An ω-complete Equational Specification of Interleaving

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ABSTRACT

We consider the process theory PA that includes an operation for parallel composition, based on the interleaving paradigm. We prove that the standard set of axioms of PA is not ω-complete by providing a set of axioms that are valid in PA, but not derivable from the standard ones. We prove that extending PA with this set yields an ω-complete specification, which is finite in a setting with finitely many actions.

1991 Mathematics Subject Classification: 68Q10; 68Q65; 68Q70
1991 ACM Computing Classification System: D.1.3; F.1.1; F.1.2
Keywords and Phrases: Process Algebra, Algebraic Specification, Interleaving, ω-completeness.
Note: Research supported by the Netherlands Organization for Scientific Research (NWO) under contract SION 612-33-008. Work carried out under project SEN 2.1 Process Specification and Analysis.

1. Introduction

The interleaving paradigm consists of the assumption that two atomic actions cannot happen at the same time, so that concurrency reduces to nondeterminism. To express the concurrent execution of processes, many process theories have been accomodated with an operation for parallel composition that behaves according to the interleaving paradigm. For instance, CCS (see, e.g., Milner (1989)) has a binary operation for parallel composition —we shall denote it by $\_ \,||\, \_$— that satisfies the so-called *Expansion Law*:

$$\text{if } p = \sum_{i=1}^{m} a_i.p_i \text{ and } q = \sum_{j=1}^{n} b_j.q_j, \text{ then } p \,||\, q \approx \sum_{i=1}^{m} a_i.(p_i \,||\, q) + \sum_{j=1}^{n} b_j.(q_j \,||\, p);$$

here the $a_i.\_$ and the $b_j.\_$ are unary operations that prefix a process with an atomic action, and summation denotes a nondeterministic choice between its arguments.

The Expansion Law generates an infinite set of equations, one for each pair of processes p and q. Bergstra and Klop (1984) enhanced the equational characterisation of interleaving. They replaced action prefixing with a binary operation $\_\cdot\_$ for sequential composition and added an auxiliary operation $⌊⌊$ (the *left merge*; it is similar to $||$, except that it must start execution with a step from its left argument). Their axiomatisation is finite for settings with finitely many atomic actions. Moller (1990) proved that interleaving is not finitely axiomatisable without an auxiliary operation such as the left merge.

The axioms of Bergstra and Klop (1984) form a ground-complete axiomatisation of bisimulation equivalence; ground terms p and q are provably equal if, and only if, they are bisimilar. Thus, it reflects for a large part our intuition about interleaving. On the other hand, it is not optimal. For instance, it can be shown by means of structural induction that every ground instance of the axiom

$x \,||\, (y \,||\, z) \approx (x \,||\, y) \,||\, z$ is derivable (see Baeten and Weijland (1990)); however, the axiom itself is not derivable.

If an equational specification E has the property that $\mathsf{E} \vdash t^\sigma \approx u^\sigma$ for all ground substitutions σ implies that $\mathsf{E} \vdash t \approx u$, then E is called *ω-complete* (or: *inductively closed*). To derive any equation from such an equational specification it is never needed to use additional proof techniques such as structural induction. Therefore, in applications dealing with theorem proving, ω-completeness is a desirable property to have (see Lazrek *et al.* (1990)). In Heering (1986) it was argued that ω-completeness is desirable for the partial evaluation of programs.

Moller (1989) obtained an ω-complete axiomatisation for CCS without communication, by adding a law for *standard concurrency*:

$$(x \,⌊⌊\, y) \,⌊⌊\, z \approx x \,⌊⌊\, (y \,||\, z).$$

In this paper we shall address the question whether PA, the subtheory of ACP without communication and encapsulation, is ω-complete. While the algebra studied by Moller (1989) has sequential composition in the form of prefix multiplication, PA incorporates the (more general) binary operation $\cdot$ for sequential composition. Having this operation, it is no longer sufficient to add the law for standard concurrency to arrive at an ω-complete axiomatisation. However, surprisingly, it is sufficient to add this law and the set of axioms generated by a single scheme:

$$(x \cdot \alpha \,⌊⌊\, \alpha) \approx (x \,⌊⌊\, \alpha) \cdot \alpha,$$

where α ranges over alternative compositions of distinct atomic actions; if the set of atomic actions is finite, then this scheme generates finitely many axioms.

An important part of our proof has been inspired by the excellent work of Hirshfeld and Jerrum (1999) on the decidability of bisimulation equivalence for normed process algebra. In particular, they distinguish two kinds of *mixed equations*, in which a parallel composition is equated to a sequential composition. The first kind consists of equations

$$(t \cdot \alpha^k) \,||\, \alpha^l \approx (t \,||\, \alpha^l) \cdot \alpha^k$$

for positive natural numbers k and l, and for sums of atomic actions α. These equations can be derived using standard concurrency and our new axioms. The second kind of mixed equations are the so-called *pumpable equations*, which are of a more complex nature (see p. 419 of Hirshfeld and Jerrum (1999)). Basically, we show that there cannot exist pumpable equations that contain variables by associating with every candidate $t \approx u$ a ground substitution σ such that $t^\sigma \not\approx u^\sigma$.

The notion of ω-completeness is related to *action refinement*, where each atomic action may be refined to an arbitrary process. That is, in a theory with action refinement, the actions take over the role played by variables in our theory; the actions, as they occur in our theory, are not present in theories for action refinement. Aceto and Hennessy (1993) presented a complete axiomatisation for PA (including a special constant `nil`, being a hybrid of deadlock and empty process) with action refinement, modulo timed observational equivalence from Hennessy (1988). In this setting, laws such as $a \,⌊⌊\, x \approx a \cdot x$, which hold in standard PA, are no longer valid, as the atomic action a can be refined into any other process.

This paper is set up as follows. In §2 we introduce the standard axioms of interleaving, and we prove that they do not form an ω-complete specification by proving that all ground substitution instances of our new axioms are derivable, while the axioms themselves are not. In §3 we state some basic facts about the theory of interleaving that we shall need in our proof of ω-completeness. In §4 we collect some results on certain mixed equations, and in §5 we investigate a particular kind of terms that consist of nestings of parallel and sequential compositions. In §6 we prove our main theorem, that the standard theory of interleaving enriched with the law for standard concurrency and our new axioms is ω-complete.

Table 1: The axioms of PA_A, with $a \in A$ and α any sum of distinct elements of A.

(A1)	$x + y$	$\approx y + x$	(M1)	$x \parallel y$	$\approx x \mathbin{⌊⌊} y + y \mathbin{⌊⌊} x$
(A2)	$x + (y + z)$	$\approx (x + y) + z$	(M2$_a$)	$a \mathbin{⌊⌊} x$	$\approx a \cdot x$
(A3)	$x + x$	$\approx x$	(M3$_a$)	$a \cdot x \mathbin{⌊⌊} y$	$\approx a \cdot (x \parallel y)$
(A4)	$(x + y) \cdot z$	$\approx x \cdot z + y \cdot z$	(M4)	$(x + y) \mathbin{⌊⌊} z$	$\approx x \mathbin{⌊⌊} z + y \mathbin{⌊⌊} z$
(A5)	$(x \cdot y) \cdot z$	$\approx x \cdot (y \cdot z)$	(M5)	$(x \mathbin{⌊⌊} y) \mathbin{⌊⌊} z$	$\approx x \mathbin{⌊⌊} (y \parallel z)$
			(M6$_\alpha$)	$x \cdot \alpha \mathbin{⌊⌊} \alpha$	$\approx (x \mathbin{⌊⌊} \alpha) \cdot \alpha$

2. Interleaving

A *process algebra* is an algebra that satisfies the axioms A1–A5 of Table 1. Suppose that A is a set of constant symbols and suppose that $\parallel$ and $⌊⌊$ are binary operation symbols; a process algebra with interpretations for the constant symbols in A and the operations $\parallel$ and $⌊⌊$ satisfying M1, M4, M5, M2$_a$ and M3$_a$ for all $a \in A$, and M6$_\alpha$ for all sums of distinct elements of A, we shall call an *A-merge algebra*; the variety of A-merge algebras we denote by PA_A.

The axioms A1–A5 together with the axioms M1–M4 form the standard axiomatisation of interleaving. Consider the single-sorted signature Σ with the elements of A as constants and the binary operations $+$, $\cdot$, $⌊⌊$ and $\parallel$. In writing terms we shall often omit the operation $\cdot$ for sequential composition; we assume that sequential composition binds strongest and that the operation $+$ for alternative composition binds weakest.

Let $\mathcal{R}$ consist of the axioms A3–A5 and M1–M4 of Table 1 interpreted as rewrite rules by orienting them from left to right. The term rewriting system $\langle \Sigma, \mathcal{R} \rangle$ is ground terminating and ground confluent modulo associativity and commutativity of $+$ (cf. the axioms A1 and A2). Every ground normal form of $\langle \Sigma, \mathcal{R} \rangle$ is an element of the set of *basic terms*, which is inductively defined as follows:

1. each element of A is a basic term;
2. if t is a basic term and $a \in A$, then at is a basic term; and
3. if t and u are basic terms, then $t + u$ is a basic term.

If t is a basic term, then there exist disjoint finite sets I and J, elements a_i and b_j of A and basic terms t_i, for $i \in I$ and $j \in J$ such that

$$t \approx \sum_{i \in I} a_i t_i + \sum_{j \in J} b_j \qquad \text{(by A1 and A2).}$$

It is well-known that the axioms A1–A5 together with M1–M4 do not constitute an ω-complete axiomatisation; all ground substitution instances of M5 are derivable, while the axiom itself is not. Moller (1989) has shown that, in a setting with prefix sequential composition instead of the binary operation $\cdot$, it suffices to add M5 to obtain an ω-complete axiomatisation (see Groote (1990) for an alternative proof). Clearly, neither $x\alpha \mathbin{⌊⌊} \alpha$ nor $(x \mathbin{⌊⌊} \alpha)\alpha$ is an instance of any of the axioms A1–A5 and M1–M5, so M6$_\alpha$ is not derivable. However, each ground substitution instance of M6$_\alpha$ is derivable.

