

# An Equational Axiomatization for Multi-Exit Iteration

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## Abstract

This paper presents an equational axiomatization of bisimulation equivalence over the language of Basic Process Algebra (BPA) with multi-exit iteration. Multi-exit iteration is a generalization of the standard binary Kleene star operation that allows for the specification of agents that, up to bisimulation equivalence, are solutions of systems of recursion equations of the form

$$\begin{array}{l} X_1 \stackrel{\text{def}}{=} P_1 X_2 + Q_1 \\ \vdots \\ X_n \stackrel{\text{def}}{=} P_n X_1 + Q_n \end{array}$$

where  $n$  is a positive integer, and the  $P_i$  and the  $Q_i$  are process terms. The addition of multi-exit iteration to BPA yields a more expressive language than that obtained by augmenting BPA with the standard binary Kleene star (BPA\*). As a consequence, the proof of completeness of the proposed equational axiomatization for this language, although standard in its general structure, is much more involved than that for BPA\*. An expressiveness hierarchy for the family of  $k$ -exit iteration operators proposed by Bergstra, Bethke and Ponse is also offered.

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# 1 Introduction

One of the classic topics in the theory of computation is the study of axiomatic characterizations of equalities over variations on regular expressions. This field of research has been active since Kleene’s original paper [31], where regular expressions were first introduced, and has yielded a collection of very deep and beautiful mathematical results. (The interested reader is invited to consult, e.g., [43, 15, 40, 33, 13, 32] for an overview of the results that have been obtained within this line of research.) According to the point of view of formal language theory, a regular expression denotes a regular language, and two regular expressions are equal exactly when they denote the same language. This notion of semantics for regular expressions is the natural one to choose when the finite automaton associated with a regular event by Kleene’s *synthesis theorem* (cf., e.g., [31, 43]) is viewed as accepting a language. However, as first observed by Milner [35], defining the semantics of an automaton as the language it accepts is inappropriate when one views it as a *reactive system*, i.e., as a system that computes by reacting to stimuli from its environment. For this reason, a wealth of notions of equivalence over processes that, to different degrees, capture their behaviour as reactive machines have been proposed in the literature on process theory. (The interested reader is invited to consult the references [25, 27] for more details.) Amongst these, the notion of bisimulation equivalence [39] has emerged as a fundamental semantic equivalence for reactive systems, and the development of its theory has by now reached a level of maturity that is comparable to that of the standard notions from the theory of formal languages. For example, the complete axiomatizations of bisimulation equivalence and observation congruence for the regular fragment of CCS [37], provided by Milner in his classic papers [36, 38], parallel those obtained by Salomaa for regular languages [42, 43], and have contributed to the realization that the notion of *process* is as elegant and mathematically tractable as that of *language*.

Despite these successes, process theory has traditionally lacked a systematic investigation of (equational) axiomatizations of process equivalences over regular expressions—a notable exception being Milner’s seminal paper [36], where an implicational proof system was proposed for bisimulation equivalence over regular events, and the problem of its completeness raised. (To the best of the authors’ knowledge, this problem of Milner’s is still awaiting a solution.)

The study of axiomatic questions for variations on the language of regular expressions from the perspective of process theory has received new impulse after the publication of [9]. In *op. cit.* the authors have investigated the expressive power of variations on standard process description languages in which infinite behaviours are defined by means of Kleene’s star operation [31, 17] rather than by means of systems of recursion equations, and have proposed an axiom system for bisimulation equivalence over the language of Basic Process Algebra [10] with the original binary version of the Kleene star operation (BPA\*). The completeness of the axiom system proposed *ibidem* was proven by Fokkink and Zantema in [23, 22], and this result has been followed by a series of contributions in which several notions of process equivalence have been equationally axiomatized over languages incorporating variations on the Kleene star operation—see, e.g., [20, 3, 2, 1, 24, 21].

Interestingly, as already noted by Milner in [36, Sect. 6], not every process defined using finite-state systems of recursion equations can be described, up to bisimulation equivalence, using only regular expressions. (As shown in [9], any regular process can be specified in the axiom system  $ACP_\tau$  [11] with Kleene star using handshake communication.) The limited expressive power of the Kleene star operation in denoting finite automata modulo bisimulation equivalence is highlighted in [8], where it is shown that the process described by the following recursion equation

$$(1) \quad X \stackrel{\text{def}}{=} a \cdot (a \cdot X + b) + a$$

cannot be expressed in the language  $BPA^*$  modulo bisimulation equivalence. (Of course, a simple use of Arden's lemma yields that the language denoted by  $X$  is the same associated with the regular expression  $(a \cdot a)^*(a \cdot b + a)$ . However, the process denoted by  $X$  above is *not* bisimulation equivalent to that associated with the  $BPA^*$  term  $(a \cdot a)^*(a \cdot b + a)$ .)

In order to increase the expressive power of super-languages of BPA incorporating Kleene star-like operations, the use of  $k$ -exit iteration has been proposed in [8]. For every positive integer  $k$ , and process terms  $P_i$  and  $Q_i$  ( $1 \leq i \leq k$ ), the process term  $(P_1, \dots, P_k)^*(Q_1, \dots, Q_k)$  denotes a solution to the following list of recursion equations:

$$\begin{aligned} X_1 &\stackrel{\text{def}}{=} P_1 X_2 + Q_1 \\ &\vdots \\ X_k &\stackrel{\text{def}}{=} P_k X_1 + Q_k \end{aligned}$$

For example, the term  $(a, a)^*(a, b)$  uses double-exit iteration, and the reader will immediately realize that, up to isomorphism, it denotes the finite automaton associated with the variable  $X$  in (1).

The aim of this paper is to present an equational axiomatization of bisimulation equivalence over the language obtained by augmenting Basic Process Algebra with the family of  $k$ -exit iteration operations. In fact, we shall consider a slight syntactic generalization of the family of  $k$ -exit iteration operations, in that we permit the construction of terms of the form  $(P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$ , with  $m$  and  $n$  two arbitrary positive integers. The result is a purely algebraic process language that is more expressive than  $BPA^*$ .

Apart from the standard laws from the equational theory for BPA, and the adaptations of the three axioms for the binary Kleene star [23] to multi-exit iteration, the equational axiomatization that we propose contains two more axiom schemas for multi-exit iteration (cf. the axiom schemas MEI4 and MEI5 in Table 2), which equate terms with distinct exit degrees. For example, instances of these axiom schemas relate double-exit iteration with the binary Kleene star as follows:

$$\begin{aligned} (2) \quad &(x_0, x_1)^*(y, x_2((x_0(x_1 + x_2))^*y)) = (x_0(x_1 + x_2))^*y \\ (3) \quad &(x, x)^*(y, y) = x^*y . \end{aligned}$$

The second of these laws equates two terms that can repeatedly perform process  $x$ , until they exit the  $x$ -loop by executing process  $y$ . Equation 2 is more subtle, and the reader

may find the process graphs depicted in Figure 1 useful in arguing for its soundness. (The terms related by Eqn. 2 are boldface in the picture.) The interested reader is referred to Sect. 2.4 for more details on the soundness argument, and a discussion of the laws.

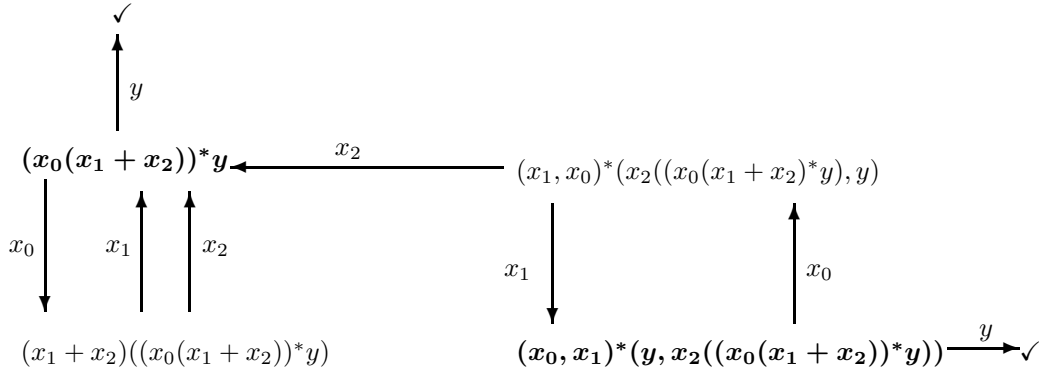


Figure 1: The process graphs for the terms in Eqn. 2

Our proof of the completeness of the proposed axiom system with respect to bisimulation equivalence over BPA with multi-exit iteration uses an adaptation of a proof technique developed by the second author in [22]. The actual details of the completeness proof are, however, more delicate and intricate than those of the proof given *ibidem*. In particular, as the reader will realize, extra care need be taken in dealing with equalities whose equational deduction makes an essential use of the new laws MEI4–5.

As remarked above, Bergstra, Bethke and Ponse [8, 9] have shown that the process described by the recursion equation (1) cannot be specified in the language BPA\* modulo bisimulation equivalence. In this paper, we generalize this result by showing that, in the presence of a non-empty set of actions, the family of  $k$ -exit iteration operations from [8] induces a strict expressiveness hierarchy of super-languages of BPA. More precisely, we prove that, for every positive integer  $k$ , the agent that corresponds to the solution of the following system of recursion equations (with  $X_1$  as the leading variable)

$$\begin{aligned}
 X_1 &\stackrel{\text{def}}{=} aX_2 + a \\
 X_2 &\stackrel{\text{def}}{=} aX_3 + aa \\
 &\vdots \\
 X_{k+1} &\stackrel{\text{def}}{=} aX_1 + \underbrace{a \dots a}_{(k+1) \text{ times}}
 \end{aligned}$$

can only be specified, modulo bisimulation equivalence, with the use of multi-exit iterations of the form  $(P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$  with  $n$  greater than  $k$ .

We conclude this introduction with a brief overview of the contents of this paper. We begin by introducing the language of Basic Process Algebra with multi-exit iteration

and its operational semantics (Sects. 2.1–2.3). The equational axiomatization of bisimulation equivalence over Basic Process Algebra with multi-exit iteration is presented in Sect. 2.4. The whole of Sect. 3 is devoted to a detailed proof of the completeness of the proposed axiomatization. The proof we present consists of three steps. First we isolate a collection of basic process terms, which cover, up to bisimulation equivalence, the whole language of process terms, and whose structure will simplify the proof of the promised completeness theorem (Sect. 3.1). We then proceed to define a well-founded ordering over basic terms which will allow for an inductive proof of our main result, and study some of its properties (Sect. 3.2). Finally, we shall show that two bisimilar basic terms can be proven equal using the equations in the proposed axiom system (Sect. 3.4). The paper concludes with an expressiveness hierarchy for the family of  $k$ -exit iteration operations (Sect. 4).

## 2 BPA with Multi-Exit Iteration

We begin by presenting the language of Basic Process Algebra with multi-exit iteration and its semantics.

### 2.1 The Syntax

We assume a non-empty alphabet  $A$  of atomic actions, with typical elements  $a, b$ , and a countably infinite set  $\mathbf{Var}$  of process variables, disjoint from  $A$ , with typical elements  $x, y, z$ . We shall use  $\alpha, \beta$  to range over  $A \cup \mathbf{Var}$ .

The language  $\mathbb{T}(\text{BPA}^{me^*}(A))$  of terms over Basic Process Algebra (BPA) with multi-exit iteration is defined inductively as follows:

- each  $\alpha \in A \cup \mathbf{Var}$  is a term;
- if  $P$  and  $Q$  are terms, then  $P + Q$  is a term;
- if  $P$  and  $Q$  are terms, then  $P \cdot Q$  is a term;
- if  $P_1, \dots, P_m$  and  $Q_1, \dots, Q_n$  ( $m, n \geq 1$ ) are terms, then  $(P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$  is a term.

We shall use  $P, Q, \dots, Y$  to range over  $\mathbb{T}(\text{BPA}^{me^*}(A))$ . The set of closed terms, i.e., terms that do not contain occurrences of process variables, is denoted by  $\mathbf{T}(\text{BPA}^{me^*}(A))$ . In writing terms over the above syntax, we shall always assume that the operation  $\cdot$  binds stronger than  $+$ . In the sequel the operation  $\cdot$  will often be omitted, so  $PQ$  denotes  $P \cdot Q$ . We shall use the symbol  $\equiv$  to stand for syntactic equality of terms. The size of a process term is the number of operations occurring in it. A (closed) substitution is a mapping from process variables to (closed) terms in the language  $\mathbb{T}(\text{BPA}^{me^*}(A))$ . For every term  $P$  and (closed) substitution  $\sigma$ , the (closed) term obtained by replacing every occurrence of a variable  $x$  in  $P$  with the (closed) term  $\sigma(x)$  will be written  $P\sigma$ .

