$ for all **A** in **MOD'**, then in particular $g\, f_F = h\, f_F$. Now $f_F = id$, so $g = h$.

Next, let any **G** be given with jointly epic homomorphisms $g_A : A \to G$ for all **A** in **MOD'**. Then $f_G : G \to F$ maps **G** to **F**. For any **A** in **MOD'**, because both $f_A, (f_G\ g_A) : A \to F$ and there is only one such homomorphism, we have $(f_G\ g_A) = f_A$.

So a final model is a model into which every other model can uniquely be mapped. It is the model constructed by closing the coproduct (or, taking the joint closure).

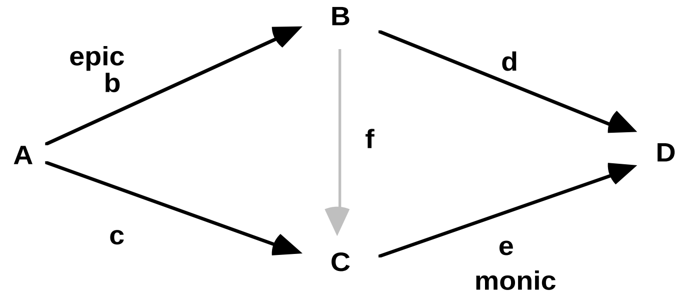
Definition 4.21 (amalgamation property)

A category has the *amalgamation property* if for every two homomorphisms $b : A \to B$ and $c : A \to C$ there exists a **D** and homomorphisms $d : B \to D$ and $e : C \to D$ such that $db = ec$.

It is not difficult to verify that the amalgamation property follows from the existence of joint closures. To establish a next result, we need an additional constraint:

Constraint 5

If four homomorphisms $b : A \to B$, $c : A \to C$, $d : B \to D$ and $e : C \to D$ are given such that $db = ec$, **b** is epic and **e** is monic, then there is a homomorphism $f : B \to C$ such that $fb = c$ and $ef = d$.



Proposition 4.22

Suppose the amalgamation property and Constraint 5 are fulfilled.

Let $\mathbf{c : A \to C}$ be given with $\mathbf{C}$ closed.

a) If $\mathbf{c}$ is monic then $\mathbf{A}$ is closed.

b) If $\mathbf{c}$ is epic, then $\mathbf{c : A \to C}$ is an epic closure of $\mathbf{A}$. In particular, $\mathbf{C}$ is its own epic closure.

c) If epic $\mathbf{a : A \to B}$ and monic $\mathbf{b : B \to C}$ are given such that $\mathbf{ba = c}$, then $\mathbf{a : A \to B}$ is an epic closure of $\mathbf{A}$.

Proof

a) Suppose $\mathbf{c}$ is monic and a homomorphism $\mathbf{d : A \to D}$ is given. Apply the amalgamation property to find an $\mathbf{E}$ and homomorphisms $\mathbf{e : C \to E}$ and $\mathbf{f : D \to E}$ with $\mathbf{ec = fd}$. Because $\mathbf{C}$ is closed $\mathbf{e}$ is monic. By Lemma 4.2 we obtain $\mathbf{ec}$ is monic, and applying it again we have $\mathbf{d}$ is monic. Therefore $\mathbf{A}$ is closed.

b) Suppose $\mathbf{c}$ is epic and an epimorphism $\mathbf{d : A \to D}$ is given. Again apply the amalgamation property to find an $\mathbf{E}$ and homomorphisms $\mathbf{e : C \to E}$ and $\mathbf{f : D \to E}$ with $\mathbf{ec = fd}$. By Constraint 5 we find a homomorphism $\mathbf{g : D \to C}$ such that $\mathbf{gd = c}$ and $\mathbf{eg = f}$. Therefore $\mathbf{c : A \to C}$ is an epic closure of $\mathbf{A}$.

c) This follows from a) and b).

In this section we have described the possibilities for incorporating a number of objects into one object (the co-product) and compacting the result (closure) to obtain one 'efficient representation' of these objects (the joint closure). We will apply these ideas to the category of temporal models in the next section.

5 Joint closure constructions for temporal models

In this section we will show that the category-theoretic machinery described in the previous section is applicable to temporal models. We will describe the category of our models, **BTM**, and verify that the constraints of Section 4 are satisfied in **BTM** (Note that the object-level language and models are fixed). In Section 6 we will present some of the implications of our framework to the model theory of temporal logic.

Definition 5.1

The objects of our *category of temporal models*, **BTM** are temporal models $\mathbf{(M, T, <)}$ such that (see Section 3):

(i) $\mathbf{(T, <)}$ is a forest (set of disjoint trees) with infinite branches;

(ii) $\mathbf{M}$ is a function from $\mathbf{T}$ to the set of object-level models $\mathcal{M}_\mathbf{0}$.

The arrows are homomorphisms between the models as defined in Section 3.

If no confusion is expected we often denote a model by **M** only.

It can easily be verified that this indeed defines a category, i.e. the composition of two homomorphisms is a homomorphism, composition is associative and the identity function on any model is a homomorphism.

Proposition 5.2

The following hold for homomorphisms in **BTM**.

a) A homomorphism is epic iff it is surjective.

b) A homomorphism is monic iff it is injective.

c) If we have objects A_i and homomorphisms $h_i : A_i \to C$ then: the h_i are jointly epic iff for every **c** in **C** there is an **i** and an **a** in A_i with $h_i(a) = c$.

Proof

a) Let $p : (M, T, <) \to (N, S, <)$ be an epimorphism, but suppose **s** in **S** is not in the image of **p**. Let T_S be the subtree of **S** rooted at **s**; by Lemma 3.2(v), T_S is not in the image of **p** either. Let $T_{S'}$ be an isomorphic tree, say with root **s'**. Now construct a model **(N', S', <)** which is the same as **(N, S, <)**, but also incorporates $T_{S'}$, with if $t < s$, then it has an extra link $t < s'$. Define two homomorphisms $f, g : (N, S, <) \to (N', S', <)$ where **f** is the embedding of **S** into **S'**, and **g** is the same except that T_S in **S** is mapped to $T_{S'}$ in **S'**. Now it holds $fp = gp$ but $f \neq g$. Contradiction, so **p** must be surjective. On the other hand, suppose **p** is surjective, let **f**, **g** be given with $fp = gp$ and suppose $f \neq g$. Then there exists **s** in **S** with $f(s) \neq g(s)$, but $s = p(t)$ for some **t** in **T**, and then $fp(t) = f(s) \neq g(s) = gp(t)$. Therefore $f = g$ so **p** is epimorphic.