Proposition 2.1 If α is a finite sum of elements of A, then, for every ground term t,

$$\mathrm{A1}, \ldots, \mathrm{A5}, \mathrm{M1}, \ldots, \mathrm{M4} \vdash t\alpha \mathbin{⌊⌊} \alpha \approx (t \mathbin{⌊⌊} \alpha)\alpha.$$

Proof. Suppose that $\alpha = a_1 + \cdots + a_n$. It suffices to prove the proposition for all basic terms; we do induction on their structure.

If $t \in A$, then

$$\begin{aligned}
t\alpha \mathbin{\|\!\!_} \alpha &\approx t(\alpha \,||\, \alpha) && \text{(by M3}_t\text{)}\\
&\approx t(\alpha \mathbin{\|\!\!_} \alpha + \alpha \mathbin{\|\!\!_} \alpha) && \text{(by M1)}\\
&\approx t(\alpha \mathbin{\|\!\!_} \alpha) && \text{(by A3)}\\
&\approx t(a_1 \mathbin{\|\!\!_} \alpha + \cdots + a_n \mathbin{\|\!\!_} \alpha) && \text{(by M4)}\\
&\approx t(a_1\alpha + \cdots + a_n\alpha) && \text{(by M2}_{a_1}, \ldots, \text{M2}_{a_n}\text{)}\\
&\approx t(\alpha\alpha) && \text{(by A4)}\\
&\approx (t \mathbin{\|\!\!_} \alpha)\alpha && \text{(by A5 and M2}_t\text{).}
\end{aligned}$$

If t is of the form bt' with $b \in A$, then

$$\begin{aligned}
(bt')\alpha \mathbin{\|\!\!_} \alpha &\approx b(t'\alpha \,||\, \alpha) && \text{(by A5 and M3}_b\text{).}\\
&\approx b(t'\alpha \mathbin{\|\!\!_} \alpha + \alpha \mathbin{\|\!\!_} t'\alpha) && \text{(by M1)}\\
&\approx b(t'\alpha \mathbin{\|\!\!_} \alpha + a_1 \mathbin{\|\!\!_} t'\alpha + \cdots + a_n \mathbin{\|\!\!_} t'\alpha) && \text{(by M4)}\\
&\approx b(t'\alpha \mathbin{\|\!\!_} \alpha + a_1t'\alpha + \cdots + a_nt'\alpha) && \text{(by M2}_{a_1}, \ldots, \text{M2}_{a_n}\text{)}\\
&\approx b((t' \mathbin{\|\!\!_} \alpha)\alpha + (\alpha \mathbin{\|\!\!_} t')\alpha) && \text{(by IH, A4, M2, and M4)}\\
&\approx b(t' \,||\, \alpha)\alpha && \text{(by A4 and M1)}\\
&\approx (bt' \mathbin{\|\!\!_} \alpha)\alpha && \text{(by M3}_b\text{).}
\end{aligned}$$

If t is of the form $t' + t''$, then we derive

$$\begin{aligned}
(t' + t'')\alpha \mathbin{\|\!\!_} \alpha &\approx (t'\alpha + t''\alpha) \mathbin{\|\!\!_} \alpha && \text{(by A4)}\\
&\approx t'\alpha \mathbin{\|\!\!_} \alpha + t''\alpha \mathbin{\|\!\!_} \alpha && \text{(by M4)}\\
&\approx (t' \mathbin{\|\!\!_} \alpha)\alpha + (t'' \mathbin{\|\!\!_} \alpha)\alpha && \text{(by IH)}\\
&\approx ((t' + t'') \mathbin{\|\!\!_} \alpha)\alpha && \text{(by A4 and M4).}
\end{aligned}$$

□

Consequently, in the case of binary sequential composition, the axioms A1–A5 together with M1–M5 do not constitute an ω-complete axiomatision. In the sequel, we shall prove that PA_A is ω-complete.

3. Basic Facts

We shall often implicitly make use of the associativity of $+$ and $||$; commutativity and associativity of $+$ are by A1 and A2, commutativity of $||$ follows from M1 and A1. To see that $||$ is associative, note that

$$\begin{aligned}
(x \,||\, y) \,||\, z &\approx (x \,||\, y) \mathbin{\|\!\!_} z + z \mathbin{\|\!\!_} (x \,||\, y) && \text{(by M1)}\\
&\approx (x \mathbin{\|\!\!_} y + y \mathbin{\|\!\!_} x) \mathbin{\|\!\!_} z + z \mathbin{\|\!\!_} (x \,||\, y) && \text{(by M1)}\\
&\approx (x \mathbin{\|\!\!_} y) \mathbin{\|\!\!_} z + (y \mathbin{\|\!\!_} x) \mathbin{\|\!\!_} z + z \mathbin{\|\!\!_} (x \,||\, y) && \text{(by M4)}\\
&\approx x \mathbin{\|\!\!_} (y \,||\, z) + y \mathbin{\|\!\!_} (x \,||\, z) + z \mathbin{\|\!\!_} (x \,||\, y) && \text{(by M5),}
\end{aligned}$$

and that $x||(y||z) \approx x \mathbin{\|\!\!\_} (y||z) + y \mathbin{\|\!\!\_} (z||x) + z \mathbin{\|\!\!\_} (y||x)$ by a similar derivation, so that $x||(y||z) \approx (x||y)||z$ follows with two applications of commutativity. The statement $\mathsf{PA}_A \vdash u \approx u + t$ we shall frequently abbreviate by $t \preccurlyeq u$; if $t \preccurlyeq u$, then we call t a *summand* of u. Note that $\preccurlyeq$ is a partial order on the set of terms modulo $\approx$; in particular, if $t \preccurlyeq u$ and $u \preccurlyeq t$, then $t \approx u$.

Lemma 3.1 Let a be an element of A and let t, u and v be ground terms. If $at \preccurlyeq u + v$, then $at \preccurlyeq u$ or $at \preccurlyeq v$.

Proof. Suppose, without loss of generality, that at, u and v are normal forms of the term rewriting system $\langle \Sigma, \mathcal{R} \rangle$.
If $u \preccurlyeq v$ or $v \preccurlyeq u$, then the proposition is immediate, so let us assume that $u \not\preccurlyeq v$ and $v \not\preccurlyeq u$. If $u + v$ is not a normal form, then it should contain a redex for A3; by contraction of this redex $u + v \rightarrow u' + v$, with u' a ground term such that $u' \preccurlyeq u$. Note that $u' \not\preccurlyeq v$ and $v \not\preccurlyeq u'$, so this procedure may be repeated until we find a normal form $u^* + v$ of $u + v$ such that $u^* \preccurlyeq u$. Since $u^* + v \approx u + v$, $at \preccurlyeq u^* + v$. Since $at + u^* + v$ is not a normal form, either $at + u^*$ or $at + v$ must contain a redex for A3; hence $at \preccurlyeq u^* \preccurlyeq u$ or $at \preccurlyeq v$. □

Lemma 3.2 If α is a finite sum of elements of A, then

$$\mathsf{PA}_A \vdash x\alpha \mathbin{\|\!\_} \alpha^n \approx (x \mathbin{\|\!\_} \alpha^n)\alpha, \text{ and } \mathsf{PA}_A \vdash x\alpha \,||\, \alpha^n \approx (x \,||\, \alpha^n)\alpha.$$

Proof. It is straightforward to show by induction on n that the identity (*) $\alpha^{n+1} \approx \alpha^n \,||\, \alpha$ is derivable from PA_A; we shall use it in the proof of the first set of equations (**) $x\alpha \mathbin{\|\!\_} \alpha^n \approx (x \mathbin{\|\!\_} \alpha^n)\alpha$, which is by induction on n. If $n = 1$, then (**) is an instance of M6$_\alpha$, and for the induction step we have the following derivation:

$$\begin{aligned}
x\alpha \mathbin{\|\!_} \alpha^{n+1} &\approx x\alpha \mathbin{\|\!_} (\alpha^n \,||\, \alpha) && \text{(by *)}\\
&\approx (x\alpha \mathbin{\|\!_} \alpha^n) \mathbin{\|\!_} \alpha && \text{(by M5)}\\
&\approx (x \mathbin{\|\!_} \alpha^n)\alpha \mathbin{\|\!_} \alpha && \text{(by IH)}\\
&\approx ((x \mathbin{\|\!_} \alpha^n) \mathbin{\|\!_} \alpha)\alpha && \text{(by M6}_\alpha\text{)}\\
&\approx (x \mathbin{\|\!_} (\alpha^n \,||\, \alpha))\alpha && \text{(by M5)}\\
&\approx (x \mathbin{\|\!_} \alpha^{n+1})\alpha && \text{(by *).}
\end{aligned}$$

The second set of equations is also derived by induction on n, using (**). We start with the case $n = 1$:

$$\begin{aligned}
x\alpha \,||\, \alpha &\approx x\alpha \mathbin{\|\!_} \alpha + \alpha \mathbin{\|\!_} x\alpha && \text{(by M1)}\\
&\approx (x \mathbin{\|\!_} \alpha)\alpha + (\alpha \mathbin{\|\!_} x)\alpha && \text{(by M6}_\alpha\text{, M4, M2, A5 and A4)}\\
&\approx (x \,||\, \alpha)\alpha && \text{(by A4 and M1).}
\end{aligned}$$

The induction step proceeds as follows:

$$\begin{aligned}
x\alpha \,||\, \alpha^{n+1} &\approx x\alpha \mathbin{\|\!_} \alpha^{n+1} + \alpha^{n+1} \mathbin{\|\!_} x\alpha && \text{(by M1)}\\
&\approx (x \mathbin{\|\!_} \alpha^{n+1})\alpha + \alpha(x\alpha \,||\, \alpha^n) && \text{(by **, A4, M4 and M3)}\\
&\approx (x \mathbin{\|\!_} \alpha^{n+1})\alpha + \alpha((x \,||\, \alpha^n)\alpha) && \text{(by IH)}\\
&\approx (x \mathbin{\|\!_} \alpha^{n+1})\alpha + (\alpha^{n+1} \mathbin{\|\!_} x)\alpha && \text{(by A5, A4, M4 and M3)}\\
&\approx (x \,||\, \alpha^{n+1})\alpha && \text{(by M1).}
\end{aligned}$$
□

Milner and Moller (1993) proved that if t, u and v are ground terms such that $t \,||\, v$ and $u \,||\, v$ are bisimilar, then t and u are bisimilar (a similar result was obtained earlier by Castellani and Hennessy (1989) in the context of distributed bisimulation). Also, they proved that every finite process has, up to bisimulation equivalence, a unique decomposition into prime components. Since PA_A is a sound and complete axiomatisation for bisimulation equivalence (Bergstra and Klop, 1984), the following two results are consequences of theirs.