Intuitively, closed terms stand for agents whose behaviour is completely specified, whereas terms containing occurrences of process variables denote agents with partially specified behaviour. For example, an atomic action  $a$  stands for a process that can only

perform itself in one computational step and terminate in doing so; on the other hand, the term  $a + x$  denotes a partially specified process, whose behaviour depends in part on that of the process term that is substituted for the variable  $x$ .

Apart from actions and variables, the signature of the language  $\mathbb{T}(\text{BPA}^{me*}(A))$  includes the binary operations of alternative composition  $+$  and sequential composition  $\cdot$  familiar from the theory of Basic Process Algebra [10, 7], and a variation on the original binary version of the Kleene star operation [31], that will be referred to as multi-exit iteration. For positive integers  $m$  and  $n$ , the process term  $(P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$  stands for an agent whose behaviour is specified by the following defining equation:

$$(P_1, \dots, P_m)^*(Q_1, \dots, Q_n) = P_1 \cdot (P_2, \dots, P_m, P_1)^*(Q_2, \dots, Q_n, Q_1) + Q_1 \cdot$$

Multi-exit iteration is a mild syntactic generalization of the family of  $k$ -exit iteration operations introduced in [8]—the only difference being that in *op. cit.* the number of the  $P_i$  must always be equal to that of the  $Q_j$ . As we shall see, multi-exit iteration and the family of  $k$ -exit iteration operations have the same expressive power modulo bisimulation equivalence. (Cf. Sect. 4 for some remarks on the expressive power of the  $k$ -exit iteration operations.)

In order to simplify notation in the presentation of the operational semantics and of the axiomatization for  $\mathbb{T}(\text{BPA}^{me*}(A))$ , we shall use the notion of ‘vectors of processes’. A vector of processes is a tuple  $(P_1, \dots, P_m)$ , where  $m \geq 0$ . We shall use  $\vec{P}, \vec{Q}, \vec{R}, \vec{S}$  to denote such vectors of processes. In multi-exit iteration, the expressions at the left- and right-hand side of the star are non-empty vectors of processes. The length of a vector  $\vec{P}$  will be written  $\text{length}(\vec{P})$ .

In the sequel,  $(Q, \vec{P})$  represents the vector that is obtained by concatenating the process term  $Q$  in front of vector  $\vec{P}$ , and  $(\vec{P}, Q)$  represents the vector that is obtained by appending the process term  $Q$  at the rear of vector  $\vec{P}$ . Furthermore, we assume the following features for vectors:

- Multiplication with a process term:  $(P_1, \dots, P_m) \cdot Q$  equals  $(P_1Q, \dots, P_mQ)$ .
- Power of a vector: for a positive integer  $n$ ,  $(P_1, \dots, P_m)^n$  equals

$$\underbrace{(P_1, \dots, P_m, \dots, P_1, \dots, P_m)}_{n \text{ times}}$$

Enclosing parentheses will always be omitted from vectors of length one, i.e.,  $(P)$  will be written  $P$ .

## 2.2 Operational Semantics

The operational semantics for the language  $\mathbb{T}(\text{BPA}^{me*}(A))$  is given by the labelled transition system [30, 41]

$$\left( \mathbb{T}(\text{BPA}^{me*}(A)), \left\{ \xrightarrow{\alpha} \mid \alpha \in A \cup \text{Var} \right\}, \left\{ \xrightarrow{\alpha} \checkmark \mid \alpha \in A \cup \text{Var} \right\} \right)$$

where the transition relations  $\xrightarrow{\alpha}$  and the unary predicates  $\xrightarrow{\alpha} \checkmark$  are, respectively, the least subsets of  $\mathbb{T}(\text{BPA}^{me*}(A)) \times \mathbb{T}(\text{BPA}^{me*}(A))$  and  $\mathbb{T}(\text{BPA}^{me*}(A))$  satisfying the rules

in Table 1. Intuitively, a transition  $P \xrightarrow{\alpha} Q$  means that the system represented by the term  $P$  can perform the action  $\alpha$ , thereby evolving into  $Q$ , whereas  $P \xrightarrow{x} P'$  means that the initial behaviour of  $P$  may depend on the term that is substituted for the process variable  $x$ . The special symbol  $\checkmark$  stands for (successful) termination; therefore the interpretation of the statement  $P \xrightarrow{\alpha} \checkmark$  is that the process term  $P$  can terminate by performing  $\alpha$ , if  $\alpha$  is an atomic action, or by executing to completion the term that is substituted for the process variable  $x$ , if  $\alpha = x$ .

$$\begin{array}{c}
\frac{}{\alpha \xrightarrow{\alpha} \checkmark} \\
\\
\frac{P \xrightarrow{\alpha} \checkmark}{P + Q \xrightarrow{\alpha} \checkmark} \quad \frac{Q \xrightarrow{\alpha} \checkmark}{P + Q \xrightarrow{\alpha} \checkmark} \quad \frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'} \quad \frac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'} \\
\\
\frac{P \xrightarrow{\alpha} \checkmark}{P \cdot Q \xrightarrow{\alpha} Q} \quad \frac{P \xrightarrow{\alpha} P'}{P \cdot Q \xrightarrow{\alpha} P' \cdot Q} \\
\\
\frac{P \xrightarrow{\alpha} \checkmark}{(P, \vec{Q})^*(R, \vec{S}) \xrightarrow{\alpha} (\vec{Q}, P)^*(\vec{S}, R)} \quad \frac{P \xrightarrow{\alpha} P'}{(P, \vec{Q})^*(R, \vec{S}) \xrightarrow{\alpha} P' \cdot (\vec{Q}, P)^*(\vec{S}, R)} \\
\\
\frac{R \xrightarrow{\alpha} \checkmark}{(P, \vec{Q})^*(R, \vec{S}) \xrightarrow{\alpha} \checkmark} \quad \frac{R \xrightarrow{\alpha} R'}{(P, \vec{Q})^*(R, \vec{S}) \xrightarrow{\alpha} R'}
\end{array}$$

Table 1: Transition Rules

**Remark:** The flow of a control of a process described using multi-exit iteration bears strong similarity to that of a flowchart schema [47, 34] in normal form [16, 18]. For the sake of completeness, we recall that a flowchart schema is said to be in normal form if it is a finite tree-like schema whose branches may only bend back to ancestor nodes.

**Definition 2.1** *The term  $P'$  is a derivative of  $P$  if  $P$  can evolve into  $P'$  by zero or more transitions. A derivative  $P'$  of  $P$  is proper if  $P$  can evolve into  $P'$  by performing at least one transition.*

The following basic fact can be easily shown by structural induction on terms:

**Fact 2.2** *For every  $P \in \mathbb{T}(\text{BPA}^{me*}(A))$ , the set of derivatives of  $P$  is finite.*

Process terms are considered modulo bisimulation equivalence from Park [39]. Intuitively, two process terms are bisimilar if they have the same branching structure.

**Definition 2.3** *Two process terms  $P$  and  $Q$  are bisimilar, denoted by  $P \leftrightarrow Q$ , if there exists a symmetric binary relation  $\mathcal{B}$  on process terms which relates  $P$  and  $Q$ , such that:*

- if  $R \mathcal{B} S$  and  $R \xrightarrow{\alpha} R'$ , then there is a transition  $S \xrightarrow{\alpha} S'$  such that  $R' \mathcal{B} S'$ ,
- if  $R \mathcal{B} S$  and  $R \xrightarrow{\alpha} \checkmark$ , then  $S \xrightarrow{\alpha} \checkmark$ .

Such a relation  $\mathcal{B}$  will be called a bisimulation (witnessing the equivalence  $P \Leftrightarrow Q$ ). The relation  $\Leftrightarrow$  will be referred to as bisimulation equivalence.

Note that if  $P$  is bisimilar to  $Q$ , then every (proper) derivative of  $P$  is bisimilar to some (proper) derivative of  $Q$ , and vice versa.

The transition rules in Table 1 are in the ‘path’ format of Baeten and Verhoef [6]. Hence, bisimulation equivalence is a congruence with respect to all the operations in the signature of  $\mathbb{T}(\text{BPA}^{me*}(A))$ . The interested reader is invited to consult [6, 19] for the definition of the path format for operational rules, and for a proof of the aforementioned congruence result.

The reader might have noticed that we have defined notions of operational semantics and bisimulation equivalence that apply to open terms directly, following [38, 26] for process algebra with abstraction, and [1] for process algebra with the prefix iteration operation from [20], which is a restricted version of the Kleene star. This approach deviates from the standard practice in process theory, which prescribes to define operational semantics and bisimulation equivalence for closed terms only, and to give meaning to open terms thus:

$$P \Leftrightarrow Q \triangleq P\sigma \Leftrightarrow Q\sigma \text{ for all closed substitutions } \sigma : \text{Var} \rightarrow \mathbb{T}(\text{BPA}^{me*}(A)).$$

(Note that this amounts to stipulating that two open terms are equivalent exactly when the equation  $P = Q$  holds in the algebra of closed terms modulo bisimulation equivalence.) The following result shows that both approaches yield the same notion of bisimulation equivalence over  $\mathbb{T}(\text{BPA}^{me*}(A))$ , that is, in our setting, two open terms are bisimilar if and only if all their closed instantiations are.

**Lemma 2.4** *For every  $P, Q \in \mathbb{T}(\text{BPA}^{me*}(A))$ ,*

$$P \Leftrightarrow Q \Leftrightarrow P\sigma \Leftrightarrow Q\sigma \text{ for all closed substitutions } \sigma : \text{Var} \rightarrow \mathbb{T}(\text{BPA}^{me*}(A)).$$

**Proof:** The result can be proven following the strategy that was employed in [1] for prefix iteration. We remark, however, that the technical details are considerably more complicated in the case of multi-exit iteration. In particular, in the proof of the ‘if implication’, the choice of a canonical closed substitution  $\sigma_{(P,Q)}$  with the property that

$$P\sigma_{(P,Q)} \Leftrightarrow Q\sigma_{(P,Q)} \Rightarrow P \Leftrightarrow Q$$

requires more care than in the aforementioned reference. The interested reader is invited to consult [44, Theorems 3 and 4], where this type of result is proven in the more general setting of a simply typed lambda calculus, which captures multi-exit iteration.  $\square$

Lemma 2.4 implies that any complete axiomatization of bisimulation equivalence over the language  $\mathbb{T}(\text{BPA}^{me*}(A))$  is also a complete axiomatization of the algebra of closed terms modulo bisimulation. Such an axiomatization will, *a fortiori*, be  $\omega$ -complete in the sense of, e.g., [28].

### 2.3 Norm and L-value of Process Terms

Process terms in  $\mathbb{T}(\text{BPA}^{me*}(A))$  are *normed*, which means that they are able to terminate by embarking in a finite sequence of transitions. We call such a sequence a

termination trace. The *norm* of a process term  $P$ , denoted by  $|P|$ , is the length of its shortest termination trace; this notion stems from [4]. Note that bisimilar process terms have the same norm. The following lemma, which is due to Caucal [14], is typical for normed processes. The interested reader is referred to [22] for its proof.

**Lemma 2.5** *Let  $P, Q, R, S \in \mathbb{T}(\text{BPA}^{me^*}(A))$  be such that  $PQ \underline{\leftrightarrow} RS$ . Then the following statements hold:*

- If  $|Q| = |S|$ , then  $P \underline{\leftrightarrow} R$  and  $Q \underline{\leftrightarrow} S$ .
- If  $|Q| < |S|$ , then there is a proper derivative  $P'$  of  $P$  such that  $P \underline{\leftrightarrow} RP'$  and  $P'Q \underline{\leftrightarrow} S$ .

The notion of norm may be used as a measure of the complexity of terms which is useful in inductive proofs. (Cf., e.g., the proof of the above lemma in [22].) However, this measure of term complexity does not lend itself to use in completeness proofs like the one presented in *op. cit.* because it does not respect term size. For example, the term  $aa + a$  has smaller norm than its sub-term  $aa$ . For this reason, Fokkink and Zantema [23] introduced the notion of *L-value*, which does not have this drawback.