b) Suppose **p** is injective, but not monic, then there exists **f**, **g** with $pf = pg$ but $f \neq g$. Then there is **s** with $f(s) \neq g(s)$, but then $pf(s) \neq pg(s)$. Therefore **p** is monomorphic. On the other hand, suppose $p : (M, T, <) \to (N, S, <)$ is not injective, so there are **s**, **t** in **T** with $p(s) = p(t)$. But then **p** maps the path P_s from the root to **s** and the path P_t from the root to **t** onto the same path from the root to **p(s)**. Now construct a model **(M', T', <)** induced by P_s and P_t, and let **f** be the embedding of this model into **(M, T, <)**, and let **g** map P_s into P_t and vice versa. Now it holds that $pf = pg$ but $f \neq g$, and therefore **p** is not monic.

c) Suppose h_i are jointly epic, but there is a **c** which is in the image of no h_i. Then the subtree T_c rooted at **c** is in no image either. Now construct a model **D** which is a copy of **C** except that the tree T_c has a copy T_c' in which both **c** and **c'** have the same immediate predecessor. Let **f** be the embedding of **C** into **D** , and g maps T_c onto T_c'. Then for all **i** $fh_i = gh_i$ but $f \neq g$. So as this was not to be the case, every $\mathbf{c}$ is in the image of some $\mathbf{h_i}$. Conversely, suppose every $\mathbf{c}$ in $\mathbf{C}$ is in the image of some $\mathbf{h_i}$. Take $\mathbf{f}$ and $\mathbf{g}$ with $\mathbf{fh_i = gh_i}$ for all $\mathbf{i}$, but suppose $\mathbf{f \neq g}$, so there is a $\mathbf{c}$ in $\mathbf{C}$ with $\mathbf{f(c) \neq g(c)}$, then $\mathbf{c}$ is in the image of $\mathbf{h_i}$ for some $\mathbf{h_i}$, suppose $\mathbf{c = h_i(s)}$, but then $\mathbf{fh_i(s) = f(c) \neq g(c) = gh_i(c)}$, which was not to be the case. Therefore $\mathbf{f = g}$, so the $\mathbf{h_i}$ are jointly epic.

In the category **BTM** we can give a characterization of closed models which renders the intuition of what closed is better:

Proposition 5.3

Let $\mathbf{M}$ be a model, then the following are equivalent:

(i) $\mathbf{M}$ is closed.

(ii) For all $\mathbf{s}$, $\mathbf{t}$ and $\mathbf{t'}$ with $\mathbf{s < t}$, $\mathbf{s < t'}$ and $\mathbf{M_t \equiv M_{t'}}$ it holds that $\mathbf{t = t'}$, and for minimal elements $\mathbf{r}$, $\mathbf{r'}$ with $\mathbf{M_r \equiv M_{r'}}$ it holds that $\mathbf{r = r'}$.

Proof

Suppose $\mathbf{M}$ is closed but there are $\mathbf{s, t \neq t'}$ with $\mathbf{s < t}$, $\mathbf{s < t'}$ and $\mathbf{M_t \equiv M_{t'}}$. Define a homomorphism $\mathbf{f}$ on $\mathbf{T}$ which is identity except that $\mathbf{f(t) = f(t')}$. Let the successor relation on $\mathbf{T' = f[T]}$ be defined by $\mathbf{u < v}$ iff there are $\mathbf{u'}$ and $\mathbf{v'}$ in $\mathbf{T}$ with $\mathbf{u = f(u')}$ and $\mathbf{v = f(v')}$ and $\mathbf{u' < v'}$. Let $\mathbf{M'}$ be defined by $\mathbf{M'_{f(s)} \equiv M_s}$. Since $\mathbf{f}$ is surjective and identity except on $\mathbf{t}$ and $\mathbf{t'}$ where $\mathbf{M_t = M_{t'}}$ this is well-defined. It is easy to check that the model $\mathbf{M'}$ based on $\mathbf{(T', <)}$ is a forest and therefore in **BTM**. Now $\mathbf{f}$ is not injective, so not monic and then $\mathbf{M}$ is not closed, which it was supposed to be. If there are roots $\mathbf{r}$, $\mathbf{r'}$ with $\mathbf{M_r \equiv M_{r'}}$ then this same construction can be applied. Therefore condition (ii) must be satisfied.

Conversely, suppose condition (ii) is satisfied, but $\mathbf{M}$ is not closed. Then there must be a homomorphism $\mathbf{f : M \to M'}$ which is not monic, so not injective. Therefore there must be $\mathbf{t \neq t'}$ with $\mathbf{f(t) = f(t')}$. One can take these points at a minimal distance from their roots. If they are roots, then $\mathbf{M_t \equiv M'_{f(t)} \equiv M'_{f(t')} \equiv M_{t'}}$ and then condition (ii) is violated. If they are not roots, then still $\mathbf{M_t \equiv M_{t'}}$. Furthermore they have immediate predecessors $\mathbf{s < t}$ and $\mathbf{s' < t'}$. Then $\mathbf{f(s) < f(t)}$ and $\mathbf{f(s') < f(t') = f(t)}$. Then it must hold that $\mathbf{f(s) = f(s')}$, and as above then also $\mathbf{M_s \equiv M_{s'}}$. But as $\mathbf{t}$ and $\mathbf{t'}$ were chosen at minimal depth it must hold that $\mathbf{s = s'}$, and then condition (ii) is violated again. Therefore $\mathbf{M}$ must be closed.

So in a closed model there are no two different minimal elements with equivalent object-level model and any two different successors of a point have a non-equivalent object-level model (again, in $\mathbf{CTL^*}$ a structure with this property is called *deterministic*). We will verify which constraints are satisfied in **BTM**:

Constraint 1

For any $\mathbf{p : M \to L}$ there exists $\mathbf{N}$ and homomorphisms $\mathbf{q : M \to N}$ epic and $\mathbf{r : N \to L}$ monic such that $\mathbf{p = rq}$.

Proof

Let $\mathbf{M, L}$ be based on $\mathbf{(S, <), (P, <)}$ respectively. Let $\mathbf{T}$ consist of all $\mathbf{s}$ in $\mathbf{P}$ which are in the image of $\mathbf{p}$. Let the successor relation on $\mathbf{T}$ be defined by $\mathbf{s < t}$, where $\mathbf{s = p(s')}$ and $\mathbf{t = p(t')}$ iff $\mathbf{p(s') < p(t')}$ in $\mathbf{P}$, and let $\mathbf{N}$ be defined by, if $\mathbf{s = p(s')}$: $\mathbf{N(s) \equiv M(s')}$. Let $\mathbf{q}$ be equal to $\mathbf{p}$ on all $\mathbf{s}$ in $\mathbf{S}$, then $\mathbf{q}$ is surjective and therefore epic. Let $\mathbf{r}$ be the embedding of $\mathbf{T}$ in $\mathbf{P}$. Then $\mathbf{r}$ is injective and therefore monic. Now it needs to be checked that $\mathbf{(N, T, <)}$ is indeed an element of **BTM:** every component $\mathbf{C}$ of $\mathbf{T}$ has to be a tree:

(i): rooted

Take $\mathbf{s}$ in $\mathbf{C}$, then $\mathbf{s = p(s')}$, let $\mathbf{r'}$ be the root of the component of $\mathbf{s'}$, then $\mathbf{p(r') = r}$ is the root of $\mathbf{C}$. Take $\mathbf{t}$ in $\mathbf{C}$ and suppose $\mathbf{t \neq r}$. Then $\mathbf{t = p(t')}$, so there is a path $\mathbf{r' \ll t'}$, which is mapped into a path $\mathbf{r \ll t}$

(ii): Left Linear:

Take $\mathbf{s}$ in $\mathbf{C}$ then the set of $\mathbf{t}$ with $\mathbf{t \ll s}$ is the path $\mathbf{r \ll s}$, which is isomorphically mapped from $\mathbf{S}$, where that path is totally ordered.