Lemma 3.3 If t, u and v are ground terms such that $\mathsf{PA}_A \vdash t \,||\, v \approx u \,||\, v$, then $\mathsf{PA}_A \vdash t \approx u$.

Definition 3.4 A ground term t we shall call *parallel prime* if there do not exist ground terms u and v such that $\mathsf{PA}_A \vdash t \approx u \,||\, v$.

Theorem 3.5 (Unique factorisation) Any ground term can be expressed uniquely as a parallel composition of parallel prime components.

We associate to each term t a *norm* $\lfloor t \rfloor$ and a *depth* $\lceil t \rceil$ as follows:

$$\begin{array}{rlrll} \lfloor x \rfloor = \lfloor a \rfloor &= 1 & \lceil a \rceil = \lceil x \rceil &= 1 & (a \in A \text{ and } x \text{ a variable}); \\ \lfloor x * y \rfloor &= \lfloor x \rfloor + \lfloor y \rfloor & \lceil x * y \rceil &= \lceil x \rceil + \lceil y \rceil & \text{if } * \in \{\cdot, \lfloor\!\lfloor, ||\}; \text{ and} \\ \lfloor x + y \rfloor &= \min\{\lfloor x \rfloor, \lfloor y \rfloor\} & \lceil x + y \rceil &= \max\{\lceil x \rceil, \lceil y \rceil\}. & \end{array}$$

Notice that if $t \approx u$, then t and u must have equal norm and depth.

Lemma 3.6 If t, t', u and u' are ground terms such that $\lfloor t \rfloor = \lfloor t' \rfloor$, $\lfloor u \rfloor = \lfloor u' \rfloor$ and $\mathsf{PA}_A \vdash tu \approx t'u'$, then $\mathsf{PA}_A \vdash t \approx t'$ and $\mathsf{PA}_A \vdash u \approx u'$.

Proof. By induction on the structure of t. □

Suppose t is a ground normal form of the system $\langle \Sigma, \mathcal{R} \rangle$ and suppose that

$$t \approx \sum_{i \in I} a_i t_i + \sum_{j \in J} b_j;$$

then the *degree* $d(t)$ of t is defined by $d(t) = |I| + |J|$. We let the degree of an arbitrary ground term be the degree of its unique normal form in $\langle \Sigma, \mathcal{R} \rangle$. Note that $d(tu) = d(t \lfloor\!\lfloor u) = d(t)$ and that $\max\{d(t), d(u)\} \leq d(t+u), d(t \,||\, u) \leq d(t) + d(u)$. Moreover, if t and u do not have summands in common, then $d(t+u) = d(t) + d(u)$.

By $d_{\max}(t)$ we shall denote the maximal degree that occurs in t, i.e.,

$$d_{\max}(t) = \max(\{d(t)\} \cup \{d_{\max}(t') \mid \text{there exists an } a \in A \text{ such that } at' \preccurlyeq t\}).$$

Definition 3.7 Let t and t' be ground terms; we shall write $t \longrightarrow t'$ if there exists $a \in A$ such that $at' \preccurlyeq t$ and $\lfloor t' \rfloor < \lfloor t \rfloor$. We define the set $\mathsf{red}(t)$ of *reducts* of t as the least set that contains t and is closed under $\longrightarrow$; if $t \longrightarrow t'$, then we call t' an *immediate reduct* of t.

Lemma 3.8 Let t, u and v be ground terms;

i. if t is a reduct of uv and $\lfloor t \rfloor \leq \lfloor v \rfloor$, then t is a reduct of v; and

ii. if t is a reduct of $u \,||\, v$ and t is parallel prime, then t is a reduct of u, or t is a reduct of v.

Lemma 3.9 Let t be a ground term. If $t \longrightarrow t'$ and $t \longrightarrow t''$ implies that $\mathsf{PA}_A \vdash t' \approx t''$ for all ground terms t' and t'', then there exists a parallel prime ground term t^* such that $\mathsf{PA}_A \vdash t \approx t^* \,||\, \ldots \,||\, t^*$.

Proof. First, suppose that u and v are parallel prime, and let u' and v' be such that $u \longrightarrow u'$ and $v \longrightarrow v'$; then, $u \,||\, v \longrightarrow u' \,||\, v$ and $u \,||\, v \longrightarrow u \,||\, v'$. So, if $u' \,||\, v \approx u \,||\, v'$, then since $\lfloor u' \rfloor < \lfloor u \rfloor$, u cannot be a component of the prime decomposition of u'; hence, by Theorem 3.5, $u \approx v$.
Suppose $t \approx t_1 \,||\, \ldots \,||\, t_n$, with t_i parallel prime for all $1 \leq i \leq n$ and $\lfloor t_1 \rfloor \leq \cdots \leq \lfloor t_n \rfloor$.
If $\lfloor t_1 \rfloor = 1$, then $\lfloor t_i \rfloor = 1$ for all $1 \leq i \leq n$; for suppose that t_i' is a ground term such that $t_i \longrightarrow t_i'$, then from $t_2 \,||\, \cdots \,||\, t_n \approx t_1 \,||\, \cdots \,||\, t_{i-1} \,||\, t_i' \,||\, t_{i+1} \,||\, \cdots \,||\, t_n$, we get by Lemma 3.3 that $t_i \approx t_1 \,||\, t_i'$, but t_i is parallel prime. From $t_1 \,||\, \cdots \,||\, t_{i-1} \,||\, t_{i+1} \,||\, \cdots \,||\, t_n \approx t_1 \,||\, \cdots \,||\, t_{j-1} \,||\, t_{j+1} \,||\, \cdots \,||\, t_n$, we conclude by Lemma 3.3 that $t_i \approx t_j$.
The remaining case is that $\lfloor t_i \rfloor > 1$ for all $1 \leq i \leq n$. Let t_i' and t_j' be ground terms such that $t_i \longrightarrow t_i'$ and $t_j \longrightarrow t_j'$ for some $1 \leq i < j \leq n$; then by Lemma 3.3 $t_i' \,||\, t_j \approx t_i \,||\, t_j'$. Since $\lfloor t_i' \rfloor < \lfloor t_i \rfloor$, t_i cannot be a component of the prime decomposition of t_i', so by Theorem 3.5 $t_i \approx t_j$. □

4. Mixed Equations

We shall collect some results about *mixed equations*; these are equations of the form $tu \approx v \,||\, w$.

Lemma 4.1 If t, u and v are ground terms such that $\mathsf{PA}_A \vdash tu \approx u \,||\, v$, then there exists a finite sum α of elements of A such that $\mathsf{PA}_A \vdash u \approx \alpha^k$ for some $k \geq 1$.

Proof. Note that $\lceil t \rceil = \lceil v \rceil$; we shall first prove the following
Claim: if $\lceil t \rceil, \lceil v \rceil = 1$, then there exists a $k \geq 1$ such that $u \approx t^k$ and $t \approx v$.
Let $t = a_1 + \ldots + a_m$ with $a_1, \ldots, a_m \in A$; we proceed by induction on $\lfloor u \rfloor$.
If $\lfloor u \rfloor = 1$, then there exists $a \in A$ such that $a \preccurlyeq u$, whence $at \preccurlyeq u \,||\, v$. Since $a_1 u + \cdots + a_m u \approx u \,||\, v$, there exists by Lemma 3.1 an i such that $a_i u \approx av$; hence by Lemma 3.6 $u \approx v$. Since $\lceil v \rceil = 1$ it follows that $tu \approx v \,||\, v \approx vv \approx vu$, hence by Lemma 3.6 $t \approx v$.
If $\lfloor u \rfloor > 1$, then there exist $b_1, \ldots, b_n \in A$ and ground terms $u_1, \ldots, u_n$ such that $u \approx b_1 u_1 + \ldots + b_n u_n$. Then $a_1 u + \ldots + a_m u \approx tu \approx u \,||\, v \approx b_1(u_1 \,||\, v) + \ldots + b_n(u_n \,||\, v) + vu$, so by Lemma 3.1 $u_i \,||\, v \approx u$, for all $1 \leq i \leq n$. By Lemma 3.3 there exists u' such $u_i \approx u'$ for all $1 \leq i \leq n$, and, by A4, $(b_1 + \ldots + b_n)u' \approx u \approx u' \,||\, v$. Hence by the induction hypothesis $v \approx b_1 + \ldots + b_n$ and $u' \approx v^k$ for some $k \geq 1$. So $u \approx v^{k+1}$, and from

$$\begin{aligned} tu &\approx vu + b_1(u' \,||\, v) + \cdots + b_n(u' \,||\, v) \\ &\approx vu + vu && \text{(by A4)} \\ &\approx vu && \text{(by A3)} \end{aligned}$$

it follows, by Lemma 3.6, that $t \approx v$. This completes the proof of our claim.
The proof of the lemma is by induction on $\lceil v \rceil$. If $\lceil t \rceil, \lceil v \rceil = 1$ then t is a finite sum of elements of A and by our claim $u \approx t^k$ for some $k \geq 1$. If $\lceil t \rceil, \lceil v \rceil > 1$, then there exists $a \in A$ and ground terms t' and v' such that $av' \preccurlyeq v$ and $t'u \approx u \,||\, v'$; hence, by the induction hypothesis, there exists a finite sum α of elements of A such that $u \approx \alpha^k$, for some $k \geq 1$. □

Lemma 4.1 has the following consequence.