**Definition 2.6** *The L-value of a process term  $P$ , notation  $L(P)$ , is defined thus:*

$$L(P) \triangleq \sup\{|P'| \mid P' \text{ a proper derivative of } P\} .$$

Note that, as the set of derivatives of  $P$  is finite for every term  $P \in \mathbb{T}(\text{BPA}^{me^*}(A))$  (Fact 2.2), the *L-value* of a process term is a well-defined natural number. For example,  $L(\alpha) = \sup \emptyset = 0$ .

The following basic properties of the notion of *L-value* will be useful in the technical developments to follow.

**Lemma 2.7**

1. If  $P \underline{\leftrightarrow} Q$ , then  $L(P) = L(Q)$ ;
2.  $L(P) < L(PQ)$ ;
3.  $L(P_i) < L((P_1, \dots, P_m)^*(Q_1, \dots, Q_n))$  for  $i = 1, \dots, m$ .

**Remark:** Note that, in general, the *L-values* of  $Q$  and  $Q_i$  are *not* smaller than those of  $PQ$  and  $(P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$ , respectively. For instance, if  $P \equiv a$  and  $Q \equiv a + aa$ , then  $L(PQ) = L(Q) = 1$ . The reader will find the construction of a similar example for multi-exit iteration an easy exercise.

## 2.4 The Axiom System

Table 2 contains our axiom system  $\mathcal{E}$  for  $\mathbb{T}(\text{BPA}^{me^*}(A))$  modulo bisimulation equivalence. It consists of the standard axioms A1–5 from Basic Process Algebra (cf. [10, 7]) together with five axiom schemas MEI1–5 for multi-exit iteration. In these axiom schemas,

A1	$x + y = y + x$
A2	$(x + y) + z = x + (y + z)$
A3	$x + x = x$
A4	$(x + y)z = xz + yz$
A5	$(xy)z = x(yz)$
MEI1	$x(\vec{v}, x)^*(\vec{w}, y) + y = (x, \vec{v})^*(y, \vec{w})$
MEI2	$(x, \vec{v})^*(y, \vec{w})z = (x, \vec{v})^*(yz, \vec{w}z)$
MEI3	$(x_1, \vec{v})^*(y + x_2(\vec{v}, x_1 + x_2)^*(\vec{w}, y), \vec{w}) = (x_1 + x_2, \vec{v})^*(y, \vec{w})$ $(\text{length}(\vec{v}) = \text{length}(\vec{w}))$
MEI4	$(x_0, x_1, \vec{v})^*(y, x_2(\vec{v}, x_0(x_1 + x_2))^*(\vec{w}, y), \vec{w}) = (x_0(x_1 + x_2), \vec{v})^*(y, \vec{w})$ $(\text{length}(\vec{v}) = \text{length}(\vec{w}))$
MEI5	$((x, \vec{v})^m)^*((y, \vec{w})^n) = (x, \vec{v})^*(y, \vec{w})$

Table 2: The Axiom System  $\mathcal{E}$

$\vec{v}$  and  $\vec{w}$  denote meta-variables which range over vectors of processes, and  $n, m$  range over the set of positive integers.

**Remark:** For the sake of clarity, we remark that a meta-variable  $\vec{v}$  is just syntactic sugar for an arbitrary vector of process variables. Therefore each of the axiom schemas MEI1–5 stands for an infinite family of equations, viz. one for each instantiation of the meta-variables ranging over vectors of process variables and of those ranging over positive integers.

Laws MEI1–3 are modifications of the standard axioms for the binary Kleene star which, together with A1–5, have been shown to be complete for bisimulation equivalence over the language  $\text{BPA}^*$  in [23, 22]. Axiom MEI1 is the defining equation for multi-exit iteration, axiom MEI2 is the adaptation of the associativity law A5 to multi-exit iteration, and axiom MEI3 is the multi-exit version of a law which originates from Troeger’s work [45].

The axiom schemas MEI3 and MEI4, when viewed as rewrite rules from right to left, may be both seen as putting the list of recursion equations associated with their right-hand sides into ‘standard form’ in the sense of Milner [38]. Consider, for example, the instance of MEI4 obtained by taking  $\vec{v} = (v_1, \dots, v_n)$  and  $\vec{w} = (w_1, \dots, w_n)$  for some positive integer  $n$ . The list of recursion equations associated with the right-hand side of the resulting instance of MEI4 is

$$\begin{aligned} X_0 &\stackrel{\text{def}}{=} (x_0(x_1 + x_2))X_1 + y \\ X_i &\stackrel{\text{def}}{=} v_i X_{(i+1) \bmod (n+1)} + w_i \quad (1 \leq i \leq n) . \end{aligned}$$

An equivalent way of expressing the above list of recursion equations is to bring it into a form that exactly encodes the process graph for the term  $(x_0(x_1 + x_2), \vec{v})^*(y, \vec{w})$ , i.e., in

what Milner calls standard form in [38]. This can be done by introducing the auxiliary recursion variable  $X'_0$ , with defining equation

$$X'_0 \stackrel{\text{def}}{=} x_1X_1 + x_2X_1,$$

and modifying the defining recursion equation for  $X_0$  thus:

$$X_0 \stackrel{\text{def}}{=} x_0X'_0 + y .$$

The resulting list of recursion equations is, modulo the identification of the recursion variables  $Y_2$  and  $X_1$ , the one associated with the left-hand side of this instance of MEI4, which is

$$\begin{aligned} Y_0 &\stackrel{\text{def}}{=} x_0Y_1 + y \\ Y_1 &\stackrel{\text{def}}{=} x_1Y_2 + x_2X_1 \\ Y_2 &\stackrel{\text{def}}{=} v_1Y_3 + w_1 \\ &\vdots \\ Y_{n+1} &\stackrel{\text{def}}{=} v_nY_0 + w_n . \end{aligned}$$

Indeed, the use we shall make of the axiom schemas MEI3–4 in the proof of the completeness theorem may be seen as mimicking two of the key steps in Milner's completeness proof for the regular fragment of CCS [38], viz. the association of standard equation lists with regular processes, and the proof that two expressions associated with the same equation list are provably equal.

Finally, we remark that axiom MEI5 enables to reduce repetitive patterns in the left- and right-hand side of multi-exit terms. Axioms MEI4 and MEI5, which typically relate terms of distinct exit degrees, are, to the best of our knowledge, new.

**Notation 2.8** For a term  $X \equiv (P_1, \dots, P_n)^*(Q_1, \dots, Q_m)$ , the expression  $\text{shift}(X)$  stands for the term  $(P_2, \dots, P_n, P_1)^*(Q_2, \dots, Q_m, Q_1)$ . For a non-negative integer  $k$ ,  $\text{shift}^k(X)$  denotes the result of applying the function  $\text{shift}$  to the term  $X$   $k$  times. Note that  $X \Leftrightarrow P_1\text{shift}(X) + Q_1$ .

**Proposition 2.9 (Soundness)** If  $\mathcal{E} \vdash P = Q$ , then  $P \Leftrightarrow Q$ .

**Proof:** Since bisimulation equivalence is a congruence, the claim can be proven by checking soundness for each axiom separately. Below, we shall confine ourselves to presenting bisimulation relations that show the soundness of axiom schemas MEI3–5.

- AXIOM SCHEMA MEI3.

Let  $n \geq 0$ , and consider the vectors of variables  $\vec{v} = (v_1, \dots, v_n)$  and  $\vec{w} = (w_1, \dots, w_n)$ . For every  $i \in \{1, \dots, n+1\}$ , define the process terms  $P_i$  and  $Q_i$  as follows:

$$\begin{aligned} P_i &\triangleq (v_i, \dots, v_n, x_1 + x_2, v_1, \dots, v_{i-1})^*(w_i, \dots, w_n, y, w_1, \dots, w_{i-1}) \\ Q_i &\triangleq (v_i, \dots, v_n, x_1, v_1, \dots, v_{i-1})^*(w_i, \dots, w_n, y + x_2P_1, w_1, \dots, w_{i-1}) . \end{aligned}$$

Note that the equation  $Q_{n+1} = P_{n+1}$  is an instance of the axiom schema MEI3. To establish the soundness of this equation, check that the symmetric closure of the relation

$$\mathcal{B}_3 \triangleq \{(Q_i, P_i) \mid 1 \leq i \leq n+1\} \cup \{(P_i, P_i) \mid 1 \leq i \leq n+1\}$$

is a bisimulation.

- AXIOM SCHEMA MEI4.

Let  $n \geq 0$ , and consider the vectors of variables  $\vec{v} = (v_1, \dots, v_n)$  and  $\vec{w} = (w_1, \dots, w_n)$ . For every  $i \in \{1, \dots, n+1\}$ , define the process terms  $P_i$  and  $Q_i$  as follows:

$$\begin{aligned} P_i &\triangleq (v_i, \dots, v_n, x_0(x_1 + x_2), v_1, \dots, v_{i-1})^*(w_i, \dots, w_n, y, w_1, \dots, w_{i-1}) \\ Q_i &\triangleq (v_i, \dots, v_n, x_0, x_1, v_1, \dots, v_{i-1})^*(w_i, \dots, w_n, y, x_2 P_1, w_1, \dots, w_{i-1}) . \end{aligned}$$

Note that the equation  $Q_{n+1} = P_{n+1}$  is an instance of the axiom schema MEI4. To establish the soundness of this equation, check that the symmetric closure of the relation

$$\begin{aligned} \mathcal{B}_4 &\triangleq \{(Q_i, P_i) \mid 1 \leq i \leq n+1\} \cup \{(P_i, P_i) \mid 1 \leq i \leq n+1\} \cup \\ &\quad \{((x_1, \vec{v}, x_0)^*(x_2 P_1, \vec{w}, y), (x_1 + x_2) P_1)\} \end{aligned}$$

is a bisimulation.

- AXIOM SCHEMA MEI5.

Let  $i, j \geq 0$ , and consider the vectors of variables  $\vec{v} = (v_0, \dots, v_i)$  and  $\vec{w} = (w_0, \dots, w_j)$ . Let  $n$  and  $m$  be two positive integers. Consider the terms

$$\begin{aligned} X &\equiv (v_0, \dots, v_i)^*(w_0, \dots, w_j) \\ Y &\equiv ((v_0, \dots, v_i)^m)^*((w_0, \dots, w_j)^n) . \end{aligned}$$

To establish the soundness of the instance  $Y = X$  of the axiom schema MEI5, it is sufficient to check that the symmetric closure of the relation

$$\mathcal{B}_5 \triangleq \{(\text{shift}^k(X), \text{shift}^k(Y)) \mid k \geq 0\}$$

is a bisimulation. To this end, note that, for every  $k \geq 0$ , the only transitions afforded by the term  $\text{shift}^k(X)$  are

- $\text{shift}^k(X) \xrightarrow{v_h} \text{shift}^{k+1}(X)$ , where  $h \triangleq k \bmod (i+1)$ , and
- $\text{shift}^k(X) \xrightarrow{w_l} \checkmark$ , where  $l \triangleq k \bmod (j+1)$ .

Similarly, the only transitions afforded by the term  $\text{shift}^k(Y)$  are

- $\text{shift}^k(Y) \xrightarrow{v_p} \text{shift}^{k+1}(Y)$ , where  $p \triangleq (k \bmod m(i+1)) \bmod (i+1)$ , and
- $\text{shift}^k(Y) \xrightarrow{w_q} \checkmark$ , where  $q \triangleq (k \bmod n(j+1)) \bmod (j+1)$ .

A simple calculation in modulo arithmetic now yields that  $h = p$  and  $l = q$ . It follows immediately that the relation  $\mathcal{B}_5$  is indeed a bisimulation, which was to be shown. □

**Remark:** The restriction on the length of the vectors of variables  $\vec{v}$  and  $\vec{w}$  is necessary for the soundness of the axiom schemas MEI3 and MEI4. For example, if  $\vec{v} \triangleq v_1$  and  $\vec{w}$  is the empty vector, then the process term  $(x_1, v_1)^*(y + x_2(v_1, x_1 + x_2)^*y)$  can perform the sequence  $x_1 x_2$ , but  $(x_1 + x_2, v_1)^*y$  cannot. It follows that a general version of MEI3 does not hold modulo trace equivalence [29], and, *a fortiori*, modulo bisimulation equivalence.