The idea in the proof is that $\mathbf{q}$ maps $\mathbf{M}$ onto its image (in $\mathbf{L}$) and $\mathbf{r}$ embeds this image into $\mathbf{L}$.

Constraint 2

For every $\mathbf{f, g : L \to M}$ there exists a coequalizer $\mathbf{h : M \to N}$

Proof

Let $\mathbf{L}$ and $\mathbf{M}$ be based on $\mathbf{(P, <)}$ and $\mathbf{(T, <)}$ respectively.

Define a relation $=\sim$ on $\mathbf{T}$ as follows: $\mathbf{s =\sim t}$ iff there is a $\mathbf{u}$ in $\mathbf{P}$ with $\mathbf{s = f(u)}$ and $\mathbf{t = g(u)}$ or vice versa, and let $\sim$ be the reflexive and transitive closure of $=\sim$.

Let $\mathbf{S}$ consist of the equivalence classes $\mathbf{T/\sim}$. For $\mathbf{k_1, k_2}$ in $\mathbf{S}$, define $\mathbf{k_1 < k_2}$ iff there is $\mathbf{s}$ in $\mathbf{k_1}$ and $\mathbf{t}$ in $\mathbf{k_2}$ with $\mathbf{s < t}$. Set $\mathbf{N([s]) \equiv M(s)}$, and let $\mathbf{h(s) = [s]}$ (the class of $\mathbf{s}$). First we will check if we have constructed a correct model $\mathbf{(N, S, <)}$.

1. It can be easily checked that $<$ on $\mathbf{S}$ is irreflexive, antisymmetric and antitransitive.

2. $\mathbf{N}$ is well-defined: if $\mathbf{s =\sim t}$, then $\mathbf{s = f(u)}$, $\mathbf{t = g(u)}$, so $\mathbf{M(s) \equiv L(u) \equiv M(t)}$. Since this is transitive, $\mathbf{M}$ is the same on all members of a class.

3. S is a forest:

It can be easily seen that if $\mathbf{s \sim t}$ then $\mathbf{s}$ and $\mathbf{t}$ are at the same depth in $\mathbf{T}$. Therefore we can associate a depth to every class, and if $\mathbf{[s] < [t]}$ then the associated depth of $\mathbf{[s]}$ is one lower then the depth associated with $\mathbf{[t]}$. If the depth of a class is zero then it can have no predecessor because all of its elements are at depth zero and therefore have no predecessors. Thus, no infinite descending chains can exist, so **S** is well-founded. The proof of left linearity is straightforward.

Now we will see that **h** is a homomorphism: if $\mathbf{s < t}$ then by definition $\mathbf{[s] < [t]}$. Also by definition $\mathbf{N([s]) \equiv M(s)}$. Suppose **s** is minimal, then if $\mathbf{t \sim s}$, **t** must be at the same depth as **s** so it is minimal too, therefore **[s]** can have no predecessor, so is minimal too.

Let's check that $\mathbf{hf = hg}$: take **s** in **P** then $\mathbf{f(s) \sim g(s)}$ so that $\mathbf{hf(s) = hg(s)}$. Suppose we have a homomorphism $\mathbf{k : (MAT, <) \rightarrow (K, R, <)}$ with $\mathbf{kf = kg}$. Then define $\mathbf{e: (N, S, <) \rightarrow (K, R, <)}$ by $\mathbf{e([s]) = k(s)}$. This is well-defined since if $\mathbf{s \sim t}$ then $\mathbf{s = f(u)}$ and $\mathbf{t = g(u)}$ and then $\mathbf{k(s) = k(f(u)) = k(g(u)) = k(t)}$. If $\mathbf{[s] < [t]}$ then there are **u** in **[s]** and **v** in **[t]** with $\mathbf{u < v}$ so then $\mathbf{k(u) < k(v)}$ so $\mathbf{e([s]) < e([t])}$. Also, $\mathbf{N([s]) \equiv M(s) \equiv K(k(s)) \equiv K(e[s])}$. If **[s]** is a minimal element, then **s** is minimal, and therefore **k(s)** is minimal, so **e** maps minimal elements to minimal elements.

By definition it holds that $\mathbf{eh = k}$.

Uniqueness:

If for all **s** it must hold that $\mathbf{e([s]) = k(s)}$ then by definition this **e** is unique.

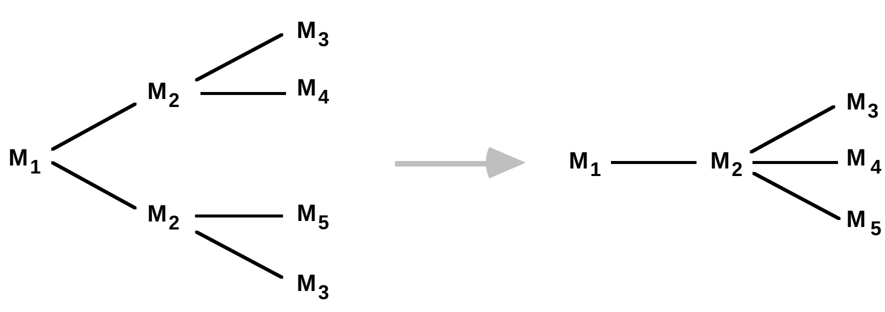
The coequalizer identifies points which are the image of a point of **L** under **f**, respectively **g**.

Constraint 3

For every **D** in **BTM** there exists an epic closure of **D**.

Proof

The idea in the construction of this closure is to identify common initial subbranches (up to object-level equivalence) with each other as much as possible.



Let **D** be **(M, T, <)**.

Define an equivalence relation on **T** as follows:

$\mathbf{s \sim t}$ iff $\mathbf{s' < s}$ and $\mathbf{t' < t}$ and $\mathbf{s' \sim t'}$ and $\mathbf{M(s) \equiv M(t)}$, or

s and **t** are root and **M(s)** $\equiv$ **M(t)**.

Now let **S** be the set of equivalence classes **T/~**, define **N** on **S** by **N([s])** $\equiv$ **M(s)** (this is well-defined), where **[s]** is the equivalence class of **s**. For two equivalence classes **k1** and **k2**, define **k1 < k2** iff there is **s** in **k1** and **t** in **k2** with **s < t**. Notice that two elements can only be equivalent if they are at the same distance from the root. It is easy to check that the ordering < on **S** is irreflexive and antitransitive.