Lemma 4.2 If t, t', u and v are ground terms such that $\mathsf{PA}_A \vdash tu \approx t'u \,||\, v$, then there exists a finite sum α of elements of A such that $\mathsf{PA}_A \vdash u \approx \alpha^k$ for some $k \geq 1$.

Proof. By induction on the norm of t'. □

Lemma 4.3 Let α be a finite sum of elements of A; if t, u and v are ground terms such that $\mathsf{PA}_A \vdash t\alpha^k \approx u \,||\, v$ for some $k \geq 1$, then $\mathsf{PA}_A \vdash u \approx \alpha^l$ for some $l \leq k$, or there exists a ground term t' such that $\mathsf{PA}_A \vdash u \approx t'\alpha^k$.

Proof. The proof is by induction on the norm of v.
If $\lfloor v \rfloor = 1$, then there exists an $a \in A$ such that $a \preccurlyeq v$, whence $au \preccurlyeq t\alpha^k$. If $a \preccurlyeq t$, then $u \approx \alpha^k$, and if there exists a ground term t' such that $at' \preccurlyeq t$, then $u \approx t'\alpha^k$.
Suppose that $\lfloor v \rfloor > 1$ and let v' be a ground term such that $\lfloor v' \rfloor < \lfloor v \rfloor$ and $av' \preccurlyeq v$, whence $a(u \,||\, v') \preccurlyeq t\alpha^k$. If $a \preccurlyeq t$, then $u \,||\, v' \approx \alpha^k$, hence there exists an $l < k$ such that $u \approx \alpha^l$. Otherwise, suppose that t^* is a ground term such that $at^* \preccurlyeq t$ and $u \,||\, v' \approx t^*\alpha^k$; by induction hypothesis $u \approx \alpha^l$ for some $l \leq k$, or there exists a ground term t' such that $u \approx t'\alpha^k$. □

Hirshfeld and Jerrum (1998) give a thorough investigation of a particular kind of mixed equations; we shall adapt some of their theory to our setting.

Let α be a finite sum of elements of A. A ground term t we shall call α-*free* if $t \not\approx \alpha$ and there exists no ground term t' such that $t \approx t' \,||\, \alpha$. We shall call a ground term t an α-*term* if $t \approx \alpha^k$ for some $k \geq 1$. The α-*norm* $\lfloor t \rfloor_\alpha$ of a ground term t is the length of the shortest reduction of t to an α-term, or the norm of t if such a reduction does not exist. Note that if $t \approx u$, then $\lfloor t \rfloor_\alpha = \lfloor u \rfloor_\alpha$; the α-norm of an equation is the α-norm of both sides. We shall write $t \longrightarrow_\alpha t'$ if $t \longrightarrow t'$ and $\lfloor t' \rfloor_\alpha < \lfloor t \rfloor_\alpha$; if

$\lfloor u \rfloor_\alpha = 1$, then we say that u is an *α-unit*. In line with Definition 3.7, a ground term t' is an *α-reduct* of a ground term t if t' is reachable from t by an α-reduction.

It is easy to see that $\lfloor t \,||\, u \rfloor_\alpha = \lfloor t \rfloor_\alpha + \lfloor u \rfloor_\alpha$, so we have the following lemma.

Lemma 4.4 Let α be a finite sum of elements of A; any α-free α-unit is parallel prime.

Lemma 4.5 If t is α-free, then $t\alpha$ is α-free.

Proof. Suppose that there exists a ground term t' such that $t\alpha \approx t' \,||\, \alpha$; we prove that t is not α-free. If $\lfloor t' \rfloor = 1$, then by Lemma 4.3 $t' \approx \alpha$ and hence $t \approx \alpha$. If $\lfloor t' \rfloor > 1$, then by Lemma 4.3 there exists a ground term t^* such that $t' \approx t^*\alpha$, hence by Lemma 3.2 $t\alpha \approx (t^* \,||\, \alpha)\alpha$ and by Lemma 3.6 $t \approx t^* \,||\, \alpha$. $\square$

Hirshfeld and Jerrum (1998) proved a variant of Lemma 3.9.

Lemma 4.6 Let α be a finite sum of elements of A, and let t be an α-free ground term. If $t \longrightarrow_\alpha t'$ and $t \longrightarrow_\alpha t''$ implies that $\mathsf{PA}_A \vdash t' \approx t''$ for all ground terms t' and t'', then there exists a parallel prime ground term t^* such that $\mathsf{PA}_A \vdash t \approx t^* \,||\, \ldots \,||\, t^*$.

A *pumpable equation* is a mixed equation of the form

$$(t_1 \,||\, \cdots \,||\, t_m)\alpha^k \approx u_1\alpha^k \,||\, \cdots \,||\, u_n\alpha^k,$$

where α is a finite sum of elements of A, $k \geq 1$, $m, n \geq 2$ and t_i and u_j are α-free ground terms for $1 \leq i \leq m$ and $1 \leq j \leq n$. The following lemma occurs in Hirshfeld and Jerrum (1998) as Lemma 7.2.

Lemma 4.7 There are no pumpable equations with α-norm less than three.

Proposition 4.8 Let t, u, u' and v be ground terms such that t and v are α-free and

$$\mathsf{PA}_A \vdash (t \,||\, u)\alpha^k \approx v\alpha^k \,||\, u'\alpha^k. \tag{4.1}$$

If u and u' are α-units, then $\mathsf{PA}_A \vdash u \approx u'$.

Proof. If there exists a ground term u^* such that $u \approx u^* \,||\, \alpha$, then by Lemma 3.2 $v\alpha^k \,||\, u'\alpha^k \approx (t \,||\, u^* \,||\, \alpha)\alpha^k \approx (t \,||\, u^*)\alpha^k \,||\, \alpha$; by Lemma 4.5 $v\alpha^k$ is α-free, hence there exists a ground term u^{**} such that $u' \approx u^{**} \,||\, \alpha$. Vice versa, from $u' \approx u^{**} \,||\, \alpha$ we obtain the existence of a u^* such that $u \approx u^* \,||\, \alpha$. In both cases $(t \,||\, u^*)\alpha^k \,||\, \alpha \approx v\alpha^k \,||\, u^{**}\alpha^k \,||\, \alpha$, whence

$$(t \,||\, u^*)\alpha^k \approx v\alpha^k \,||\, u^{**}\alpha^k.$$

Hence, we may assume without loss of generality that the α-units u and u' are α-free, so that (4.1) is a *pumpable equation*. By Lemma 4.7 there are no pumpable equations with α-norm less than three, so $\lfloor t \rfloor_\alpha, \lfloor v \rfloor_\alpha \geq 2$; we prove the lemma by induction on $\lfloor t \rfloor_\alpha$.
If there exist ground terms t' and v' such that $t \longrightarrow_\alpha t'$, $v \longrightarrow_\alpha v'$ and $(t' \,||\, u)\alpha^k \approx v'\alpha^k \,||\, u'\alpha^k$, then we may conclude $u \approx u'$ from the induction hypothesis. Since the α-units $u'\alpha^k$ and u have unique immediate α-reducts, in the case that remains, t and v have unique immediate α-reducts t' and v', respectively; hence, by Lemma 4.6 there exists a parallel prime ground term v^* such that $v \approx v^* \,||\, \cdots \,||\, v^*$. By Lemma 3.2

$$(t' \,||\, u)\alpha^k \approx v\alpha^k \,||\, \alpha^{k+i} \approx (v \,||\, \alpha^{k+i})\alpha^k, \text{ for some } i \geq 0,$$

so $t' \,||\, u \approx v^* \,||\, \cdots \,||\, v^* \,||\, \alpha^{k+i}$. Since u is α-free, whence parallel prime by Lemma 4.5, it follows that $u \approx v^*$; hence

$$(t \,||\, u)\alpha^k \approx (u \,||\, \cdots \,||\, u)\alpha^k \,||\, u'\alpha^k \text{ and } t' \approx u \,||\, \cdots \,||\, u \,||\, \alpha^{k+i}.$$

Clearly, there exists a $j \geq k$ such that $u'\alpha^k \,||\, \alpha^j$ is an α-reduct of $v\alpha^k \,||\, u'\alpha^k$ (α-reduce $v\alpha^k$ to α^j). Hence, by Lemma 3.2, $(u' \,||\, \alpha^j)\alpha^k$ is an α-reduct of $(t \,||\, u)\alpha^k$, so $u' \,||\, \alpha^j$ is an α-reduct of $t \,||\, u$.