The reader will find the construction of a similar example disproving the soundness of a general version of the axiom schema MEI4 an easy exercise.

**Remark:** The fact that multi-exit iteration and the family of  $k$ -exit iteration operations ( $k \geq 1$ ), as defined in [8], have the same expressive power modulo bisimulation equivalence is witnessed

by the soundness of axiom MEI5. This axiom may be used to turn every instance of a multi-exit iteration into an equivalent one that uses  $k$ -exit iteration for an appropriate  $k$ . In fact, an application of axiom MEI5 yields the equality

$$(P_1, \dots, P_m)^*(Q_1, \dots, Q_n) = ((P_1, \dots, P_m)^n)^*((Q_1, \dots, Q_n)^m)$$

whose left- and right-hand sides are terms with  $(mn)$ -exit iteration, as defined in [8].

In the proof of the completeness of the axiom system  $\mathcal{E}$  for bisimulation equivalence over the language  $\mathbb{T}(\text{BPA}^{me^*}(A))$ , we shall find it useful to have more general variants of the axioms MEI3,4.

Let us assume that  $\vec{v}_i$  and  $\vec{w}_i$  ( $i \in \{0, 1\}$ ) are vectors of process variables such that  $\vec{v}_i$  and  $\vec{w}_i$  ( $i \in \{0, 1\}$ ) have the *same* length. The following equation, which we shall refer to as MEI3', can be derived easily from MEI3 using MEI1:

$$(\vec{v}_0, x_1, \vec{v}_1)^*(\vec{w}_0, y + x_2(\vec{v}_1, \vec{v}_0, x_1 + x_2)^*(\vec{w}_1, \vec{w}_0, y), \vec{w}_1) = (\vec{v}_0, x_1 + x_2, \vec{v}_1)^*(\vec{w}_0, y, \vec{w}_1) .$$

Similarly, the following equation, which we shall refer to as MEI4', can be derived easily from MEI4 using MEI1:

$$(\vec{v}_0, x_0, x_1, \vec{v}_1)^*(\vec{w}_0, y, x_2(\vec{v}_1, \vec{v}_0, x_0(x_1 + x_2))^*(\vec{w}_1, \vec{w}_0, y), \vec{w}_1) = (\vec{v}_0, x_0(x_1 + x_2), \vec{v}_1)^*(\vec{w}_0, y, \vec{w}_1) .$$

Note that the soundness of these generalized versions of equations MEI3–4 depends crucially on the assumption that  $\vec{v}_0$  and  $\vec{w}_0$  are vectors of process variables of the same length, and their use in the proof of the completeness theorem will be restricted to situations in which this requirement is met.

Henceforth process terms will be considered modulo associativity and commutativity of the  $+$ -operation, i.e., modulo axioms A1–2.

**Notation 2.10** *For an axiom system  $\mathcal{T}$ , we write  $\mathcal{T} \vdash P = Q$  iff the equation  $P = Q$  is provable from the axioms in  $\mathcal{T}$  using the rules of equational logic. For a collection of equations  $X$  over the signature of  $\mathbb{T}(\text{BPA}^{me^*}(A))$ , we write  $P \stackrel{X}{=} Q$  as a short-hand for  $A1, A2, X \vdash P = Q$ .*

*For  $I = \{i_1, \dots, i_n\}$  a finite, non-empty index set, we write  $\sum_{i \in I} P_i$  for  $P_{i_1} + \dots + P_{i_n}$ .*

*For notational convenience, and in order to reduce the number of cases in the completeness proof, in what follows we shall identify a term  $P$  with the meta-notations  $\sum_{i \in \emptyset} Q_i + P$  and  $(\sum_{i \in \emptyset} Q_i)Q + P$ .*

The collection of possible transitions of each process term  $P$  is non-empty and finite, say  $\{P \xrightarrow{\alpha_i} P_i \mid i = 1, \dots, m\} \cup \{P \xrightarrow{\beta_j} \checkmark \mid j = 1, \dots, n\}$ . We call the term

$$\sum_{i=1}^m \alpha_i P_i + \sum_{j=1}^n \beta_j$$

the *expansion* of  $P$ . The terms  $\alpha_i P_i$  and  $\beta_j$  will be referred to as the *summands* of  $P$ .

**Lemma 2.11** *Each process term is provably equal to its expansion.*

**Proof:** Straightforward, by structural induction on terms, using axioms A4, A5 and MEI1.  $\square$

### 3 The Completeness Proof

The remainder of the paper is devoted to a proof of completeness of the axiom system  $\mathcal{E}$  with respect to bisimulation equivalence over the language  $\mathbb{T}(\text{BPA}^{me^*}(A))$ . The proof of this result will be given by adapting a proof technique developed by the second author in [22]. A comparison between this proof technique and the one originally used in [23] to solve the completeness problem for bisimulation equivalence over Basic Process Algebra with binary Kleene star may be found in [22].

The proof we present consists of three steps. First we isolate a collection of basic process terms, which cover, up to bisimulation equivalence, the whole language of process terms, and whose structure will simplify the proof of the promised completeness theorem. We then proceed to define a well-founded ordering over basic terms which will allow for an inductive proof of our main result, and study some of its properties. Finally, we shall show that two bisimilar basic terms can be proven equal using the equations in  $\mathcal{E}$ . This proof strategy will be familiar to readers acquainted with the literature on completeness results from process theory. The actual details of the proof that we now proceed to present are, however, rather challenging and novel.

#### 3.1 Basic Terms

As a first stepping stone towards our main result, we now aim at constructing a collection of *basic* process terms, with the property that each process term is provably equal to a basic one. We shall then prove the completeness theorem by showing that bisimilar basic terms are provably equal.

Let  $\mathbb{G}$  denote the collection of process terms in  $\mathbb{T}(\text{BPA}^{me^*}(A))$  which only use action prefixing, in the sense of Milner [37], in lieu of general sequential composition, and in which vectors at the left- and right-hand side of multi-exit iteration have the same length. That is,  $\mathbb{G}$  is defined inductively as follows:

- each  $\alpha \in A \cup \text{Var}$  is in  $\mathbb{G}$ ;
- if  $P$  and  $Q$  are in  $\mathbb{G}$ , then  $P + Q$  is in  $\mathbb{G}$ ;
- if  $\alpha \in A \cup \text{Var}$  and  $P$  is in  $\mathbb{G}$ , then  $\alpha P$  is in  $\mathbb{G}$ ;
- if  $P_1, \dots, P_n$  and  $Q_1, \dots, Q_n$  ( $n \geq 1$ ) are in  $\mathbb{G}$ , then  $(P_1, \dots, P_n)^*(Q_1, \dots, Q_n)$  is in  $\mathbb{G}$ .

**Lemma 3.1** *Every process term can be proven equal to a term in  $\mathbb{G}$  using axioms  $A4, 5$  and  $MEI2, 5$ .*

The import of the above lemma is that, without loss of generality, we may restrict ourselves to considering terms in  $\mathbb{G}$ . However,  $\mathbb{G}$  is not yet our desired set of basic terms. In fact, one of the most fundamental properties that we require of a collection of basic terms for use in the proof of the completeness theorem is that it be closed under transitions. This is a property that  $\mathbb{G}$  does *not* enjoy. For example, the term  $(a^*a)^*a$  is clearly in  $\mathbb{G}$ , and

$$(a^*a)^*a \xrightarrow{a} (a^*a)((a^*a)^*a) .$$

However, the right-hand side of the above transition is a term that is not contained in  $\mathbb{G}$ . In order to overcome this complication, we introduce the following collection  $\mathbb{H}$  of process terms:

- if a term  $(P_1, \dots, P_n)^*(Q_1, \dots, Q_n)$  is in  $\mathbb{G}$ , then it is also in  $\mathbb{H}$ ;
- if a term  $(P_1, \dots, P_n)^*(Q_1, \dots, Q_n)$  is in  $\mathbb{G}$ , and if  $P'$  is a proper derivative of  $P_n$ , then  $P'(P_1, \dots, P_n)^*(Q_1, \dots, Q_n)$  is in  $\mathbb{H}$ .

For example, the term  $(a^*a)((a^*a)^*a)$  is included in  $\mathbb{H}$  because  $a^*a$  is a proper derivative of itself.

The set  $\mathbb{B}$  of *basic terms* is the union of  $\mathbb{G}$  and  $\mathbb{H}$ .

**Lemma 3.2** *If  $P \in \mathbb{B}$  and  $P \xrightarrow{\alpha} P'$ , then  $P' \in \mathbb{B}$ .*

**Proof:** Let  $P \xrightarrow{\alpha} P'$ . If  $P \in \mathbb{H} \setminus \mathbb{G}$ , then it is easy to see that  $P' \in \mathbb{H}$ . If  $P \in \mathbb{G}$ , then a straightforward structural induction yields that  $P' \in \mathbb{B}$ .  $\square$

For later use, we now define an equivalence relation  $\cong$  on  $\mathbb{H}$  as the least one satisfying axiom MEI5 and the equivalences

- $(P_1, \dots, P_n)^*(Q_1, \dots, Q_n) \cong (P_2, \dots, P_n, P_1)^*(Q_2, \dots, Q_n, Q_1)$ ;
- $(P_n, P_1, \dots, P_{n-1})^*(Q_n, Q_1, \dots, Q_{n-1}) \cong P'(P_1, \dots, P_n)^*(Q_1, \dots, Q_n)$  if  $P'$  is a proper derivative of  $P_n$ .

The key properties of the relation  $\cong$  needed in the technical developments to follow will be studied in the following section.

### 3.2 An Ordering on Terms

Having identified a set of basic terms closed under transitions, we now proceed to define a well-founded ordering on this set that will allow us to give an inductive proof of the main result of this paper.

**Definition 3.3** *We say that a term  $P$  is an exit derivative of a term  $Q$  iff  $P$  is a derivative of  $Q$ , but not vice versa.*

Let  $\prec$  denote the least transitive relation over the set of process terms satisfying:

- $P \prec Q$  if  $L(P) < L(Q)$ ,
- $P \prec Q$  if  $P$  is an exit derivative of  $Q$ .

Intuitively, if  $P \prec Q$ , then either the set of derivatives of  $P$  is properly included in that of  $Q$ , or the  $L$ -value of  $P$  is strictly smaller than that of  $Q$ .

**Lemma 3.4**  *$\prec$  is a well-founded ordering on  $\mathbb{T}(\text{BPA}^{me^*}(A))$ .*

**Proof:** First of all, observe that if  $P \prec Q$  then  $L(P) \leq L(Q)$ . This follows because if  $P$  is a derivative of  $Q$ , then  $L(P) \leq L(Q)$  since all proper derivatives of  $P$  are also proper derivatives of  $Q$ .

Suppose now, towards a contradiction, that  $\prec$  is not well-founded. This means that there exists an infinite descending chain  $P_0 \succ P_1 \succ P_2 \succ \dots$ . As  $L(P_n) \geq L(P_{n+1})$  for all  $n$ , there is an  $N$  such that  $L(P_N) = L(P_n)$  for all  $n > N$ . Since  $P_m \succ P_n$  for  $m, n > N$  with  $m < n$ , it follows that  $P_n$  is an exit derivative of  $P_m$  for each such  $m, n$ . By Fact 2.2, each process term has only finitely many derivatives, so there are  $m, n > N$  with  $m < n$  and  $P_m \equiv P_n$ . Then  $P_m \not\succ P_n$ , and we have found a contradiction. Hence,  $\prec$  is well-founded.  $\square$

Following [22], we now proceed to study the interaction between the operational semantics of process terms and the above-defined ordering. A technical tool we shall use below is a weight function  $g$  that associates a natural number to each process term. This is defined thus:

$$\begin{aligned} g(\alpha) &\triangleq 0 \\ g(P + Q) &\triangleq \max\{g(P), g(Q)\} + 1 \\ g(PQ) &\triangleq \max\{g(P), g(Q)\} \\ g((P_1, \dots, P_m)^*(Q_1, \dots, Q_n)) &\triangleq \max\{g(P_1), \dots, g(P_m), g(Q_1) + 1, \dots, g(Q_n) + 1\} . \end{aligned}$$

The basic property of this weight function that we shall need is expressed in the lemma below, which follows by a straightforward structural induction.