The proof of well-foundedness of **(N, S, <)** is similar to the proof of constraint 3. Left-linearity is also straightforward.

Now we will check that the mapping **h** which takes an element **s** of **T** to its class in **S** is a homomorphism:

1. if **s < t** in **T** then **[s] < [t]** by definition.
2. **N([s])** $\equiv$ **M(s)** by definition
3. Take an **s** in **T** which is a root, and suppose **[s]** is not a root, then

there is a **t ~ s** which is not a root, but then it can not hold that **s ~ t**. By definition **h** is surjective and therefore epic. Now suppose we have **g : (M, T, <)** $\rightarrow$ **(L, P, <)** epic. Define **e : (L, P, <)** $\rightarrow$ **(N, S, <)** as follows:

if we have **t** in **P**, then **t = g(s)** for some **s**. Let **e(t) = [s]**. We have to show that this is well-defined: suppose **g(s) = g(t)**, then we have to show that **s ~ t**. If **g(s) = g(t)** then **M(s)** $\equiv$ **N(g(s))** $\equiv$ **N(g(t))** $\equiv$ **M(t)**. If **s** and **t** are root then we are ready. If not then they are both not root (because **g** is a homomorphism). So let **s' < s** and **t' < t** then it must hold that **g(s') = g(t')** and therefore **M(s')** $\equiv$ **M(t')**, so now we have to check if **s' ~ t'**. This process can be iterated until both **s'** and **t'** are root, then we are ready. So **e** is well-defined. If **s < t** in **P** then because **g** is epic we have **s = g(s')**, **t = g(t')** and then **s' < t'** so **[s'] < [t']**. Take **s** in **P**, then **s = f(t)**, so **L(s)** $\equiv$ **M(t)** $\equiv$ **N([t])** $\equiv$ **N(e(s))**.

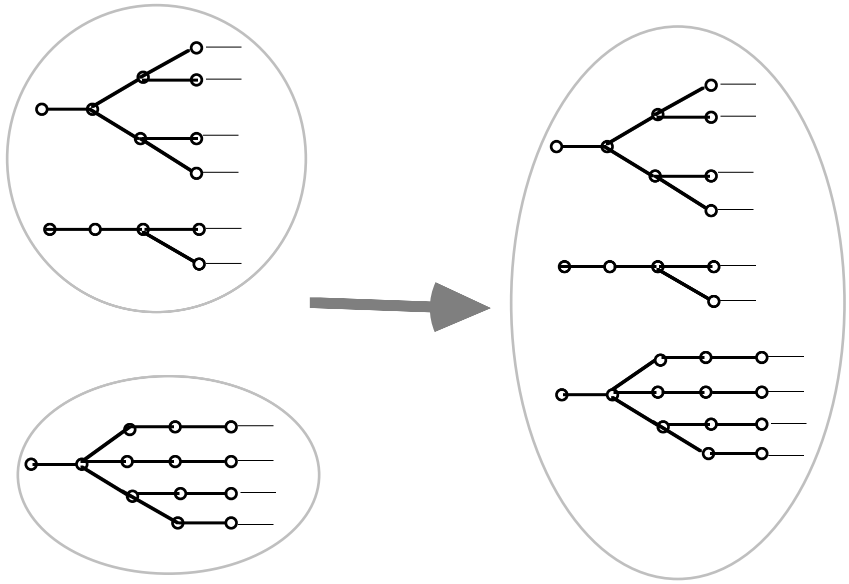
Let **s** in **P** be a root, then **s = f(t)** with **t** a root in **T** and then **e(s) = [t]** is a root. Thus, **e** is a homomorphism.

Using Proposition 5.3 the closure must identity minimal elements with equivalent object-level models and points with a common predecessor which have equivalent object-level models.

Constraint 4

Every set of models has a coproduct.

This coproduct is just the disjoint union of the set of models.



Proof

Let $(A_i)_{i \in I}$ be a collection of models, where the flows of time can be taken disjoint. Then the coproduct C is the model which is the disjoint union of models in A_i. This is then always a forest, and therefore again in our category. For every i the associated homomorphism $h_i : A_i \to C$ is just the embedding of A_i into the part of C taken from A_i. Now suppose we have a model B and homomorphisms $g_i : A_i \to B$ for every i. Set $h_i(A_i)$ to be the part of C coming from A_i. Now define a homomorphism $f : C \to B$ by $f(s) = g_i(t)$ if $h_i(t) = s$. As $h_i(A_i)$ is disjoint from $h_j(A_j)$ if j is unequal to i, and every s in C is in $h_i(A_i)$ for some i, this mapping is well-defined. Because it is a homomorphism on every component of C, it is a homomorphism, and moreover $g_i = fh_i$. Now suppose we have another homomorphism $f' : C \to B$. If we look at $f'|h_i(A_i)$ then it must hold that $g_i = f'|h_i(A_i)\ h_i$, and as h_i is an embedding, $f'|h_i(A_i) = g_i$, and thus on A_i, $f' = f$. As i is arbitrary, it must hold that $f = f'$. So f is unique.

Constraint 5

If four homomorphisms $b : A \to B$, $c : A \to C$, $d : B \to D$ and $e : C \to D$ are given such that $db = ec$, b is epic and e is monic, then there is a homomorphism $f : B \to C$ such that $fb = c$ and $ef = d$.

Proof

Construct $f : B \to C$ as follows. Take $s \in B$, which has (b is surjective) a b-inverse $s' \in A$ ($b(s') = s$) and $d(b(s')) = e(c(s'))$. Define $f(s) = c(s')$.

1. This is well-defined:

Suppose $s', t' \in A$ with $b(s') = b(t') = s$. Then

$$e(c(s')) = d(b(s')) = d(b(t')) = e(c(t')),$$

but as e is injective we have $c(s') = c(t')$.

2. f is a homomorphism:

i) Suppose $s < t$ in B then there exist s', t' in A with $s' < t'$ and $b(s') = s$ and $b(t') = t$. Therefore $c(s') < c(t')$ and $f(s) = c(s')$, $f(t) = c(t')$, so $f(s) < f(t)$.

ii) Take $s \in B$ with $b(s') = s$, then $B_s \equiv A_{s'} \equiv C_{c(s')} \equiv C_{f(s)}$.

iii) If $s \in B$ is minimal and $b(s') = s$ then s' is minimal and therefore $c(s') = f(s)$ is minimal;

3. By construction $fb = c$. Now take s in B, $s = b(s')$ then

$$d(s) = d(b(s')) = e(c(s')) = e(f(s)),$$

so $d = ef$.

As **BTM** satisfies the constraints of the previous section, we have that there exist general constructions for the joint closure and final model of a set of models.