If $u' \mid\mid \alpha^j$ is obtained by reducing t to an α-term, then $u \approx u'$ follows, since u and u' are α-free. Otherwise, there exists $j' \leq j$ such that $u' \mid\mid \alpha^{j'}$ is an α-reduct of t, hence of the unique immediate α-reduct t' of t. Every α-reduct of t' with α-norm 1 is of the form $u \mid\mid \alpha^{j''}$. Since u and u' are parallel prime, $u \approx u'$ follows. □

5. Mixed Terms

We shall now define the set of *head normal forms*, thus restricting the set of terms that we need to consider in our proof that PA_A is ω-complete. The syntactic form of head normal form motivate our investigation of a particular kind of terms that we shall call *mixed terms* (nestings of parallel and sequential compositions). We shall work towards a theorem that certain instantiations of mixed terms are either parallel prime or a parallel composition of a parallel prime and α^k for some finite sum α of elements of A.

Let x be a variable, suppose $t = x$ or $t = xt'$ for some term t', and suppose $\bar{u} = u_1, \ldots, u_j$ and $\bar{v} = v_1, \ldots, v_j$ are sequences of terms; we define the set of *x-prefixes* $L_j[t, \bar{u}, \bar{v}]$ inductively as follows:

$$L_0[t] = t; \text{ and}$$
$$L_{j+1}[t, \bar{u}, u_{j+1}, \bar{v}, v_{j+1}] = (L_j[t, \bar{u}, \bar{v}] \mathbin{\lfloor\!\lfloor} u_{j+1})v_{j+1}.$$

Definition 5.1 We define the set of *head normal forms* as follows:

1. if $a \in A$ and t is a term, then a and at are head normal forms;
2. if x is a variable, v is an x-prefix and t is any term, then v and $v \mathbin{\lfloor\!\lfloor} t$ are head normal forms; and
3. if t and u are head normal forms, then $t + u$ is a head normal form.

If t is a head normal form, then there exist finite sets I, J, K and L such that

$$t \approx \sum_{i \in I} a_i t_i + \sum_{j \in J} b_j + \sum_{k \in K} v_k \mathbin{\lfloor\!\lfloor} u_k + \sum_{l \in L} w_l \qquad \text{(by A1 and A2)}$$

with the a_i and b_j elements of A, the t_i and u_k arbitrary terms and each v_k and w_l an x-prefix for some variable x.

Lemma 5.2 For each term t there exists a head normal form t^* such that $\mathsf{PA}_A \vdash t \approx t^*$.

Proof. The proof is by induction on the depth of t. The elements of A and the variables are head normal forms by definition.

Now, suppose by way of induction hypothesis that t' and t'' are head normal forms, and in particular that

$$t' \approx \sum_{i \in I} a_i t_i + \sum_{j \in J} b_j + \sum_{k \in K} v_k \mathbin{\lfloor\!\lfloor} u_k + \sum_{l \in L} w_l.$$

If $t = t' + t''$, then t is a head normal form.
If $t = t't''$, then by A4

$$t \approx \sum_{i \in I} (a_i t_i)t'' + \sum_{j \in J} b_j t'' + \sum_{k \in K} (v_k \mathbin{\lfloor\!\lfloor} u_k)t'' + \sum_{l \in L} w_l t''.$$

and with applications of A5 to the $(a_i t_i)t''$ and the $w_l t''$ the left-hand side becomes a head normal form.

If $t = t' \mathbin{\|\!\_} t''$, then by M4

$$t' \approx \sum_{i\in I} a_i t_i \mathbin{\|\!\_} t'' + \sum_{j\in J} b_j \mathbin{\|\!\_} t'' + \sum_{k\in K} (v_k \mathbin{\|\!\_} u_k) \mathbin{\|\!\_} t'' + \sum_{l\in L} w_l \mathbin{\|\!\_} t''.$$

and with an application of $\mathrm{M3}_{a_i}$ to each $a_i t_i \mathbin{\|\!\_} t''$, an application of $\mathrm{M2}_{b_j}$ to each $b_j \mathbin{\|\!\_} t''$, and an application of M5 to each $(v_k \mathbin{\|\!\_} u_k) \mathbin{\|\!\_} t''$, the left-hand side of becomes a head normal form.
If $t = t' \,||\, t''$, then $t \approx t' \mathbin{\|\!\_} t'' + t'' \mathbin{\|\!\_} t'$ by M1 and we may proceed as in the previous case. □

We shall associate with every equation $t \approx u$ a substitution σ such that $t^\sigma \approx u^\sigma$ implies that $t \approx u$. The main idea of our ω-completenss proof is to substitute for every variable in t or u a ground term that has a subterm φ_n of degree n, where, intuitively, n is large compared to the degrees already occurring in t and u. Let a be an element of A and let $n \geq 1$; we define

$$\varphi_n = a^n + a^{n-1} + \ldots + a.$$

Lemma 5.3 If $n \geq 2$ and t is a ground term, then $\varphi_n t$ is parallel prime.

Proof. Clearly, t is the unique immediate reduct of $\varphi_n t$, so by Lemma 3.9 there exists a parallel prime ground term u^* such that $\varphi_n t \approx u^* \,||\cdots||\, u^*$. Moreover, if $n \geq 2$, then $aat \preccurlyeq u^* \,||\cdots||\, u^*$. So there exists a u' such that $at \approx u' \,||\, u^* \,||\cdots||\, u^*$, and since t is the unique immediate reduct of at, $u' \,||\, u^* \,||\cdots||\, u^*$ must be a parallel composition of equivalent parallel components. Since $\lceil u' \rceil < \lceil u^* \rceil$ we have that u' is not of the form $u^* \,||\cdots||\, u^*$, hence $\varphi_n t \approx u^*$. □

Suppose t is a term, and let $\bar{u} = u_1, \ldots, u_j$ and $\bar{v} = v_1, \ldots, v_j$ be sequences of terms; we define the set of *mixed terms* $M_j[t, \bar{u}, \bar{v}]$ inductively as follows:

$$M_0[t] = t; \text{ and}$$
$$M_{j+1}[t, \bar{u}, u_{j+1}, \bar{v}, v_{j+1}] = (M_j[t, \bar{u}, \bar{v}] \,||\, u_{j+1}) v_{j+1}.$$

Let t be a ground term; we denote by $d^{\rightarrow}_{\max}(t)$ the least upperbound for the degrees of all the reducts of t, i.e.,

$$d^{\rightarrow}_{\max}(t) = \max\{d(t') \mid t' \in \mathsf{red}(t)\}.$$

Definition 5.4 A mixed term $M_j[\varphi_n t, \bar{u}, \bar{v}]$ we shall call a *generalised φ_n-term* if

$$d^{\rightarrow}_{\max}(M_j[t, \bar{u}, \bar{v}]) < n.$$

Note that there are no generalised φ_1-terms.

Lemma 5.5 Let $M_j[\varphi_n t, \bar{u}, \bar{v}]$ be a generalised φ_n-term and let u be a ground term such that

$$\mathsf{PA}_A \vdash M_j[\varphi_n t, \bar{u}, \bar{v}] \approx \varphi_n t v_1 \cdots v_j \,||\, u.$$

Then there exists a finite sum α of elements of A such that

$$\mathsf{PA}_A \vdash v_1 \cdots v_j \approx \alpha^k \text{ and } \mathsf{PA}_A \vdash u \approx u_1 \,||\cdots||\, u_j \approx \alpha^l \text{ for some } k, l \geq 1.$$

Proof. From $M_j[\varphi_n t, \bar{u}, \bar{v}] \longrightarrow M_j[t, \bar{u}, \bar{v}]$ and $d(M_j[t, \bar{u}, \bar{v}]) < n$ it follows that $M_j[t, \bar{u}, \bar{v}] \approx t v_1 \cdots v_j \,||\, u$; hence

$$d^{\rightarrow}_{\max}(t v_1 \cdots v_j \,||\, u) < n. \tag{5.1}$$

Note that $\lfloor u \rfloor = \lfloor u_1 \,||\cdots||\, u_j \rfloor$; we shall prove the lemma by induction on $\lfloor u \rfloor$.
If $\lfloor u \rfloor = 1$, then $j = 1$ and $\lfloor u_1 \rfloor = 1$. By Lemma 4.2 there exists a finite sum α of elements of A such that $v_1 \approx \alpha^k$ for some $k \geq 1$, and hence by Lemma 4.3 $u \approx \alpha$. By Lemma 3.2 $(\varphi_n t \,||\, u_1)\alpha^k \approx \varphi_n t \alpha^k \,||\, \alpha \approx (\varphi_n t \,||\, \alpha)\alpha^k$, so $u_1 \approx \alpha$.