**Lemma 3.5** *If  $P'$  is a derivative of  $P$ , then  $g(P') \leq g(P)$ . Moreover, if*

- $P \equiv P_1 + P_2$  for some terms  $P_1$  and  $P_2$ , and  $P'$  is a proper derivative of  $P$ , or
- $P \equiv (P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$ , for some terms  $P_i$  ( $1 \leq i \leq m$ ) and  $Q_j$  ( $1 \leq j \leq n$ ), and  $P'$  is a proper derivative of some  $Q_j$ ,

then  $g(P') < g(P)$ .

The following two lemmas will play a major rôle in the proof of the main result of this paper. The first states that, intuitively, escaping from a loop reduces the complexity of a process term, as measured by the above defined ordering. This is due to the fact that once a process term has exited a loop, it can never return to it.

**Lemma 3.6** *If  $Q'$  is a proper derivative of  $Q_i$  for some  $i = 1, \dots, n$ , then*

$$Q' \prec (P_1, \dots, P_m)^*(Q_1, \dots, Q_n) .$$

**Proof:** If  $Q'$  is a proper derivative of some  $Q_i$ , then Lem. 3.5 gives that

$$g(Q') < g((P_1, \dots, P_m)^*(Q_1, \dots, Q_n)) .$$

Again using Lem. 3.5, we infer from this inequality that  $(P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$  is not a derivative of  $Q'$ . Since  $Q'$  is a derivative of  $(P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$ , it follows that  $Q' \prec (P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$ .  $\square$

As witnessed by the proof of the above lemma, the exit derivatives of a term of the form  $(P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$  are exactly the proper derivatives of the terms  $Q_i$ . As an immediate consequence of this observation, we note that:

**Fact 3.7** *If  $P, Q \in \mathbb{H}$  and  $P \cong Q$ , then  $P$  and  $Q$  have the same  $L$ -value and exit derivatives.*

The observations collected in the above result imply that  $\cong$ -equivalent basic terms dominate the same process terms with respect to the ordering  $\prec$ , i.e., that if  $P \prec R \cong S$ , then  $P \prec S$ .

The following lemma is part of the crux of the proof of the completeness theorem; intuitively, it states that the only transitions between basic terms that do not decrease the complexity of terms, as measured by  $\prec$ , are those belonging to some loop.

**Lemma 3.8** *If  $P \in \mathbb{B}$  and  $P \xrightarrow{\alpha} P'$ , then either  $P' \prec P$ , or  $P, P' \in \mathbb{H}$  and  $P \cong P'$ .*

**Proof:** We begin by establishing two facts that we shall use in the proof of the statement of the lemma.

A. If  $P \in \mathbb{B}$ ,  $P' \notin \mathbb{H}$  and  $P \xrightarrow{\alpha} P'$ , then  $P'$  has smaller size than  $P$ .

*Proof.* First of all, note that the claim is vacuously true if  $P \in \mathbb{H} \setminus \mathbb{G}$ , because, in that case, it follows that  $P' \in \mathbb{H}$ . That the claim holds for  $P \in \mathbb{G}$  can be shown by a simple structural induction on  $P$ .

B. If  $P \in \mathbb{H}$  and  $P \xrightarrow{\alpha} P'$ , then either  $g(P) > g(P')$ , or  $P' \in \mathbb{H}$  and  $P \cong P'$ .

*Proof.* Since  $P \in \mathbb{H}$ , it follows that, for some terms  $P_i$  and  $Q_i$  ( $1 \leq i \leq n$ ),

- either  $P \equiv (P_1, \dots, P_n)^*(Q_1, \dots, Q_n)$ ,
- or  $P \equiv P'_n(P_1, \dots, P_n)^*(Q_1, \dots, Q_n)$ , for some proper derivative  $P'_n$  of  $P_n$ .

Hence,  $P'$  can have one of the following three forms:

1.  $P' \equiv (P_2, \dots, P_n, P_1)^*(Q_2, \dots, Q_n, Q_1)$ ,
2.  $P' \equiv P''_n(P_1, \dots, P_n)^*(Q_1, \dots, Q_n)$ , for some proper derivative  $P''_n$  of  $P_n$ , or
3.  $P' \equiv Q'_1$ , for some proper derivative of  $Q_1$ .

In the first two cases  $P' \in \mathbb{H}$  and  $P \cong P'$ , and in the last case  $g(P') = g(Q'_1) < g(P)$  (Lem. 3.5).

We are now in a position to prove the lemma. Assume that, for some basic term  $P$ ,  $P \xrightarrow{\alpha} P'$  and  $P' \not\prec P$ . We prove that  $P, P' \in \mathbb{H}$  and  $P \cong P'$ .

To this end, note, first of all, that, since  $P'$  is a proper derivative of  $P$  and  $P' \not\prec P$ , it must be the case that  $P$  is a proper derivative of  $P'$ . So there exists a sequence of transitions

$$P_0 \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} P_n, \quad n \geq 2,$$

where  $P_0 \equiv P \equiv P_n$  and  $P_1 \equiv P'$ . Note that each term  $P_k$ ,  $0 \leq k \leq n$ , is basic as  $P$  is (Lem. 3.2).

We claim that  $P_l \in \mathbb{H}$  for some  $0 \leq l \leq n$ . In fact, assume, towards a contradiction, that  $P_k \notin \mathbb{H}$  for all  $k$ . Then fact A implies that  $P_{k+1}$  has smaller size than  $P_k$  for  $k = 0, \dots, n-1$ . Therefore  $P$  has smaller size than itself, which is impossible.

Hence,  $P_l \in \mathbb{H}$  for some  $l$ . Since each  $P_k$  is a proper derivative of each  $P_{k'}$ , we have  $g(P_k) \leq g(P_{k'})$  for all  $k$  and  $k'$  (Lem. 3.5). Therefore  $g(P_k)$  must be the same for all  $k$ . Now fact B and  $P_l \in \mathbb{H}$  imply that  $P_k \in \mathbb{H}$  for all  $k$ , and  $P_0 \cong P_1 \cong \dots \cong P_n$ .  $\square$

Elements of  $\mathbb{B} \times \mathbb{B}$  are considered modulo commutativity. The well-founded ordering  $\prec$  on  $\mathbb{B}$  induces an ordering on  $\mathbb{B} \times \mathbb{B}$  as the least transitive relation satisfying:

$$(P, Q) \sqsubset (R, S) \quad \text{if} \quad P \prec R \text{ and } Q \cong S .$$

It is immediate to see that the above-defined ordering on  $\mathbb{B} \times \mathbb{B}$  is also well-founded.

### 3.3 A Lemma

In the proof of the completeness theorem to follow, we shall often need to analyze equivalences between terms of the form  $P'(P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$ . We now aim at establishing a lemma that will be a useful tool in the study of these equivalences. The following lemma will be applied in the proof of Lem. 3.10.

**Lemma 3.9** *Let  $X \equiv (P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$  and  $Y \equiv (R_1, \dots, R_n)^*(S_1, \dots, S_n)$ . Assume that  $TX \Leftrightarrow UY$  for some  $T$  and  $U$ , and that  $Q_1$  does not have a proper derivative that is bisimilar to  $Y$ . Then we have the following two possibilities:*

1. either  $T \Leftrightarrow U$ ;
2. or  $T \Leftrightarrow U(V + W)$  for some terms  $V$  and  $W$ , whose proper derivatives are contained in those of  $T$ , such that  $VX \Leftrightarrow R_1\text{shift}(Y)$  and  $WX \Leftrightarrow S_1$ .

**Proof:** Lem. 2.5 applied to the equivalence  $TX \Leftrightarrow UY$  yields three possibilities:

- A. either  $T \Leftrightarrow U$  and  $X \Leftrightarrow Y$ ;
- B. or  $T \Leftrightarrow UT'$  and  $T'X \Leftrightarrow Y$  for some proper derivative  $T'$  of  $T$ ;
- C. or  $TU' \Leftrightarrow Y$  and  $X \Leftrightarrow U'Y$  for some proper derivative  $U'$  of  $U$ .

Case A agrees with the first statement in the lemma.

If case B holds, then  $T'X \Leftrightarrow Y \Leftrightarrow R_1\text{shift}(Y) + S_1$ . Then clearly there exist terms  $V$  and  $W$ , whose proper derivatives are contained in those of  $T'$ , such that  $T' \Leftrightarrow V + W$ ,  $VX \Leftrightarrow R_1\text{shift}(Y)$ , and  $WX \Leftrightarrow S_1$ . This agrees with the second statement in the lemma.

Finally, case C contradicts one of the assumptions. In fact, as  $X \Leftrightarrow U'Y$  implies  $P_1\text{shift}(X) + Q_1 \Leftrightarrow U'Y$ , in that case  $Q_1$  has a proper derivative that is bisimilar to  $Y$ .  $\square$

The following technical lemma will be used repeatedly in the proof of the promised completeness theorem.

**Lemma 3.10** *Let  $X \equiv (P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$  and  $Y \equiv (R_1, \dots, R_n)^*(S_1, \dots, S_n)$ . Assume that  $T_1X \Leftrightarrow R_1\text{shift}(Y)$  for some term  $T_1$ , and that  $Q_1$  has no proper derivative that is bisimilar to  $Y$ . Then we have the following two possibilities:*

- I. either there are terms  $U_1, \dots, U_n$ , whose proper derivatives are contained in those of  $T_1$ , such that:

$$\begin{aligned} T_1 &\Leftrightarrow R_1(R_2, \dots, R_n, R_1)^*(U_2, \dots, U_n, U_1) \\ U_iX &\Leftrightarrow S_i \quad i = 1, \dots, n \end{aligned}$$

- II. or there is a  $k \in \{1, \dots, n\}$  and there are terms  $T_2, \dots, T_k$  and  $U_2, \dots, U_k$ , whose proper derivatives are contained in those of  $T_1$ , such that:

$$\begin{aligned} T_i &\Leftrightarrow R_i(T_{i+1} + U_{i+1}) \quad i = 1, \dots, k-1 \\ T_k &\Leftrightarrow R_k \\ T_iX &\Leftrightarrow R_i\text{shift}^i(Y) \quad i = 2, \dots, k \\ U_iX &\Leftrightarrow S_i \quad i = 2, \dots, k . \end{aligned}$$

**Proof:** Assume that the proviso of the lemma holds for some terms  $X, Y$  and  $T_1$  of the required form. Suppose that the following equivalence  $A_i$  holds for some  $i \leq n$  and for some term  $T_i$ :

$$A_i \quad T_i X \Leftrightarrow R_i \text{shift}^i(Y).$$

Recall that we assumed that it holds for  $i = 1$ .

As  $Q_1$  has no proper derivative that is bisimilar to  $Y$ , and  $Y$  is a proper derivative of  $\text{shift}^i(Y)$  for every non-negative integer  $i$ , it follows that  $Q_1$  has no proper derivative that is bisimilar to  $\text{shift}^i(Y)$ . Therefore we can apply Lem. 3.9 to equivalence  $A_i$  to obtain the following possibilities; either

$$B_i \quad T_i \Leftrightarrow R_i,$$

or, if  $i < n$ , then there exist terms  $T_{i+1}$  and  $U_{i+1}$ , whose proper derivatives are contained in those of  $T_i$ , such that

$$C_i \quad T_i \Leftrightarrow R_i(T_{i+1} + U_{i+1}), \quad T_{i+1}X \Leftrightarrow R_{i+1}\text{shift}^{i+1}(Y) \quad \text{and} \quad U_{i+1}X \Leftrightarrow S_{i+1},$$

or, if  $i = n$ , then there exist terms  $T_{n+1}$  and  $U_1$ , whose proper derivatives are contained in those of  $T_n$ , such that

$$C_n \quad T_n \Leftrightarrow R_n(T_{n+1} + U_1), \quad T_{n+1}X \Leftrightarrow R_1\text{shift}(Y) \quad \text{and} \quad U_1X \Leftrightarrow S_1.$$

Note that, by transitivity, the proper derivatives of the  $T_i$  and the  $U_i$  are all contained in those of  $T_1$ .

In light of the above discussion, Lem. 3.9 gives that condition  $A_i$  implies either  $B_i$  or  $C_i$  for  $i = 1, \dots, n$ . Furthermore,  $C_i$  clearly implies  $A_{i+1}$  for  $i = 1, \dots, n-1$ . So, since we assumed that  $A_1$  holds, we can distinguish the following two cases:

- I. either  $C_1, \dots, C_n$  hold;
- II. or there is a  $k \in \{1, \dots, n\}$  such that  $C_1, \dots, C_{k-1}$  and  $B_k$  hold.