6 Logical connections

In this section we present some of the results that can be obtained by applying the algebraic techniques developed above to the model theory of branching time temporal models. First we will summarize some persistency results of formulae under coproducts and joint closures. Next we will discuss four specific classes of models and introduce associated semantic consequence relations, and their mutual connections. One of these classes is the class of linear time models. We will discuss the connections with that class in some more detail.

Proposition 6.1

Suppose T is a disjoint union of isolated T_i .

Then for t in T_i

$$(M, t) \vDash \varphi \quad \Leftrightarrow \quad (M|T_i, t) \vDash \varphi$$

$$M \vDash \varphi \quad \Leftrightarrow \quad \{ M|T_i \mid i \in I \} \vDash \varphi$$

Proof

Straightforward.

As an immediate consequence we have persistency under the coproduct construction.

Corollary 6.2

Let $(M_i)_{i \in I}$ be a set of temporal models and P their coproduct.

a) For any formula φ it holds

$$(P, t) \vDash \varphi \quad \Leftrightarrow \quad (M_i, t) \vDash \varphi$$

$$P \vDash \varphi \quad \Leftrightarrow \quad \text{for all } i \text{ in } I \text{ it holds } M_i \vDash \varphi$$

b) Let **Th** be a temporal theory, then **P** is a model of **Th** if and only if for every **i** in **I** the model $\mathbf{M_i}$ is a model of **Th**.

Because we have shown that the joint closure can be built from a coproduct followed by a surjective homomorphism, and the coproduct construction behaves well under persistency, we have persistency under joint closure in the following sense.

Corollary 6.3

Let **C** be the joint closure of the indexed set of models $(\mathbf{M_i})_{\mathbf{i} \in \mathbf{I}}$, where $\mathbf{M_i}$ is based on $\mathbf{T_i}$. For $\mathbf{t} \in \mathbf{T_i}$, **t'** denotes the corresponding time point in the joint closure.

Every formula φ that is forward persistent under surjective homomorphisms satisfies

$$(\mathbf{M_i}, \mathbf{t}) \models \varphi \quad \Rightarrow \quad (\mathbf{C}, \mathbf{t'}) \models \varphi$$

$$\mathbf{M_i} \models \varphi \quad \text{for all } \mathbf{i} \text{ in } \mathbf{I} \quad \Rightarrow \quad \mathbf{C} \models \varphi$$

Every formula φ that is backward persistent under surjective homomorphisms satisfies

$$(\mathbf{C}, \mathbf{t'}) \models \varphi \quad \Rightarrow \quad (\mathbf{M_i}, \mathbf{t}) \models \varphi$$

$$\mathbf{C} \models \varphi \quad \Rightarrow \quad \mathbf{M_i} \models \varphi \quad \text{for all } \mathbf{i} \text{ in } \mathbf{I}$$

Every formula φ that is two-sided persistent under surjective homomorphisms satisfies

$$(\mathbf{C}, \mathbf{t'}) \models \varphi \quad \Leftrightarrow \quad (\mathbf{M_i}, \mathbf{t}) \models \varphi$$

$$\mathbf{C} \models \varphi \quad \Leftrightarrow \quad \mathbf{M_i} \models \varphi \quad \text{for all } \mathbf{i} \text{ in } \mathbf{I}$$

In the class **BT** of all branching time models we distinguish two subclasses, namely **LT**, the class of linear time models and **CL**, the class of closed models. Since it is easy to establish that linear time models are closed we have

$$\mathbf{LT} \subset \mathbf{CL} \subset \mathbf{BT}$$

There are other connections as well. Every branching time model can be mapped by a surjective homomorphism onto a closed one. Moreover, all branches in a branching time model are linear models, and together they cover the whole flow of time. For any set of models **S** its joint closure, denoted by **jcl(S)**, can be constructed. Under certain conditions this joint closure is a final model for a class of models. In principle, from these classes of models we can define corresponding satisfaction relations.

Definition 6.4

Let **S** be a class of branching time temporal models. For any temporal formula φ, define:

$$\mathbf{S} \models_{\mathbf{BT}} \varphi \quad \Leftrightarrow \quad (\forall \mathbf{M} \in \mathbf{BT}: \ \mathbf{M} \in \mathbf{S} \ \Rightarrow \ \mathbf{M} \models \varphi)$$

$$\mathbf{S} \models_{\mathbf{CL}} \varphi \quad \Leftrightarrow \quad (\forall \mathbf{M} \in \mathbf{CL}: \ \mathbf{M} \in \mathbf{S} \ \Rightarrow \ \mathbf{M} \models \varphi)$$

$$S \models_{LT} \varphi \quad \Leftrightarrow \quad (\forall M \in LT: \; M \in S \;\Rightarrow\; M \models \varphi)$$
$$S \models_{JCL} \varphi \quad \Leftrightarrow \quad jcl(S) \models \varphi$$

Obviously, we have that, for instance, $S \models_{LT} \varphi \;\Leftrightarrow\; S \cap LT \models_{BT} \varphi$. These definitions may seem rather unusual. An interesting case is when **S** is the set of all branching time models of a temporal theory **Th**. Then $S \models_{BT} \varphi$ means that φ is a branching time semantical consequence of **Th**. Furthermore, $S \models_{LT} \varphi$ means that φ is a linear time consequence of **Th**.

There are some apparent logical relations between these notions:

Proposition 6.5

$$S \models_{BT} \varphi \quad \Rightarrow \quad S \models_{CL} \varphi \quad \Rightarrow \quad S \models_{LT} \varphi$$
$$\Rightarrow \quad S \models_{JCL} \varphi$$

A main question is how different they are, and, in general, what the relations are. There is a real difference between linear time models and the others because they satisfy the following axioms expressing indistinguishable future:

$$\exists X\, \varphi \;\leftrightarrow\; \forall X\, \varphi$$
$$\exists F\, \varphi \;\leftrightarrow\; \forall F\, \varphi$$
$$\exists G\, \varphi \;\leftrightarrow\; \forall G\, \varphi$$

The final model of a set of (linear) time models will not in general satisfy these axioms. Any branching time model satisfying these axioms can in fact be mapped uniquely to a linear one.

First we need the following connection between a model **M** and its collection of linear time submodels (its branches) **Br(M)**. This notion is extended to a class of models **S**: the set of all branches of models in **S** is denoted by **Br(S)**. By **cl(M)** we denote the (epic) closure of **M**.

Theorem 6.6

Let **M** be any model. Then $jcl(Br(M)) \equiv cl(M)$.

In particular, **M** is closed if and only if $jcl(Br(M)) \equiv M$.

Proof

We will apply Proposition 4.18. To this end it is required that the canonical embeddings $a_L : L \rightarrow M$ for **L** in **Br(M)** are jointly surjective. Since **M** is covered by its maximal branches and each maximal branch is an element of **Br(M)** this is indeed the case.