If $\lfloor u \rfloor > 1$, then there are three cases: $\lfloor u_j \rfloor = 1$ and $j > 1$, $\lfloor u_j \rfloor > 1$ and $j = 1$, and $\lfloor u_j \rfloor > 1$ and $j > 1$. We shall only treat the last case; for the other two cases the proof is similar.
Let u'_j be an immediate reduct of u_j; by (5.1) there exists an immediate reduct u' of u such that

$$M_j[\varphi_n t, u_1, \ldots, u_{j-1}, u'_j, \bar{v}] \approx \varphi_n t v_1 \cdots v_j \mid\mid u'.$$

Since $M_j[\varphi_n t, u_1, \ldots, u_{j-1}, u'_j, \bar{v}]$, there exists by the induction hypothesis a finite sum α of elements of A such that $v_1 \cdots v_j \approx \alpha^k$ and $u_1 \mid\mid \cdots \mid\mid u_{j-1} \mid\mid u'_j \approx \alpha^l$ for some $k, l \geq 1$. By means of $j-1$ applications of Lemma 3.2, we find that

$$M_j[\varphi_n t, \bar{u}, \bar{v}] \approx (\varphi_n t \alpha^{k - \lfloor v_j \rfloor} \mid\mid u_j)\alpha^{\lfloor v_j \rfloor} \mid\mid \alpha^l \approx \varphi_n t \alpha^k \mid\mid u.$$

Since $\varphi_n t \alpha^k$ is parallel prime by Lemma 5.3, there exists a ground term u^* with $\lfloor u^* \rfloor = \lfloor u_j \rfloor$ and $\lfloor u^* \rfloor_\alpha = \lfloor u_j \rfloor_\alpha$ such that $u \approx u^* \mid\mid \alpha^l$ and $(\varphi_n t \alpha^{k-\lfloor v_j \rfloor} \mid\mid u_j)\alpha^{\lfloor v_j \rfloor} \approx \varphi_n t \alpha^k \mid\mid u^*$. If $\lfloor u^* \rfloor \leq \lfloor v_j \rfloor$, then by Lemma 4.3 u^* is an α-term, which implies that u and u_j are also α-terms. So suppose that $\lfloor u^* \rfloor > \lfloor v_j \rfloor$, and let $u^\dagger$ be a ground term such that $u^* \approx u^\dagger \alpha^{\lfloor v_j \rfloor}$. Since $\lfloor u_j \rfloor > \lfloor u^\dagger \rfloor$ and $\lfloor u_j \rfloor_\alpha = \lfloor u^\dagger \rfloor_\alpha$, by Proposition 4.8, u_j and $u^\dagger$ are not α-units. Since u'_j is an α-term and $u_j \longrightarrow u'_j$, u_j must be an α-term, so also $u^\dagger$ is an α-term. Consequently, $u \approx u^* \mid\mid \alpha^l \approx u^\dagger \alpha^{\lfloor v_j \rfloor} \mid\mid \alpha^l$ is an α-term. □

Lemma 5.6 Let $M_j[\varphi_n t, \bar{u}, \bar{v}]$ be a generalised φ_n-term. If $M_j[\varphi_n t, \bar{u}, \bar{v}]$ is not α-free, then there exists $i \leq j$ such that $\mathsf{PA}_A \vdash v_i \cdots v_j \approx \alpha^k$ and u_i is not α-free.

Proof. Let t^* be a ground term such that

$$M_j[\varphi_n t, \bar{u}, \bar{v}] \approx t^* \mid\mid \alpha. \tag{5.2}$$

Since $\varphi_n t v_1 \cdots v_j$ is a reduct of $M_j[\varphi_n t, \bar{u}, \bar{v}]$ and by Lemma 5.3 $\varphi_n t v_1 \cdots v_j$ is parallel prime, $\varphi_n t v_1 \cdots v_j$ must be a reduct of t^*. Then $\varphi_n t v_1 \cdots v_j \mid\mid \alpha$ is a reduct of $M_j[\varphi_n t, \bar{u}, \bar{v}]$, so there exist sequences of ground terms $\bar{u}'$ and $\bar{v}'$ such that for some $1 \leq j' \leq j$ and $1 \leq i \leq j$,

$$M_{j'}[\varphi_n t v_1 \cdots v_{i-1}, \bar{u}', \bar{v}'] \approx \varphi_n t v_1 \cdots v_j \mid\mid \alpha, \text{ where } v'_1 \cdots v'_{j'} \approx v_i \cdots v_j.$$

By Lemma 5.5 $v_i \cdots v_j \approx \alpha^k$, so in particular $v_j \approx \alpha^l$ for some $l \leq k$. Clearly, $\lfloor t^* \rfloor > l$, so by Lemma 4.3 there exists $t^\dagger$ such that $t^* \approx t^\dagger \alpha^l$. We apply Lemma 3.2 to the right-hand side of (5.2) and cancel the α^l-tail on both sides to obtain

$$M_{j-1}[\varphi_n t, u_1, \ldots, u_{j-1}, v_1, \ldots, v_{j-1}] \mid\mid u_j \approx t^\dagger \mid\mid \alpha.$$

The remainder of the proof is by induction on j. If $j = 1$, then $\varphi_n t \mid\mid u_j \approx t^\dagger \mid\mid \alpha$ implies that u_j is not α-free and we are done. If $j > 1$ and u_j is α-free, then $M_{j-1}[\varphi_n t, u_1, \ldots, u_{j-1}, v_1, \ldots, v_{j-1}]$ is not α-free, so by the induction hypothesis there exists some $1 \leq i' \leq j-1$ such that $u_{i'}$ is not α-free and $v_{i'} \cdots v_{j-1} \approx \alpha^{k'}$, whence $v_{i'} \cdots v_j \approx \alpha^{k'+l}$. □

Proposition 5.7 If a generalised φ_n-term t^* is not parallel prime, then there exists a finite sum α of elements of A and a parallel prime generalised φ_n-term $t^\dagger$ such that $t^* \approx t^\dagger \mid\mid \alpha^k$ for some $k \geq 1$.

Proof. Let $t^* \approx M_j[\varphi_n t, \bar{u}, \bar{v}]$ and let $t_1, \ldots, t_o$ be parallel prime ground terms such that

$$M_j[\varphi_n t, \bar{u}, \bar{v}] \approx t_1 \mid\mid \ldots \mid\mid t_o.$$

Since $\varphi_n t v_1 \cdots v_j$ is a reduct of $M_j[\varphi_n t, \bar{u}, \bar{v}]$ and parallel prime, $\varphi_n t v_1 \cdots v_j$ must be a reduct of some t_i $(1 \leq i \leq o)$; assume without loss of generality that it is a reduct of t_1.

Suppose that $M_j[\varphi_n t, \bar{u}, \bar{v}]$ is not parallel prime and let $u \approx t_2 \,||\, \cdots \,||\, t_o$. Since $\varphi_n t v_1 \cdots v_j \,||\, u$ is a reduct of $M_j[\varphi_n t, \bar{u}, \bar{v}]$, there exist sequences of ground terms $\bar{u}'$ and $\bar{v}'$ such that, for some $1 \leq j' \leq j$ and $1 \leq i \leq j$,

$$M_{j'}[\varphi_n t v_1 \cdots v_{i-1}, \bar{u}', \bar{v}'] \approx \varphi_n t v_1 \cdots v_j \,||\, u, \text{ where } v'_1 \cdots v'_{j'} \approx v_i \cdots v_j.$$

So by Lemma 5.5, there exists a finite sum α of elements of A such that $u \approx \alpha^k$, for some $k \geq 1$. It remains to prove that if $M_j[\varphi_n t, \bar{u}, \bar{v}] \approx t^\dagger \,||\, \alpha$, then $t^\dagger$ is a generalised φ_n-term, for then it follows that t_1 is a generalised φ_n-term by induction on k. Since $M_j[\varphi_n t, \bar{u}, \bar{v}]$ is not α-free, there exists by Lemma 5.6 an $i \leq j$ such that u_i is not α-free and $v_i \cdots v_j \approx \alpha^l$ for some $l \geq 1$. So either $u_i \approx \alpha$ or there exists u'_i such that $u_i \approx u'_i \,||\, \alpha$; we only consider the second possibility, as the other can be dealt with similarly. By Lemma 3.2 we obtain

$$M_j[\varphi_n t, \bar{u}, \bar{v}] \approx M_{j-1}[\varphi_n t, u_1, \ldots, u_{i-1}, u'_i, u_{i+1}, \ldots, u_j, \bar{v}] \,||\, \alpha$$

Hence $t^\dagger \approx M_{j-1}[\varphi_n t, u_1, \ldots, u_{i-1}, u'_i, u_{i+1}, \ldots, u_j, \bar{v}]$ is a generalised φ_n-term. □

6. ω-completeness

Let A be a nonempty set; we shall now prove that PA_A is ω-complete. We shall assume that the variables used in an equation $t \approx u$ are enumerated by

$$x_1, x_2, \ldots, x_k, \ldots$$

Let x_i be a variable and let m be a natural number; the particular kind of substitutions σ_m that we shall use in our proof satisfy

$$\sigma_m(x_i) = a(a\varphi_{i+m} + a)a.$$

We want to choose m large compared to the degrees already occurring in t and u; with every term t we associate a natural number $d^\sigma_{\max}(t)$ that denotes the maximal degree that occurs in t after applying a substitution of the form described above, treating the terms φ_{i+m} as fresh constants.

Definition 6.1 Suppose $\Xi = \{\xi_1, \xi_2, \ldots, \xi_k, \ldots\}$ is a countably infinite set of constant symbols such that $\Xi \cap A = \emptyset$. Let t be a term and let σ be a substitution such that

$$\sigma(x_i) = a(a\xi_i + a)a.$$

We define $d^\sigma_{\max}(t)$ as the maximal degree that occurs in t^σ, i.e., $d^\sigma_{\max}(t) = d_{\max}(t^\sigma)$.

Lemma 6.2 If t is a term and let m be a natural number, then

i. $d^{\rightarrow}_{\max}(t^{\sigma_m}) \leq d^\sigma_{\max}(t)$; and

ii. if $a \in A$ and t' is a ground term such that $at' \preccurlyeq t^{\sigma_m}$, then $d(t') \leq d^\sigma_{\max}(t)$.