We consider these two cases in turn.

- I. Suppose that  $C_1, \dots, C_n$  hold, so that:

$$\begin{array}{lcl} T_i & \Leftrightarrow & R_i(T_{i+1} + U_{i+1}) \quad i = 1, \dots, n-1 \\ T_n & \Leftrightarrow & R_n(T_{n+1} + U_1) \\ U_i X & \Leftrightarrow & S_i \quad i = 1, \dots, n \\ T_{n+1} X & \Leftrightarrow & R_1 \text{shift}(Y) . \end{array}$$

By assumption  $T_1 X \Leftrightarrow R_1 \text{shift}(Y)$ , so the last equivalence implies  $T_{n+1} X \Leftrightarrow T_1 X$ . Then Lem. 2.5 yields  $T_{n+1} \Leftrightarrow T_1$ . To complete the proof for this case, it is sufficient to show that the equivalences

$$\begin{array}{lcl} T_i & \Leftrightarrow & R_i(T_{i+1} + U_{i+1}) \quad i = 1, \dots, n-1 \\ T_n & \Leftrightarrow & R_n(T_1 + U_1) \end{array}$$

together yield the following equivalence:

$$T_1 \quad \Leftrightarrow \quad R_1(R_2, \dots, R_n, R_1)^*(U_2, \dots, U_n, U_1) .$$

The construction of the required bisimulation relation is not hard, and is left to the reader.

- II. Suppose that there is a  $k \in \{1, \dots, n\}$  such that  $C_1, \dots, C_{k-1}$  and  $B_k$  hold. Then

$$\begin{array}{lcl} T_i & \Leftrightarrow & R_i(T_{i+1} + U_{i+1}) \quad i = 1, \dots, k-1 \\ T_k & \Leftrightarrow & R_k \\ T_i X & \Leftrightarrow & R_i \text{shift}^i(Y) \quad i = 2, \dots, k \\ U_i X & \Leftrightarrow & S_i \quad i = 2, \dots, k \end{array}$$

and we are done.

The proof of the lemma is now complete.  $\square$

### 3.4 The Completeness Theorem

We are finally in a position to prove the desired completeness result. However, before presenting our completeness argument in full detail, we find it instructive to exemplify the use made of the axiom schemas MEI3–5 in the most difficult case of the proof. (Cf. case 2.1.2 in the proof of Thm. 3.11.)

**Example:** Consider the process terms

$$\begin{aligned} X &\equiv (a(b+c))^*d \\ Y &\equiv (a, b, a(b+c))^*(d, cX, d) . \end{aligned}$$

The terms  $X$  and  $Y$  are bisimilar, and we shall now show how to prove them equal using the laws in the axiom system  $\mathcal{E}$ . We begin by noting that, using MEI5,  $X$  may be proven equal to the term

$$X_0 \equiv (a(b+c), a(b+c))^*(d, d) .$$

Applying MEI4 and MEI5 in this order, we finally infer that

$$\begin{aligned} X_0 &= (a, b, a(b+c))^*(d, cX_0, d) \\ &= (a, b, a(b+c))^*(d, cX, d) \\ &\equiv Y . \end{aligned}$$

$\square$

**Example:** Consider the process terms

$$\begin{aligned} X &\equiv (a+b)^*c \\ Y &\equiv (a, b, a+b)^*(c+bX, c+aX, c) . \end{aligned}$$

The terms  $X$  and  $Y$  are bisimilar, and we shall now show how to prove them equal using the laws in the axiom system  $\mathcal{E}$ . We begin by noting that, using MEI5,  $X$  may be proven equal to the term

$$X_0 \equiv (a+b, a+b, a+b)^*(c, c, c) .$$

Applying MEI3, MEI3' and MEI5 in this order, we finally infer that

$$\begin{aligned} X_0 &= (a, a+b, a+b)^*(c+bX_0, c, c) \\ &= (a, b, a+b)^*(c+bX_0, c+aX_0, c) \\ &= (a, b, a+b)^*(c+bX, c+aX, c) \\ &\equiv Y . \end{aligned}$$

$\square$

**Theorem 3.11 (Completeness)** *Let  $X, Y \in \mathbb{T}(\text{BPA}^{me^*}(A))$ . If  $X \underline{\leftrightarrow} Y$ , then  $\mathcal{E} \vdash X = Y$ .*

**Proof:** First of all, note that, as each process term is provably equal to a basic term (Lem. 3.1), it is sufficient to show that bisimilar basic terms are provably equal. This we proceed to do by induction on the well-founded ordering  $\sqsubset$  on  $\mathbb{B} \times \mathbb{B}$ . To this end, assume that  $X, Y$  are basic terms such that  $X \underline{\leftrightarrow} Y$ . Suppose furthermore, as inductive hypothesis that if two basic terms  $X', Y'$  with  $(X', Y') \sqsubset (X, Y)$  are bisimilar, then  $\mathcal{E} \vdash X' = Y'$ . We proceed to show that  $\mathcal{E} \vdash X = Y$ . For notational convenience, we shall write  $X = Y$  in lieu of  $\mathcal{E} \vdash X = Y$ . Throughout the proof, we shall use the fact that the collection of basic terms is closed under transitions (Lem. 3.2) without further mention. We prove the claim by considering the following two cases:

CASE 1.  $X \notin \mathbb{H}$  or  $Y \notin \mathbb{H}$ ;

CASE 2. Both  $X$  and  $Y$  are in  $\mathbb{H}$ .

We examine these two cases in turn.

- CASE 1. Assume that  $X \notin \mathbb{H}$  or  $Y \notin \mathbb{H}$ . By symmetry, we may suppose that  $X \notin \mathbb{H}$ . Since  $X \underline{\leftrightarrow} Y$ , by possibly using axiom A3 we can adapt the expansions of  $X$  and  $Y$  to the following forms:

$$X = \sum_{i=1}^m \alpha_i X_i + \sum_{j=1}^n \beta_j \quad Y = \sum_{i=1}^m \alpha_i Y_i + \sum_{j=1}^n \beta_j$$

where  $X_i \underline{\leftrightarrow} Y_i$  for  $i = 1, \dots, m$ . As  $X \notin \mathbb{H}$ , Lem. 3.8 gives that  $X_i \prec X$  for  $i = 1, \dots, m$ . Furthermore, again using Lem. 3.8, we infer that, for  $i = 1, \dots, m$ , either  $Y_i \prec Y$  or  $Y_i \cong Y$ . Therefore, for every index  $i$ ,  $(X_i, Y_i) \sqsubset (X, Y)$ , and we may apply the inductive hypothesis to the equivalence  $X_i \underline{\leftrightarrow} Y_i$  to derive that  $X_i = Y_i$ . Hence, by substitutivity it follows that  $X = Y$ , and we are done.

- CASE 2. Assume that  $X, Y \in \mathbb{H}$ . We proceed with the proof by considering the possible forms these bisimilar terms may take. By symmetry, it is sufficient to distinguish the following three cases:

CASE 2.1.  $X \equiv (P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$  and  $Y \equiv (R_1, \dots, R_n)^*(S_1, \dots, S_n)$ ;

CASE 2.2.  $X \equiv P'(P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$  for some proper derivative  $P'$  of  $P_m$ , and  $Y \equiv (R_1, \dots, R_n)^*(S_1, \dots, S_n)$ ;

CASE 2.3.  $X \equiv P'(P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$  for some proper derivative  $P'$  of  $P_m$ , and  $Y \equiv R'(R_1, \dots, R_n)^*(S_1, \dots, S_n)$  for some proper derivative  $R'$  of  $R_n$ .

We proceed by considering each of these cases in turn.

- \* CASE 2.1. Let  $X \equiv (P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$  and  $Y \equiv (R_1, \dots, R_n)^*(S_1, \dots, S_n)$ . By symmetry, it is sufficient to consider the following two sub-cases:

CASE 2.1.1. There exist a proper derivative  $Q'$  of some  $Q_i$  with  $Q' \underline{\leftrightarrow} Y$ , and a proper derivative  $S'$  of some  $S_j$  with  $S' \underline{\leftrightarrow} X$ ;

CASE 2.1.2. No proper derivative of any  $Q_i$  is bisimilar to  $Y$ .

We consider these two sub-cases in turn.

- o CASE 2.1.1. Assume that there exist a proper derivative  $Q'$  of some  $Q_i$  with  $Q' \underline{\leftrightarrow} Y$ , and a proper derivative  $S'$  of some  $S_j$  with  $S' \underline{\leftrightarrow} X$ .

Note that, by transitivity, we may infer that  $Q' \leftrightarrow S'$ . As  $Q' \prec X$  and  $S' \prec Y$  (Lem. 3.6), we may apply the inductive hypothesis to each of the aforementioned equivalences to derive that

$$X = S' = Q' = Y$$

and we are done.

- o CASE 2.1.2. Assume that no proper derivative of any  $Q_i$  is bisimilar to  $Y$ . We begin the proof for this case by adapting  $X$  and  $Y$ .

**Definition 3.12** For  $Z \equiv (T_1, \dots, T_j)^*(U_1, \dots, U_j)$ , let  $K(Z)$  denote the number of  $i$ 's in  $\{1, \dots, j\}$  for which  $\text{shift}^i(Z) \leftrightarrow Z$ . Note that  $K(Z) \geq 1$ , as  $\text{shift}^j(Z) \equiv Z$ .

Put

$$\begin{aligned} X_0 &\equiv ((P_1, \dots, P_m)^{K(Y)})^*((Q_1, \dots, Q_m)^{K(Y)}) \\ Y_0 &\equiv ((R_1, \dots, R_n)^{K(X)})^*((S_1, \dots, S_n)^{K(X)}) . \end{aligned}$$

It is not hard to see that, for every  $i \in \{1, \dots, m\}$  and  $k \in \{0, \dots, K(Y) - 1\}$ ,

$$\text{shift}^i(X) \leftrightarrow \text{shift}^{i+km}(X_0) .$$

Similarly, for every  $j \in \{1, \dots, n\}$  and  $k \in \{0, \dots, K(X) - 1\}$ ,

$$\text{shift}^j(Y) \leftrightarrow \text{shift}^{j+kn}(Y_0) .$$

Since  $X_0 \leftrightarrow X$  and  $Y_0 \leftrightarrow Y$ , it follows that both  $K(X_0)$  and  $K(Y_0)$  are equal to  $K(X) \cdot K(Y)$ . Owing to axiom MEI5, the equalities  $X = X_0$  and  $Y = Y_0$  are provable. The rest of the proof for this case will be devoted to proving  $X_0 = Y_0$ . For notational convenience, put  $X_0 \equiv (P_1, \dots, P_M)^*(Q_1, \dots, Q_M)$  and  $Y_0 \equiv (R_1, \dots, R_N)^*(S_1, \dots, S_N)$ .

We shall now prove that there exists an increasing sequence of integers  $0 = c_0 < \dots < c_M = N$  such that for  $l = 0, \dots, M$ :

$$A_l \text{ shift}^l(X_0) \leftrightarrow \text{shift}^{c_l}(Y_0);$$

$$B_l \text{ shift}^i(Y_0) \not\leftrightarrow Y_0 \text{ for } c_{l-1} < i < c_l \text{ (where, by convention, } c_{-1} \triangleq 0);$$

$$C_l \text{ } X_0 = (R_1, \dots, R_{c_l}, P_{l+1}, \dots, P_M)^*(S_1, \dots, S_{c_l}, Q_{l+1}, \dots, Q_M).$$

In fact, we shall show that the conjunction of the statements  $A_l$ ,  $B_l$  and  $C_l$  implies the conjunction of  $A_{l+1}$ ,  $B_{l+1}$  and  $C_{l+1}$  for  $l < M$ . Since  $A_0$  (viz.  $X_0 \leftrightarrow Y_0$ ),  $B_0$  (viz. a vacuous statement) and  $C_0$  (viz.  $X_0 = X_0$ ) hold, we can then conclude that  $C_M$  (viz.  $X_0 = Y_0$ ) holds, which is what we want to prove. Throughout the proof, we shall make use of the fact that, as  $X \cong X_0$  and  $Y \cong Y_0$ , these terms dominate the same basic terms with respect to the well-founded ordering  $\prec$ . This remark will allow us to apply the inductive hypothesis to all pairs  $(X', Y')$  of bisimilar terms such that  $(X', Y') \sqsubset (X_0, Y_0)$ , and will be used without further mention.