In this theorem isomorphism ($\equiv$) is meant in the category-theoretic way. Here it means that the flows of time are isomorphic with corresponding (equivalent) object-level. Moreover, we have the following result on the existence of final models of a theory. For a class of models **S** the class of models **S*** is defined by

$$\mathbf{S^* \;=\; \{jcl(S')\;\;|\;S'\;a\;set\;of\;models\;with\;\;S' \subseteq S\;\}}.$$

In particular, **S*** contains the joint closure **jcl(S)** of all models in **S**, but it also contains the closure of each individual model: for $\mathbf{M \in S}$, we have that $\mathbf{jcl(\{M\}) = cl(M)}$.

Theorem 6.7

Let **Th** be a temporal theory that is forward persistent under surjections and **S** a set of models of **Th**. Then **S*** is a set of models of **Th** and the joint closure **jcl(S)** of all models of **S** is a final model of **Th** in **S***.

Proof

Because **Th** is forward persistent under surjections, we have $\mathbf{jcl(S) \models Th}$. Now from Theorem 4.20 it follows that **jcl(S)** is a final model in **S***.

A class of models **S** is *closed under submodels* if for each model in the class, all of its submodels are also in the class. In particular, in that case we have $\mathbf{Br(S) \subseteq S}$. A class S is *closed under surjections* if whenever **M** is in **S** and $\mathbf{f : M \to N}$ is a surjective homomorphism, **N** is also in **S**. After these preparations we are able to establish the following theorem that gives more precise connections between the different satisfaction relations.

Theorem 6.8

Let **S** be a class of models, and φ any formula.

a) If **S** is closed under submodels, and φ is forward persistent under surjections, then

$$\mathbf{S \models_{BT} \varphi \quad\Leftrightarrow\quad S \models_{LT} \varphi}$$

b) If **S** is closed under surjections and φ is backward persistent under surjections, then

$$\mathbf{S \models_{BT} \varphi \quad\Leftrightarrow\quad S \models_{CL} \varphi}$$

c) If **S** is a set, then

$$\mathbf{S^* \models_{BT} \varphi \quad\Rightarrow\quad S \models_{JCL} \varphi}$$

If, moreover, φ is backward persistent, then

$$\mathbf{S^* \models_{BT} \varphi \quad\Leftrightarrow\quad S \models_{JCL} \varphi}$$

d) If φ is forward persistent, then

$$\mathbf{Br(S) \models_{LT} \varphi \quad\Rightarrow\quad S \models_{BT} \varphi}$$

If, moreover, $\mathbf{Br(S)} \subseteq \mathbf{S}$, then

$$\mathbf{S} \models_{\mathbf{LT}} \varphi \quad \Leftrightarrow \quad \mathbf{Br(S)} \models_{\mathbf{LT}} \varphi \quad \Leftrightarrow \quad \mathbf{S} \models_{\mathbf{BT}} \varphi$$

If, in addition, $\mathbf{Br(S^*)} = \mathbf{Br(S)}$, then

$$\mathbf{S} \models_{\mathbf{LT}} \varphi \quad \Leftrightarrow \quad \mathbf{S} \models_{\mathbf{BT}} \varphi \quad \Leftrightarrow \quad \mathbf{S^*} \models_{\mathbf{BT}} \varphi$$

e) If $\mathbf{S}$ is a set and $\mathbf{Br(S)} = \mathbf{Br(S^*)} \subseteq \mathbf{S}$ and φ is both forward and backward persistent, then

$$\mathbf{S} \models_{\mathbf{LT}} \varphi \quad \Leftrightarrow \quad \mathbf{S} \models_{\mathbf{JCL}} \varphi$$

Proof

a) Assume $\mathbf{S} \models_{\mathbf{LT}} \varphi$. Suppose an $\mathbf{M}$ in $\mathbf{S}$ is given. Because $\mathbf{S}$ is closed under submodels, $\mathbf{Br(M)} \models \varphi$. By forward persistency of φ we also have $\mathbf{M} \models \varphi$. We have proven

$$\mathbf{S} \models_{\mathbf{BT}} \varphi \quad \Leftrightarrow \quad \mathbf{S} \models_{\mathbf{LT}} \varphi$$

b) Assume $\mathbf{S} \models_{\mathbf{CL}} \varphi$. Let $\mathbf{M}$ in $\mathbf{BT}$ be given with $\mathbf{M}$ in $\mathbf{S}$. Then there is a surjective homomorphism of $\mathbf{M}$ onto its closure $\mathbf{cl(M)}$ in $\mathbf{CL}$. Because $\mathbf{S}$ is closed under surjective homomorphisms we have $\mathbf{cl(M)}$ in $\mathbf{S}$, and therefore $\mathbf{cl(M)} \models \varphi$. By persistency of φ we have $\mathbf{M} \models \varphi$. We have proven

$$\mathbf{S} \models_{\mathbf{BT}} \varphi \quad \Leftrightarrow \quad \mathbf{S} \models_{\mathbf{CL}} \varphi$$

c) The first implication is trivial. Assume φ is backward persistent and $\mathbf{S} \models_{\mathbf{JCL}} \varphi$. Let $\mathbf{M}$ be any model in $\mathbf{S^*}$. Then we can map $\mathbf{M}$ in $\mathbf{jcl(S)}$. Since $\mathbf{jcl(S)} \models \varphi$, and φ is backward persistent we have $\mathbf{M} \models \varphi$. Therefore

$$\mathbf{S} \models_{\mathbf{BT}} \varphi \quad \Leftrightarrow \quad \mathbf{S} \models_{\mathbf{FI}} \varphi$$

d) Suppose φ is forward persistent, and $\mathbf{Br(S)} \models_{\mathbf{LT}} \varphi$ then every model in $\mathbf{S}$ has a branch and this branch satisfies φ. Therefore by persistence $\mathbf{S} \models_{\mathbf{BT}} \varphi$.

Assume, moreover, $\mathbf{Br(S)} \subseteq \mathbf{S}$, then it is trivial that $\mathbf{S} \models_{\mathbf{BT}} \varphi$ implies $\mathbf{Br(S)} \models_{\mathbf{LT}} \varphi$, and that this is equivalent to $\mathbf{S} \models_{\mathbf{LT}} \varphi$

Assume, in addition, $\mathbf{Br(S^*)} = \mathbf{Br(S)}$, then the previous result can be applied to $\mathbf{S^*}$. It follows that

$$\mathbf{S^*} \models_{\mathbf{LT}} \varphi \quad \Leftrightarrow \quad \mathbf{Br(S^*)} \models_{\mathbf{BT}} \varphi \quad \Leftrightarrow \quad \mathbf{S^*} \models_{\mathbf{BT}} \varphi$$

e) This follows from c) and d).

In Corollary 6.3 and Theorems 6.7 and 6.8, properties are established for formulae which are forward/backward persistent under surjective/injective homomorphisms. The question arises whether there are formulae persistent under these special homomorphisms but not under any homomorphism. This turns out to be not the case:

Proposition 6.9

A formula is forward (backward) persistent under surjective/injective homomorphisms if and only if it is forward (backward) persistent under any homomorphism.