Proof. We shall first show that if $t' \in \mathsf{red}(t^\sigma)$, then there does not exist $\xi \in \Xi$ such that $\xi \preccurlyeq t'$ or $\xi t'' \preccurlyeq t'$ for some ground term t''; we proceed by structural induction on t.
If $t \in A$, then t is not an element of Ξ. If t is a variable and t' is a reduct of t^σ, then $t' \approx a(a\xi + a)a$, $t' \approx (a\xi + a)a$, or $t' \approx a$, so there does not exist $\xi' \in \Xi$ such $\xi' \preccurlyeq t'$ or $\xi' t'' \preccurlyeq t'$ for some ground term t''.
If $t = t_1 \cdot t_2$, then

$$\mathsf{red}(t^\sigma) = \{t'_1 t^\sigma_2 \mid t'_1 \in \mathsf{red}(t^\sigma_1)\} \cup \mathsf{red}(t^\sigma_2),$$

so if $t' \in \mathsf{red}(t^\sigma)$, then it is immediate by the induction hypothesis that there does not exist $\xi \in \Xi$ such that $\xi \preccurlyeq t'$ or $\xi t'' \preccurlyeq t'$ for some ground term t''.
If $t = t_1 * t_2$ with $* \in \{+, ||, \lfloor\!\lfloor\}$, then

$$\mathsf{red}(t^\sigma) \subseteq \{t_1^\sigma * t_2^\sigma \mid t_1' \in \mathsf{red}(t_1^\sigma),\ t_2' \in \mathsf{red}(t_2^\sigma)\} \cup \mathsf{red}(t_1^\sigma) \cup \mathsf{red}(t_2^\sigma),$$

so if $t' \in \mathsf{red}(t^\sigma)$, then it is immediate by the induction hypothesis that there does not exist $\xi \in \Xi$ such that $\xi \preccurlyeq t'$ or $\xi t'' \preccurlyeq t'$ for some ground term t''.
Now, suppose that $t' \in \mathsf{red}(t^{\sigma_m})$; then there exist a ground normal form $t'' \in \mathsf{red}(t^\sigma)$ of the term rewriting system of §2 and a ground term t^* that is obtained from t'' by replacing every occurrence of $\xi_i \in \Xi$ by φ_{i+m} such that $t' \approx t^*$. Note that $d(t^*) \leq d(t'')$; suppose that

$$t'' \approx \sum_{i \in I} a_i t_i + \sum_{j \in J} b_j.$$

Inspection of the rules of the term rewriting system of §2 shows that reducing t^* to normal form may only decrease $|I| + |J|$, hence $d(t') \leq d^\sigma_{\max}(t)$; this proves (i).
The proof of (ii) goes in a similar fashion. □

If $L_j[t, \bar{u}, \bar{v}]$ is an x_i-prefix and m is a natural number, then

$$L_j[t, \bar{u}, \bar{v}]^{\sigma_m} \longrightarrow M_j[(a\varphi_{i+m} + a)t'', \bar{u}, \bar{v}]^{\sigma_m} \succcurlyeq aM_j[\varphi_{i+m}t'', \bar{u}, \bar{v}]^{\sigma_m}, \tag{6.1}$$

where $t'' = a$ if $t = x_i$ and $t'' = at'$ if $t = x_i t'$ for some term t'. If $m \geq d^\sigma_{\max}(L_j[t, \bar{u}, \bar{v}])$, then $M_j[\varphi_{i+m}t'', \bar{u}, \bar{v}]^{\sigma_m}$ is a generalised φ_n-term; we shall call it the *generalised φ_{i+m}-term associated with $L_j[t, \bar{u}, \bar{v}]$ by σ_m*.

For ground terms t and t^*, let us write $t \mapsto t^*$ if there exists a ground term t' and an $a \in A$ such that $t \longrightarrow t' \succcurlyeq at^*$.

Lemma 6.3 Let t be an x-prefix and suppose that $m \geq d^\sigma_{\max}(t)$. If $n > m$ and t^* and $t^\dagger$ are generalised φ_n-terms such that $t^{\sigma_m} \mapsto t^*$ and $t^{\sigma_m} \mapsto t^\dagger$, then $\mathsf{PA}_A \vdash t^* \approx t^\dagger$.

Proof. Note that the unique immediate reduct of t^{σ_m} is of the form $M_j[(a\varphi_{i+m} + a)t', \bar{u}, \bar{v}]^{\sigma_m}$. Moreover, $m \geq d^\sigma_{\max}(t)$, so $M_j[\varphi_{i+m}t', \bar{u}, \bar{v}]^{\sigma_m}$ is the unique ground term t^* such that $at^* \preccurlyeq M_j[(a\varphi_{i+m} + a)t', \bar{u}, \bar{v}]^{\sigma_m}$ and $d(t^*) > m$. Hence, if t^* is any generalised φ_n-term with $n > d^\sigma_{\max}(t)$ such that $t^{\sigma_m} \mapsto t^*$, then $t^* \approx M_j[\varphi_{i+m}t', \bar{u}, \bar{v}]^{\sigma_m}$. □

Note that if t is an x-prefix, $m \geq d^\sigma_{\max}(t)$ and t^* is the generalised φ_{i+m}-term associated with t by σ_m, then t^* has no reduct with a degree in $\{m+1, m+2, \ldots, m+i-1\}$. Lemma 6.3 has the following consequence.

Lemma 6.4 Let t be an x-prefix, let u be a y-prefix and let $m \geq \max\{d^\sigma_{\max}(t), d^\sigma_{\max}(u)\}$. If $\mathsf{PA}_A \vdash t^{\sigma_m} \approx u^{\sigma_m}$, then $x = y$.

Lemma 6.5 Let t be a term, let α be a finite sum of elements of A, and suppose $m \geq d^\sigma_{\max}(t)$. If t^{σ_m} is not α-free, then $\mathsf{PA}_A \vdash t \approx \alpha$, or there exists $a \preccurlyeq \alpha$ and a term t' such that $at' \preccurlyeq t$ and $\mathsf{PA}_A \vdash t^{\sigma_m} \approx (t')^{\sigma_m} \,||\, \alpha$.

Proof. If t is a ground term, then $t \approx t^{\sigma_m}$; so, in particular, if $t^{\sigma_m} \approx \alpha$, then $t \approx \alpha$, and if $t^{\sigma_m} \approx t^* \,||\, \alpha$, then $t \approx t^* \,||\, \alpha$.
Suppose t is not a ground term; then clearly $t^{\sigma_m} \not\approx \alpha$, so there exists $t^{\sigma_m} \approx t^* \,||\, \alpha$. We may assume by Lemma 5.2 that

$$t \approx \sum_{i \in I} a_i t_i + \sum_{j \in J} b_j + \sum_{k \in K} v_k \lfloor\!\lfloor u_k + \sum_{l \in L} w_l,$$

where the v_k and the w_l are x-prefixes.
If $a \in A$ and t' is a ground term such that $at' \preccurlyeq t^*$, then $a(t' \,||\, \alpha) \preccurlyeq t^{\sigma_m}$, so by Lemma 6.2(ii) $d(t') \leq d(t' \,||\, \alpha) \leq d^\sigma_{\max}(t) \leq m$. Consequently, t^* cannot be an immediate reduct of $(v_k \lfloor\!\lfloor u_k)^{\sigma_m}$ or of $w_l^{\sigma_m}$, so $t^* \approx t_i^{\sigma_m}$ for some $i \in I$. □

We generalise the definition of α-freeness to terms with variables: a term t is *α-free* if $t \not\approx \alpha$ and there exists no term t' such that $t \approx t' \mid\mid \alpha$.

Theorem 6.6 (**ω-completeness**) Let A be a nonempty set; then PA_A is ω-complete, i.e., for all terms t and u,

$$\text{if } \mathsf{PA}_A \vdash t^\sigma \approx u^\sigma \text{ for all ground substitutions } \sigma, \text{ then } \mathsf{PA}_A \vdash t \approx u.$$

Proof. Let $m \geq \max\{d^\sigma_{\max}(t), d^\sigma_{\max}(u)\}$.
We shall prove simultaneously by induction on the depth of t that (1) if t is α-free, then t^{σ_m} is α-free; and (2) if $t^{\sigma_m} \approx u^{\sigma_m}$, then $t \approx u$. Clearly, (2) implies the theorem. By Lemma 5.2 we may assume that t and u are head normal forms, and in particular that

$$t \approx \sum_{i\in I} a_i t_i + \sum_{j\in J} b_j + \sum_{k\in K} v_k \mathbin{\lfloor\!\lfloor} u_k + \sum_{l\in L} w_l,$$

where the v_k and the w_l are x-prefixes.
Suppose $t^{\sigma_m} \approx (t')^{\sigma_m} \mid\mid \alpha$, with $at' \preccurlyeq t$ for some $a \preccurlyeq \alpha$. To prove (1) it suffices by Lemma 6.5 to prove that $t \approx t' \mid\mid \alpha$; we show that every summand of t is a summand of $t' \mid\mid \alpha$:

($a_i t_i$) If $a_i \preccurlyeq \alpha$ and $t_i^{\sigma_m} \approx (t')^{\sigma_m}$, then $t_i \approx t'$, so

$$a_i t_i \approx a_i t' \preccurlyeq t' \mid\mid \alpha \qquad \text{(by M1, M4, M3 and A3).}$$

If $a_i \preccurlyeq t'$ and $t_i^{\sigma_m} \approx \alpha$, then by the induction hypothesis $t_i \approx \alpha$; hence

$$a_i t_i \approx a_i \alpha \preccurlyeq t' \mid\mid \alpha \qquad \text{(by M1, M4, M2}_{a_i}\text{ and A3).}$$

If there exists a ground term t^* such that $a_i t^* \preccurlyeq (t')^{\sigma_m}$ and $a_i t_i^{\sigma_m} \approx a_i(t^* \mid\mid \alpha)$, then by Lemma 6.5 there exists $a \preccurlyeq \alpha$ and a term t_i' such that $at_i' \preccurlyeq t_i$ and $t_i^{\sigma_m} \approx (t_i')^{\sigma_m} \mid\mid \alpha$. Since $t^* \approx (t_i')^{\sigma_m}$ and, by Lemma 6.2, $t_i^{\sigma_m}$ cannot be an immediate reduct of a σ_m-instance of an x-prefix or a σ_m-instance of a term of the form $v \mathbin{\lfloor\!\lfloor} w$ with v an x-prefix, we conclude from the induction hypothesis that $a_i t_i' \preccurlyeq t'$. Also by the induction hypothesis $t_i \approx t_i' \mid\mid \alpha$, so

$$a_i t_i \approx a_i(t_i' \mid\mid \alpha) \preccurlyeq t' \mid\mid \alpha \qquad \text{(by M1, M4, M3}_{a_i}\text{ and A3).}$$