Assume now that, for some  $l$  with  $0 \leq l < M$ , the statements  $A_i$ ,  $B_i$  and  $C_i$  hold for all  $i \leq l$  and some increasing sequence of integers  $0 = c_0 < \dots < c_l < N$ . We set out to deduce the statements  $A_{l+1}$ ,  $B_{l+1}$  and  $C_{l+1}$  for some  $c_{l+1}$  with  $c_l < c_{l+1} \leq N$ , where  $c_{l+1}$  equals  $N$  if and only if  $l + 1 = M$ .

First, we spell out the expansions of  $P_{l+1}$  and  $R_{c_{l+1}}$ :

$$P_{l+1} = \sum_{i \in I} T_i \quad R_{c_{l+1}} = \sum_{j \in J} U_j$$

where the summands  $T_i$  and  $U_j$  are of the form either  $\alpha V$  or  $\alpha$ .

As  $A_l$  holds,  $P_{l+1}\text{shift}^{l+1}(X_0) + Q_{l+1} \Leftrightarrow R_{c_l+1}\text{shift}^{c_l+1}(Y_0) + S_{c_l+1}$ . Hence, for every  $i \in I$ , either there exists an index  $j \in J$  such that

$$(4) \quad T_i \text{shift}^{l+1}(X_0) \Leftrightarrow U_j \text{shift}^{c_l+1}(Y_0)$$

or there exists a summand  $\alpha S'$  of  $S_{c_l+1}$  such that

$$(5) \quad T_i \text{shift}^{l+1}(X_0) \Leftrightarrow \alpha S' .$$

Thus,  $I$  can be divided into the following, not necessarily disjoint, subsets:

$$\begin{aligned} I_0 &\triangleq \{i \in I \mid \exists \alpha, S' : S_{c_l+1} \xrightarrow{\alpha} S' \text{ and (5) holds}\} \\ I_1 &\triangleq \{i \in I \mid \exists j \in J \text{ such that (4) holds}\} . \end{aligned}$$

On the other hand, for every  $j \in J$  there exists an  $i \in I$  such that equivalence (4) holds. In fact, suppose, towards a contradiction, that for some  $j \in J$  this does not hold. Since  $P_{l+1}\text{shift}^{l+1}(X_0) + Q_{l+1} \Leftrightarrow R_{c_l+1}\text{shift}^{c_l+1}(Y_0) + S_{c_l+1}$ , it follows that for this  $j$  there exists a summand  $\alpha Q'$  of  $Q_{l+1}$  such that

$$\alpha Q' \Leftrightarrow U_j \text{shift}^{c_l+1}(Y_0) .$$

Since  $Y_0$  is a derivative of  $\text{shift}^{c_l+1}(Y_0)$ , then there is a proper derivative of  $Q_{l+1}$  that is bisimilar to  $Y_0$ , and therefore to  $Y$ . This contradicts the assumption for case 2.1.2.

Note that, by our previous reasoning, the index set  $I_1$  is non-empty; let  $V_1$  denote the term  $\sum_{i \in I_1} T_i$ . The following equation follows by possibly using axiom A3:

$$(6) \quad P_{l+1} = \sum_{i \in I_0} T_i + V_1 .$$

Moreover, the following equivalence is an immediate consequence of the fact that for every  $j \in J$  there exists an  $i \in I$  such that equivalence (4) holds:

$$(7) \quad V_1 \text{shift}^{l+1}(X_0) \Leftrightarrow R_{c_l+1} \text{shift}^{c_l+1}(Y_0) .$$

We proceed with the proof by deriving the following equality:

$$(8) \quad \sum_{i \in I_0} T_i \text{shift}^{l+1}(X_0) + Q_{l+1} = S_{c_l+1} .$$

*Proof of Equation (8).* We show that each summand of  $S_{c_l+1}$  is provably equal to a summand of the term at the left-hand side of the equation, and vice versa.

Let  $\alpha S'$  be a summand of  $S_{c_l+1}$ ; then the transition  $\text{shift}^{c_l}(Y_0) \xrightarrow{\alpha} S'$  holds. As

$$P_{l+1} \text{shift}^{l+1}(X_0) + Q_{l+1} \Leftrightarrow \text{shift}^{c_l}(Y_0)$$

by assumption  $A_l$ , either there exists an  $i \in I_0$  such that (5) holds, or there exists a summand  $\alpha Q'$  of  $Q_{l+1}$  such that

$$(9) \quad Q' \Leftrightarrow S' .$$

If there exists an  $i \in I_0$  such that (5) holds, then

1. either  $T_i \equiv \alpha P'$  for some proper derivative  $P'$  of  $P_{l+1}$  such that  $P' \text{shift}^{l+1}(X_0) \Leftrightarrow S'$ ,

2. or  $T_i \equiv \alpha$  and  $\text{shift}^{l+1}(X_0) \Leftrightarrow S'$ .

As  $S' \prec Y_0$  (Lem. 3.6),  $P'\text{shift}^{l+1}(X_0) \cong X_0$  and  $\text{shift}^{l+1}(X_0) \cong X_0$ , in each of these two cases we may apply the inductive hypothesis and substitutivity to infer that

$$T_i \text{shift}^{l+1}(X_0) = \alpha S' .$$

If (9) holds, then  $Q' \prec X_0$  and  $S' \prec Y_0$  (Lem. 3.6), so that we may apply the inductive hypothesis and substitutivity to infer

$$\alpha Q' = \alpha S' .$$

If  $\alpha$  is a summand of  $S_{c_{l+1}}$ , then  $\alpha$  must also be a summand of  $Q_{l+1}$ , and therefore of the term on the left-hand side of the equation. Hence, every summand of  $S_{c_{l+1}}$  is provably equal to a summand of the term at the left-hand side of (8).

By the symmetric argument it can be shown that every summand of the term at the left-hand side of (8) is provably equal to a summand of  $S_{c_{l+1}}$ . Namely, by definition of  $I_0$  each term  $T_i \text{shift}^{l+1}(X_0)$  for  $i \in I_0$  is bisimilar to a summand of  $S_{c_{l+1}}$ . Moreover, as  $P_{l+1} \text{shift}^{l+1}(X_0) + Q_{l+1} \Leftrightarrow R_{c_{l+1}} \text{shift}^{c_{l+1}}(Y_0) + S_{c_{l+1}}$ ,  $Y_0$  is a derivative of  $\text{shift}(Y_0)$ , and no derivative of  $Q_{l+1}$  is bisimilar to  $Y_0$ , every summand of  $Q_{l+1}$  must be bisimilar to a summand of  $S_{c_{l+1}}$ .

*End of Proof of Equation (8).*

We now proceed with our argument by analyzing equivalence (7) using Lem. 3.10. According to this lemma two possibilities may arise, which we consider in turn.

- CASE 2.1.2.1. If the first case of Lem. 3.10 holds with respect to equivalence (7), then, in particular, there exists a term  $W$  such that

$$W \text{shift}^{l+1}(X_0) \Leftrightarrow S_{c_{l+1}} .$$

Equality (8) implies the equivalence

$$W \text{shift}^{l+1}(X_0) \Leftrightarrow \sum_{i \in I_0} T_i \text{shift}(X_0) + Q_{l+1} .$$

Since  $X_0$  is a derivative of  $\text{shift}^{l+1}(X_0)$  and  $Y_0 \Leftrightarrow X_0$ , it follows that there is a proper derivative of  $Q_{l+1}$  which is bisimilar to  $Y_0$ , and therefore to  $Y$ . This contradicts the assumption at the start of case 2.1.2.

- CASE 2.1.2.2. In light of the previous discussion, we may assume that the second case of Lem. 3.10 holds with respect to equivalence (7). In this case, note, first of all, that, for every  $j, h \geq 0$ ,

$$\text{shift}^j(Y_0) \equiv \text{shift}^{j+hn}(Y_0) .$$

Therefore, by the second case of Lem. 3.10, there is an integer  $k \in \{1, \dots, n\}$  and there exist terms  $V_2, \dots, V_k$  and  $W_2, \dots, W_k$ , whose proper derivatives are contained in those of  $V_1$ , such that:

$$(10) \quad V_i \Leftrightarrow R_{c_l+i}(V_{i+1} + W_{i+1}) \quad i = 1, \dots, k-1$$

$$(11) \quad V_k \Leftrightarrow R_{c_l+k}$$

$$(12) \quad V_i \text{shift}^{l+1}(X_0) \Leftrightarrow R_{c_l+i} \text{shift}^{c_l+i}(Y_0) \quad i = 2, \dots, k$$

$$(13) \quad W_i \text{shift}^{l+1}(X_0) \Leftrightarrow S_{c_l+i} \quad i = 2, \dots, k .$$

Put  $c_{l+1} \triangleq c_l + k$ . We now show that  $A_{l+1}$ ,  $B_{l+1}$  and  $C_{l+1}$  hold, and that  $c_{l+1} \leq N$ , with  $c_{l+1} = N$  iff  $l + 1 = M$ .

Equivalence (11) and the instance of (12) for  $i = k$ , together with Lem. 2.5, yield the equivalence

$$\text{shift}^{l+1}(X_0) \Leftrightarrow \text{shift}^{c_{l+1}}(Y_0)$$

which corresponds to  $A_{l+1}$ .

We now show that  $B_{l+1}$  holds. Suppose, towards a contradiction, that  $\text{shift}^{c_l+i}(Y_0) \Leftrightarrow Y_0$  for some  $i \in \{1, \dots, k-1\}$ . Equivalences (12) and (13), together with A4 and MEI1, imply that  $\text{shift}^{c_l+i}(Y_0) \Leftrightarrow (V_{i+1} + W_{i+1})\text{shift}^{l+1}(X_0)$ . Therefore, as  $X_0 \Leftrightarrow Y_0$ ,

$$P_1\text{shift}(X_0) + Q_1 \Leftrightarrow Y_0 \Leftrightarrow \text{shift}^{c_l+i}(Y_0) \Leftrightarrow (V_{i+1} + W_{i+1})\text{shift}^{l+1}(X_0) .$$

As  $X_0$  is a derivative of  $\text{shift}^{l+1}(X_0)$ , and  $Y_0 \Leftrightarrow X_0$ , it follows that  $Q_1$  has a derivative bisimilar to  $Y_0$ , and therefore to  $Y$ . This contradicts the assumption for case 2.1.2. So it must be the case that  $\text{shift}^{c_l+i}(Y_0) \not\Leftrightarrow Y_0$  for  $i = 1, \dots, k-1$ , which corresponds to  $B_{l+1}$ .

As  $c_l < N$  and  $B_{l+1}$  holds, it follows that  $c_l + i \neq N$  for  $i = 1, \dots, k-1$ . Hence  $c_{l+1} \leq N$ . We now show that  $c_{l+1} = N$  iff  $l + 1 = M$ . By definition of  $K(Z)$  (Def. 3.12) we have:

- $c_{l+1} = N$  iff there are  $K(Y_0)$  indices  $i \in \{1, \dots, c_{l+1}\}$  with  $\text{shift}^i(Y_0) \Leftrightarrow Y_0$ ;
- $l + 1 = M$  iff there are  $K(X_0)$  indices  $j \in \{1, \dots, l + 1\}$  with  $\text{shift}^j(X_0) \Leftrightarrow X_0$ .

Recall that  $X_0$  and  $Y_0$  were designed such that  $K(X_0) = K(Y_0)$ . Furthermore,  $A_i$  and  $B_i$  for  $i \leq l + 1$  imply that  $\text{shift}^i(Y_0) \Leftrightarrow Y_0$  for  $1 \leq i \leq c_{l+1}$  iff  $i = c_j$  for some  $j \in \{1, \dots, l + 1\}$  with  $\text{shift}^j(X_0) \Leftrightarrow X_0$ . Hence, it follows that  $c_{l+1} = N$  iff  $l + 1 = M$ .

It remains to prove that  $C_{l+1}$  holds. Equality (6) implies that  $L(V_1) \leq L(P_{l+1})$ , so, by Lem. 2.7,  $L(V_1) < L(X_0)$ . Since the proper derivatives of the terms  $V_i$  are contained in those of  $V_1$ , it follows that  $L(V_i) < L(X_0)$  for  $i = 2, \dots, k$ . Now, invariance of  $L$ -value under bisimulation (Lem. 2.7), together with the equivalences  $X_0 \Leftrightarrow Y_0$ , (10) and (11), yields

1.  $L(R_{c_l+i}(V_{i+1} + W_{i+1})) < L(Y_0)$  for  $i = 1, \dots, k-1$  and
2.  $L(R_{c_{l+1}}) < L(Y_0)$ .