Proof

- First we will prove the case for forward persistency under surjections. Suppose φ is forward persistent under surjective homomorphisms, but not under any homomorphism. Then there is a (non-surjective) homomorphism $\mathbf{f : M \to M'}$ with $\mathbf{t \in T}$ such that $\mathbf{(M,\ t) \models \varphi}$ but $\mathbf{(M',\ f(t)) \not\models \varphi}$. Now construct the model $\mathbf{N}$ which consists of a copy of $\mathbf{M}$ and for every point $\mathbf{s}$ of $\mathbf{M'}$ not in the image of $\mathbf{f}$ a branch $\mathbf{B}$ of $\mathbf{M'}$ with $\mathbf{s \in B}$ (disjoint from the copy of $\mathbf{M}$ and disjoint from every other such branch). Then it is easy to see that $\mathbf{(N, t) \models \varphi}$. Let $\mathbf{g : N \to M'}$ be the function which maps the copy of $\mathbf{M}$ to $\mathbf{f[M]}$ in $\mathbf{M'}$, and which maps every added branch to the same branch in $\mathbf{M'}$. Then $\mathbf{g}$ is a homomorphism and $\mathbf{g}$ is surjective and $\mathbf{(M', g(t)) \not\models \varphi}$. This is in contradiction to the assumption that φ was forward persistent under surjective homomorphisms. Therefore φ is forward persistent under any homomorphism. The proof fro backward persistency under surjective homomorphisms uses the same construction.
- Now for the case of backward persistency under injective homomorphisms. Suppose φ is backward persistent under injective homomorphisms, but not under any homomorphism. Then there is a (non-injective) homomorphism $\mathbf{f : M \to M'}$ with $\mathbf{t \in T}$ such that $\mathbf{(M', f(t)) \models \varphi}$ but $\mathbf{(M, t) \not\models \varphi}$. Now construct a model $\mathbf{N}$ by taking a copy of $\mathbf{M}$ and adding the following: for every point $\mathbf{s}$ of $\mathbf{M}$, and for every branch $\mathbf{B'}$ of $\mathbf{M'}$ with $\mathbf{f(s) \in B'}$, if there is no equivalent branch $\mathbf{B}$ in $\mathbf{M}$ with $\mathbf{s \in B}$, then we add such a branch in $\mathbf{N}$. Now let $\mathbf{h : M \to N}$ be the injective homomorphism which maps $\mathbf{M}$ to its copy in $\mathbf{N}$. Let $\mathbf{g : N \to M'}$ be the homomorphism (!) which maps $\mathbf{h(s)}$ to $\mathbf{f(s)}$ and the branches from $\mathbf{M'}$ in $\mathbf{N}$ to their counterparts in $\mathbf{M'}$. Of course the model $\mathbf{N}$ and the homomorphism $\mathbf{g}$ are constructed in such a way that $\mathbf{g}$ is branch-surjective, and we can use Proposition 3.7: since $\mathbf{(M', f(t)) \models \varphi}$ and $\mathbf{f(t) = g(h(t))}$ and $\mathbf{g}$ is branch-surjective, we have $\mathbf{(N, h(t)) \models \varphi}$, but as $\mathbf{h}$ is injective and φ is backward persistent under injections we have $\mathbf{(M, t) \models \varphi}$, contradicting the assumption. Thus φ must be backward persistent under any homomorphism. The proof for forward persistency under injective homomorphisms uses the same construction.

Theorem 6.8 describes some cases in which the different satisfaction relations are equal. The question still remains whether they are not in general always equal. This is not the case:

Proposition 6.10

In general $\models_{\mathbf{BT}} \neq \models_{\mathbf{CL}}$, $\models_{\mathbf{CL}} \neq \models_{\mathbf{LT}}$, $\models_{\mathbf{CL}} \neq \models_{\mathbf{JCL}}$. Moreover, for each of these inequalities we can find a natural object-level logic (propositional logic) and a temporal theory $\mathbf{Th}$ such that for some φ we have $\mathbf{Mod(Th)} \models_X \varphi$ but not $\mathbf{Mod(Th)} \models_Y \varphi$.

Proof

We have already remarked that $\mathbf{S} \models_{\mathbf{LT}} \exists X\, \varphi \leftrightarrow \forall X\, \varphi$ for any class $\mathbf{S}$; it is easy to see that this does not hold for the other consequence relations. Let's look at $\models_{\mathbf{BT}}$ and $\models_{\mathbf{CL}}$. Take any model $\mathbf{m} \in \mathcal{M}_0$ and let $\alpha \in \mathcal{L}_0$ be such that there exist $\mathbf{k}, \mathbf{l} \in \mathcal{M}_0$ with $\mathbf{k} \models_0 \alpha$ and $\mathbf{l} \not\models_0 \alpha$ (thus the object-level logic should not be trivial). Define the following two formulae:

$\mathbf{at_0} := \mathbf{H}(\mathbf{C}\alpha \wedge \neg\, \mathbf{C}\alpha)$

$\mathbf{at_1} := \mathbf{P}(\mathbf{C}\alpha \vee \neg\, \mathbf{C}\alpha) \wedge \mathbf{HH}(\mathbf{C}\alpha \wedge \neg\, \mathbf{C}\alpha)$

It is easy to see that $\mathbf{at_0}$ is true in a point if and only if it is a minimal element and $\mathbf{at_1}$ is true in a point if and only if it is a successor of a minimal element. Now define

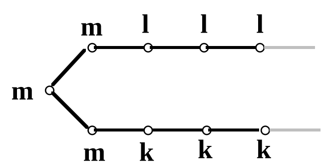
$\mathbf{Th} := \{\, \mathbf{at_i} \to \mathbf{C}\varphi \mid \mathbf{m} \models_0 \varphi,\ \varphi \in \mathcal{L}_0,\ \mathbf{i} = 0, 1 \,\} \cup$

$\{\, \mathbf{at_i} \to \neg\, \mathbf{C}\varphi \mid \mathbf{m} \not\models_0 \varphi,\ \varphi \in \mathcal{L}_0,\ \mathbf{i} = 0, 1 \,\} \cup$

$\{\, \mathbf{at_0} \to \exists \mathbf{F}(\exists \mathbf{F}(\mathbf{C}\alpha)) \,\}$ and

$\varphi := \mathbf{at_1} \to \exists \mathbf{F}(\mathbf{C}\alpha)$

If $\mathbf{M}$ is a closed model of $\mathbf{Th}$ then all initial points and their successors must be equivalent to $\mathbf{m}$, but as $\mathbf{M}$ is closed, it must have a unique root $\mathbf{r}$ with one successor $\mathbf{s}$. Then $(\mathbf{M}, \mathbf{r}) \models \exists \mathbf{F}(\exists \mathbf{F}(\mathbf{C}\alpha))$, so there exists a point $\mathbf{t}$ with $\mathbf{r} \ll \mathbf{t}$ and $(\mathbf{M}, \mathbf{t}) \models \exists \mathbf{F}(\mathbf{C}\alpha)$, but then $(\mathbf{M}, \mathbf{s}) \models \exists \mathbf{F}(\mathbf{C}\alpha)$ since $\mathbf{s}$ is the only successor of $\mathbf{r}$. Thus $\mathbf{M} \models \varphi$, and we have proved $\mathbf{Mod(Th)} \models_{\mathbf{CL}} \varphi$. Now consider the following branching time model:



In this model, the minimal element and its successors are equivalent to $\mathbf{m}$ and a $\mathbf{k}$ state is reachable from the root in at least two steps (remember that $\mathbf{k} \models_0 \alpha$), so this model is a model of $\mathbf{Th}$. But the lower successor of the root is a successor of a minimal element but has only $\mathbf{l}$ states reachable (in which $\mathbf{C}\alpha$ is not true), so it is not a model of φ. We have proven that $\mathbf{Mod(Th)} \not\models_{\mathbf{BT}} \varphi$.

Now we will look at $\models_{\mathbf{CL}}$ and $\models_{\mathbf{JCL}}$. Let $\mathbf{m} \in \mathcal{M}_0$ and let $\alpha \in \mathcal{L}_0$ be such that there exist $\mathbf{k}, \mathbf{l} \in \mathcal{M}_0$ with $\mathbf{k} \models_0 \alpha$ and $\mathbf{l} \not\models_0 \alpha$ (thus the object-level logic should not be trivial). Define: $\mathbf{Th} := \{\, \mathbf{at_0} \to \mathbf{C}\varphi \mid \mathbf{m} \models_0 \varphi,\ \varphi \in \mathcal{L}_0 \,\} \cup$

$\{\, \mathbf{at_0} \to \neg\, \mathbf{C}\varphi \mid \mathbf{m} \not\models_0 \varphi,\ \varphi \in \mathcal{L}_0 \,\}$

Let the object-level logic be such that $\mathcal{M}_0$ is a set (for instance propositional logic). Then we can take $\mathbf{jcl(Mod(Th))}$ which has a unique root $\mathbf{r}$ (with object-level equivalent to $\mathbf{m}$), in which for each point the set of its successors consists of one state for each object-level model (up to equivalence). This model contains a branch starting at $\mathbf{r}$ in which each point has $\mathbf{k}$ as its object-level model. So

$$\mathbf{jcl(Mod(Th))} \models \mathbf{at_0} \to \exists \mathbf{F}(\mathbf{C}\alpha)$$

which gives $\mathbf{Mod(Th)} \vDash_{\mathbf{JCL}} \mathbf{at_0} \rightarrow \exists \mathbf{F(C\alpha)}$. Now consider the linear model **N** consisting of a root with object-level model equivalent to **m** and all the other points have object-level models equivalent to **l**. Then this is a closed model of **Th** but $\mathbf{N} \nvDash \mathbf{at_0} \rightarrow \exists \mathbf{F(C\alpha)}$ so $\mathbf{Mod(Th)} \nvDash_{\mathbf{CL}} \mathbf{at_0} \rightarrow \exists \mathbf{F(C\alpha)}$.

7 Conclusions

Temporal models can be used to describe the behaviour of dynamic processes. The linear models usually describe a possible behavioural pattern, and a set of such models can be used to describe multiple possible patterns. These models may be described by a temporal theory. Another way of describing possible behaviour is by a branching time process which branches at any time a pattern can continue in more than one way. These models can also be axiomatized by a temporal theory. In this article we identified a uniform algebraic manner in which to relate these different kinds of models. It was shown that the branching time models form a category with homomorphisms as arrows between objects. A number of operations like the coproduct, joint closure and epic closure which perform a kind of merging of models into a final model were defined in general for categories and it turned out that these operations can be used in the category of branching time models. Therefore, out of a set of linear models we can construct a branching time model which incorporates all the linear models. This can then be transformed by homomorphisms into a model which is final. In that model all decisions that have to be made during a process (which branch to take) are moved as far backward in time as possible.It is then interesting to identify the formulae whose truth value remains the same through these constructions, so that if models of such a formula are merged, it remains a model of such a formula. Then one can define satisfaction relations based on linear, branching time, closed or final models and investigate the connections between these relations.

In [Sp90], a reduction from linear time logic to branching time logic is given, by translating formulae from linear time logic into formulae from branching time logic. The translation replaces the **F**-operator ("sometimes in the future") by $\forall\mathbf{F}$, and forces 'linear behavior' on subformulae (meaning that $\forall\mathbf{F\alpha}$ and $\exists\mathbf{F\alpha}$ should be equivalent). The idea of viewing a linear time model as a (simple) branching time model, and the construction of the set of linear time models **Br(M)** of branches of the model **M**, occur in [Sp90].

Different papers in the literature on temporal logic discuss the usefulness of linear time temporal logic versus branching time temporal logic; for example, in [EH86], [Gl94]. In general it is argued that for applications where expressivity demands are not high, the linear time approach has (conceptual and computational) advantages, whereas in cases where certain types of path quantification are required, branching time approaches have advantages. In the application areas of reasoning processes of knowledge- and agent-based systems addressed by us, the results of this paper were successfully applied to develop the model theory of the dynamics of (nonmonotonic) reasoning processes based on default logic [Re80]. An important characteristic of default reasoning is that usually different lines of reasoning are possible, each leading to a set of conclusions. In default logic these conclusion sets are descibed by (Reiter) extensions [Re80]. In common examples this leads to a variety of extensions. In logic one is used to express semantics in terms of models that represent consistent descriptions of the world and semantic entailment relations based on a specific class of this type of models. These notions are not really adequate to describe alternative conclusion sets for default reasoning. Sometimes one introduces sceptical entailment (what is true in all conclusion sets) or credulous entailment (what is true in some conclusion set). From a semantic point of view both notions only give a limited description: they only indicate global upper and lower bounds for the conclusion set of particular lines of reasoning.

In [ET96] we integrate process aspects of the reasoning in the semantics in an explicit manner. The approach extends the one introduced in [ET93], where it was shown how one line of default reasoning corresponds to one linear time model. Each extension of a default theory is generated by a reasoning process, and therefore corresponds to a temporal model describing this process. These temporal models can be described by a temporal theory which depends on the default theory. Using the machinery of the current paper, this linear time semantics has been used to define a branching time semantics and a joint closure temporal semantics for default logic in [ET96]. Each line of reasoning corresponds to a branch in the joint closure model. It is shown how (under a particular topological condition, called extension completeness) an appropriate joint closure model can be constructed in which precisely all possible lines of reasoning (and the resulting conclusion sets) can be represented (even though they might be mutually contradictory). The semantics of the default theory can be defined on the basis of this single joint closure model. In particular, sceptical and credulous entailment relations can be defined as well on the basis of this model. For more detailks, see [ET96].

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