($v_k \mathbin{\lfloor\!\lfloor} u_k$) Since the unique immediate reduct of $(v_k \mathbin{\lfloor\!\lfloor} u_k)^{\sigma_m}$ has a summand at^* such that $d(t^*) > m$ and $d^\sigma_{\max}(t') \leq m$, it follows from Lemma 6.2(ii) that $(v_k \mathbin{\lfloor\!\lfloor} u_k)^{\sigma_m} \preccurlyeq (t' \mathbin{\lfloor\!\lfloor} \alpha)^{\sigma_m}$. So there exists a term t'' such that $t'' \preccurlyeq t'$ and $(v_k \mathbin{\lfloor\!\lfloor} u_k)^{\sigma_m} \approx (t'' \mathbin{\lfloor\!\lfloor} \alpha)^{\sigma_m}$.
If $u_k^{\sigma_m}$ is not α-free, then by Lemma 6.5 and the induction hypothesis there exists u_k' such that $u_k \approx u_k' \mid\mid \alpha$. By M5 $v_k \mathbin{\lfloor\!\lfloor} (u_k' \mid\mid \alpha) \approx (v_k \mathbin{\lfloor\!\lfloor} u_k') \mathbin{\lfloor\!\lfloor} \alpha$ and from $(t'')^{\sigma_m} \mathbin{\lfloor\!\lfloor} \alpha \approx (v_k \mathbin{\lfloor\!\lfloor} u_k')^{\sigma_m} \mathbin{\lfloor\!\lfloor} \alpha$ we conclude $(v_k \mathbin{\lfloor\!\lfloor} u_k')^{\sigma_m} \approx (t'')^{\sigma_m}$; hence by the induction hypothesis $v_k \mathbin{\lfloor\!\lfloor} u_k' \approx t''$. Consequently,

$$v_k \mathbin{\lfloor\!\lfloor} u_k \approx t'' \mathbin{\lfloor\!\lfloor} \alpha \preccurlyeq t' \mid\mid \alpha \qquad \text{(by M1, M4 and A3).}$$

So, suppose that $u_k^{\sigma_m}$ is α-free; let $v_k^* = M_j[\varphi_n t^\dagger, \bar{u}^\dagger, \bar{v}^\dagger]^{\sigma_m}$ be the unique generalised φ_n-term associated with v_k by σ_m. Observe that there exists t'' such that $(t')^{\sigma_m} \mapsto t''$ and $v_k^* \mid\mid u_k^{\sigma_m} \approx t'' \mid\mid \alpha$, so v_k^* cannot be α-free. By Lemma 5.6 there exists $i \leq j$ such that $v_i^\dagger \cdots v_j^\dagger \approx \alpha^k$ for some $k \geq 1$ and $(u_i^\dagger)^{\sigma_m}$ is not α-free, so by the induction hypothesis there exists $u_i^\ddagger$ such that $u_i^\dagger \approx u_i^\ddagger \mid\mid \alpha$. By M6$_\alpha$, M5 and commutativity of $\mid\mid$ we obtain a term v_k' such that $v_k \mathbin{\lfloor\!\lfloor} u_k \approx (v_k' \mathbin{\lfloor\!\lfloor} u_k) \mathbin{\lfloor\!\lfloor} \alpha$. From $(t'')^{\sigma_m} \mathbin{\lfloor\!\lfloor} \alpha \approx ((v_k' \mathbin{\lfloor\!\lfloor} u_k)^{\sigma_m} \mathbin{\lfloor\!\lfloor} \alpha$ we conclude $(v_k' \mathbin{\lfloor\!\lfloor} u_k)^{\sigma_m} \approx (t'')^{\sigma_m}$; hence by the induction hypothesis $v_k' \mathbin{\lfloor\!\lfloor} u_k \approx t''$. Consequently,

$$v_k \mathbin{\lfloor\!\lfloor} u_k \approx t'' \mathbin{\lfloor\!\lfloor} \alpha \preccurlyeq t' \mid\mid \alpha \qquad \text{(by M1, M4 and A3).}$$

(w_l) The proof of this case is similar to the proof for the case where $(v_k \mathbin{\|\!\!\lfloor} u_k)^{\sigma_m} \preccurlyeq (t' \,||\, \alpha)^{\sigma_m}$ and $u_k^{\sigma_m}$ is α-free.

Similarly, one shows that every summand of $t' \,||\, \alpha$ is a summand of t; hence $t \approx t' \,||\, \alpha$.
Observe that by (1) we may assume without loss of generality that the generalised φ_n-term associated by σ_m with an x-prefix is parallel prime. For, suppose $v^* = M_j[\varphi_n t', \bar{u}, \bar{v}]^{\sigma_m}$ is the generalised φ_n-term associated by σ_m to some x-prefix v. If v^* is not parallel prime, then by Proposition 5.7 v^* is not α-free. Hence, by Lemma 5.6 there exists $i \leq j$ such that $u_i^{\sigma_m}$ is not α-free and $v_i \cdots v_j \approx \alpha^k$ for some $k \geq 1$. Applying (1), M6$_\alpha$, M5 and commutativity of $||$, we conclude that there exists an x-prefix v' with a parallel prime generalised φ_n-term associated to it by σ_m, such that, by M6$_\alpha$, M5 and commutativity of $||$, $v \approx v' \mathbin{\|\!\!\lfloor} \alpha^l$ for some $l \geq 1$.
For the proof of (2) suppose that $t^{\sigma_m} \approx u^{\sigma_m}$; it suffices to show that every summand of u is a summand of t, for then by a symmetric argument it follows that every summand of t is a summand of u, whence $t \approx u$. There are four cases:

1. If $b \in A$ such that $b \preccurlyeq u$, then $b \preccurlyeq t$ since σ_m-instances of summands of one of the three other types have a norm > 1.
2. If $a \in A$ and u' is a term such that $au' \preccurlyeq u$, then by Lemma 6.2 $d(u') \leq d^\sigma_{\max}(u)$. Since $m \geq d^\sigma_{\max}(u)$, $(u')^{\sigma_m}$ cannot be an immediate reduct of a σ_m-instance of an x-prefix or of a term $v \mathbin{\|\!\!\lfloor} w$, with v an x-prefix. So there exists $i \in I$ such that $a(u')^{\sigma_m} \approx a_i t_i^{\sigma_m}$. By the induction hypothesis $u' \approx t_i$, hence $au' \preccurlyeq t$.
3. Let v be an x-prefix and let u' is a term such that $v \mathbin{\|\!\!\lfloor} u' \preccurlyeq u$. By our assumption that generalised φ_n-terms associated by σ_m with x-prefixes are parallel prime, there exists $k \in K$ such that $v^{\sigma_m} \approx v_k^{\sigma_m}$ and $(u')^{\sigma_m} \approx u_k^{\sigma_m}$; by Lemma 6.4 v_k is also an x-prefix. Hence, by the induction hypothesis $v \approx v_k$ and $u' \approx u_k$, so $v \mathbin{\|\!\!\lfloor} u \preccurlyeq t$.
4. If w is an x-prefix such that $w \preccurlyeq u$, then by our assumption that generalised φ_n-terms associated by σ_m with x-prefixes are parallel prime, there exists $l \in L$ such that $w^{\sigma_m} \approx w_l^{\sigma_m}$; by Lemma 6.4 w_l is also an x-prefix. If the generalised φ_n-term associated to w by σ_m is of the form $\varphi_n w'$, then clearly the generalised φ_n-term associated to w_l by σ_m must be of the form $\varphi_n w_l'$ and it is immediate by the induction hypothesis that $w' \approx w_l'$ and $w \approx w_l$. Let $w = (t' \mathbin{\|\!\!\lfloor} u')v'$ and let $w_l = (t'' \mathbin{\|\!\!\lfloor} u'')v''$, where t' and t'' are x-prefixes to which σ_m associates parallel prime generalised φ_n-terms $t^\dagger$ and $t^\ddagger$. If $\lceil (v')^{\sigma_m} \rceil = \lceil (v'')^{\sigma_m} \rceil$, then by the induction hypothesis $(t' \mathbin{\|\!\!\lfloor} u') \approx (t'' \mathbin{\|\!\!\lfloor} u'')$ and $v' \approx v''$, whence $w \approx w_l$. So let us assume without loss of generality that $\lceil (v')^{\sigma_m} \rceil < \lceil (v'')^{\sigma_m} \rceil$; then there is a ground term v^* such that $(t' \mathbin{\|\!\!\lfloor} u')^{\sigma_m} \approx (t'' \mathbin{\|\!\!\lfloor} u'')^{\sigma_m} v^*$ $v^*(v')^{\sigma_m} \approx (v'')^{\sigma_m}$. Note that $(t^\ddagger \,||\, u'')^{\sigma_m} v^* \approx (t^\dagger \,||\, u')^{\sigma_m}$, which is not parallel prime. So there exists by Proposition 5.7 a finite sum α of elements of A and a parallel prime generalised φ_n-term t^* such that $(t^\ddagger \,||\, (u'')^{\sigma_m})v^* \approx t^* \,||\, \alpha^k$. Hence by Lemma 5.6 $v^* \approx \alpha^l$ for some $l \geq 1$. Consequently, $v'' \approx \alpha^l v'$ and by the induction hypothesis $t' \mathbin{\|\!\!\lfloor} u' \approx (t'' \mathbin{\|\!\!\lfloor} u'')\alpha^l$; hence, $w \preccurlyeq t$. □

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