It follows that

- $V_i \prec X_0$  for  $i = 1, \dots, k$ ,
- $R_i(V_{i+1} + W_{i+1}) \prec Y_0$  for  $i = 1, \dots, k-1$  and
- $R_{c_{l+1}} \prec Y_0$ .

Thus we can apply induction to equivalences (10) and (11) to derive

$$(14) \quad V_i = R_{c_l+i}(V_{i+1} + W_{i+1}) \quad i = 1, \dots, k-1$$

$$(15) \quad V_k = R_{c_{l+1}} .$$

Furthermore, since  $S' \prec Y_0$  for every proper derivative  $S'$  of each term  $S_{c_l+i}$  (Lem. 3.6), and since each derivative in the expansion of  $W_i\text{shift}^{l+1}(X_0)$  is  $\cong$ -equivalent to  $X_0$ , we can apply the same reasoning used in the proof of (8) to equivalence (13) to obtain

$$(16) \quad W_i\text{shift}^{l+1}(X_0) = S_{c_l+i} \quad i = 2, \dots, k .$$

The equations that we derived up to now can be used to prove

$$(17) \quad X_0 = (R_1, \dots, R_{c_l+i-1}, V_i, P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_l+i-1}, S_{c_l+i}, Q_{l+2}, \dots, Q_M)$$

for  $i = 1, \dots, k - 1$ , and

$$(18) \quad X_0 = (R_1, \dots, R_{c_{l+1}}, P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_{l+1}}, Q_{l+2}, \dots, Q_M) .$$

In order to derive these equations, we apply induction on  $i$ . First, we deal with the case  $i = 1$ :

$$\begin{aligned} X_0 &\stackrel{C_l}{=} (R_1, \dots, R_{c_l}, P_{l+1}, \dots, P_M)^*(S_1, \dots, S_{c_l}, Q_{l+1}, \dots, Q_M) \\ &\stackrel{(6)}{=} (R_1, \dots, R_{c_l}, \sum_{i \in I_0} T_i + V_1, P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_l}, Q_{l+1}, Q_{l+2}, \dots, Q_M) \\ &\stackrel{\text{MEI3}'}{=} (R_1, \dots, R_{c_l}, V_1, P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_l}, \sum_{i \in I_0} T_i Z + Q_{l+1}, Q_{l+2}, \dots, Q_M) \end{aligned}$$

where

$$Z \stackrel{(6)}{=} (P_{l+2}, \dots, P_M, R_1, \dots, R_{c_l}, P_1)^*(Q_{l+2}, \dots, Q_M, S_1, \dots, S_{c_l}, Q_{l+1}) .$$

Applying MEI1 to the right-hand side of the above equation for  $M - l - 1$  times, followed by equation  $C_l$ , we obtain

$$Z = P_{l+2}(\dots(P_M X_0 + Q_M)\dots) + Q_{l+2} .$$

Again applying MEI1 to the above equation, but this time from right to left, we deduce the equality  $Z = \text{shift}^{l+1}(X_0)$ . Therefore:

$$\begin{aligned} X_0 &= (R_1, \dots, R_{c_l}, V_1, P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_l}, \sum_{i \in I_0} T_i \text{shift}^{l+1}(X_0) + Q_{l+1}, Q_{l+2}, \dots, Q_M) \\ &\stackrel{(8)}{=} (R_1, \dots, R_{c_l}, V_1, P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_l}, S_{c_{l+1}}, Q_{l+2}, \dots, Q_M) . \end{aligned}$$

If  $k > 1$ , then we have proven equation (17) for  $i = 1$ . If  $k = 1$ , then  $c_{l+1} = c_l + 1$ , and by equation (15)  $V_1 = R_{c_{l+1}}$ , which proves (18).

Next, suppose that we have proven equation (17) for  $i = j$ , where  $1 \leq j < k$ . We derive either equation (17) for  $i = j + 1$ , if  $k > j + 1$ , or equation (18), if  $k = j + 1$ . To this end, we reason as follows:

$$\begin{aligned} X_0 &\stackrel{(17),(14)}{=} (R_1, \dots, R_{c_{l+j-1}}, R_{c_{l+j}}(V_{j+1} + W_{j+1}), P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_{l+j}}, Q_{l+2}, \dots, Q_M) \\ &\stackrel{\text{MEI4}'}{=} (R_1, \dots, R_{c_{l+j}}, V_{j+1}, P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_{l+j}}, W_{j+1}Z, Q_{l+2}, \dots, Q_M) \end{aligned}$$

where

$$Z \stackrel{(14)}{=} (P_{l+2}, \dots, P_M, R_1, \dots, R_{c_{l+j-1}}, V_j)^*(Q_{l+2}, \dots, Q_M, S_1, \dots, S_{c_{l+j-1}}, S_{c_{l+j}}) .$$

Applying MEI1 for  $M - l - 1$  times, followed by equation (17), we obtain

$$Z = P_{l+2}(\dots(P_M X_0 + Q_M)\dots) + Q_{l+2} .$$

Again using MEI1, we may now deduce the equality  $Z = \text{shift}^{l+1}(X_0)$ . Therefore:

$$\begin{aligned} X_0 &= (R_1, \dots, R_{c_{l+j}}, V_{j+1}, P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_{l+j}}, W_{j+1} \text{shift}^{l+1}(X_0), Q_{l+2}, \dots, Q_M) \\ &\stackrel{(16)}{=} (R_1, \dots, R_{c_{l+j}}, V_{j+1}, P_{l+2}, \dots, P_M)^*(S_1, \dots, S_{c_{l+j}}, S_{c_{l+j+1}}, Q_{l+2}, \dots, Q_M) . \end{aligned}$$

If  $k > j + 1$ , then we have proven equation (17) for  $i = j + 1$ . If  $k = j + 1$ , then  $c_{l+1} = c_l + j + 1$ , and by equation (15)  $V_{j+1} = R_{c_{l+1}}$ , which proves equation (18).

Equation (18) corresponds to  $C_{l+1}$ . Thus, we have completed the proof that the conjunction of  $A_l$ ,  $B_l$  and  $C_l$  implies the conjunction of  $A_{l+1}$ ,  $B_{l+1}$  and  $C_{l+1}$  for  $0 \leq l < M$ . Since  $A_0$ ,  $B_0$  and  $C_0$  hold, we can then conclude  $C_M$ , that is,  $X_0 = Y_0$ . By axiom MEI5, it follows that  $X = X_0$  and  $Y = Y_0$ . Therefore  $X = Y$ , which finishes the proof of this case.

\* CASE 2.2. Assume that  $X \equiv P'(P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$  for some proper derivative  $P'$  of  $P_m$ , and  $Y \equiv (R_1, \dots, R_n)^*(S_1, \dots, S_n)$ . For notational convenience we put  $X' \equiv (P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$ . We proceed with the proof by distinguishing two sub-cases, depending on whether any of the terms  $Q_i$  has a proper derivative that is bisimilar to  $Y$  or not.

o CASE 2.2.1. Suppose that there is a proper derivative  $Q'$  of some  $Q_i$ , for  $i = 1, \dots, m$ , such that  $Q' \Leftrightarrow Y$ .

As  $P'X' \Leftrightarrow R_1\text{shift}(Y) + S_1$ , and  $X$  is a derivative of  $X'$ , it follows that  $S_1$  has a proper derivative  $S'$  that is bisimilar to  $X$ . Hence, transitivity yields  $X \Leftrightarrow S' \Leftrightarrow Q' \Leftrightarrow Y$ . Since  $S' \prec Y$  and  $Q' \prec X$  (Lem. 3.6), we can apply induction to these three equivalences to obtain

$$X = S' = Q' = Y .$$

o CASE 2.2.2. Suppose that no proper derivative of any  $Q_i$  is bisimilar to  $Y$ .

As  $P'X' \Leftrightarrow Y \Leftrightarrow R_1\text{shift}(Y) + S_1$ , there exist terms  $T_1$  and  $U_1$ , whose proper derivatives are contained in those of  $P'$ , such that:

$$(19) \quad P' \quad \Leftrightarrow \quad T_1 + U_1$$

$$(20) \quad T_1X' \quad \Leftrightarrow \quad R_1\text{shift}(Y)$$

$$(21) \quad U_1X' \quad \Leftrightarrow \quad S_1 .$$

As  $L(P') < L(X)$  (Lem. 2.7), in light of the equivalences  $X \Leftrightarrow Y$  and (19), invariance of  $L$ -value under bisimulation (Lem. 2.7) yields  $L(T_1 + U_1) < L(Y)$ . It follows that  $P' \prec X'$  and  $T_1 + U_1 \prec Y$ . Therefore, by induction, equivalence (19) can be proven:

$$(22) \quad P' = T_1 + U_1 .$$

Furthermore, since  $S' \prec Y$  for every proper derivative  $S'$  of  $S_1$  (Lem. 3.6), and since each derivative in the expansion of  $U_1X'$  is  $\cong$ -equivalent to  $X$ , we can apply the same reasoning used in the proof of (8) to equivalence (21) to obtain

$$(23) \quad U_1X' = S_1 .$$

Finally, we can apply Lem. 3.10 to equivalence (20) to obtain the following two possibilities:

- CASE 2.2.2.1. If case I of Lem. 3.10 holds, then there are terms  $V_1, \dots, V_n$ , whose proper derivatives are contained in those of  $T_1$ , such that:

$$(24) \quad T_1 \quad \Leftrightarrow \quad R_1(R_2, \dots, R_n, R_1)^*(V_2, \dots, V_n, V_1)$$

$$(25) \quad V_iX' \quad \Leftrightarrow \quad S_i \quad \quad i = 1, \dots, n .$$

By (19) and Lem. 2.7, it follows that  $L(T_1) \leq L(P') < L(X)$ . In light of the equivalences  $X \Leftrightarrow Y$  and (24), invariance of  $L$ -value under bisimulation (Lem. 2.7), yields that

$$L(R_1(R_2, \dots, R_n, R_1)^*(V_2, \dots, V_n, V_1)) < L(Y) .$$

Hence, we can apply induction to equivalence (24) to obtain

$$(26) \quad T_1 = R_1(R_2, \dots, R_n, R_1)^*(V_2, \dots, V_n, V_1) .$$

Moreover, since  $S' \prec Y$  for every proper derivative  $S'$  of  $S_i$  (Lem. 3.6), and since each derivative in the expansion of  $V_i X'$  is  $\cong$ -equivalent to  $X$ , we can apply the same reasoning used in the proof of (8) to derive equivalence (25):

$$(27) \quad V_i X' = S_i \quad i = 1, \dots, n .$$

Hence, we may conclude the proof for this case thus:

$$\begin{aligned} X \equiv P' X' & \stackrel{(22)}{=} (T_1 + U_1) X' \\ & \stackrel{A4, (26), (23)}{=} R_1(R_2, \dots, R_n, R_1)^*(V_2, \dots, V_n, V_1) X' + S_1 \\ & \stackrel{MEI2, (27)}{=} R_1(R_2, \dots, R_n, R_1)^*(S_2, \dots, S_n, S_1) + S_1 \\ & \stackrel{MEI1}{=} Y . \end{aligned}$$

- CASE 2.2.2.2. If case II of Lem. 3.10 holds, then there is a  $k \in \{1, \dots, n\}$  and there are terms  $T_2, \dots, T_k$  and  $U_2, \dots, U_k$ , whose proper derivatives are contained in those of  $T_1$ , such that:

$$(28) \quad T_i \Leftrightarrow R_i(T_{i+1} + U_{i+1}) \quad i = 1, \dots, k-1$$

$$(29) \quad T_k \Leftrightarrow R_k$$

$$(30) \quad T_k X' \Leftrightarrow R_k \text{shift}^k(Y)$$

$$(31) \quad U_i X' \Leftrightarrow S_i \quad i = 2, \dots, k .$$

By (19) and Lem. 2.7, it follows that  $L(T_1) \leq L(P') < L(X)$ . Since the proper derivatives of the terms  $T_i$  are contained in those of  $T_1$ , it follows that  $L(T_i) < L(X)$  for  $i = 2, \dots, k$ . In light of the equivalences  $X \Leftrightarrow Y$  and (28)–(29), invariance of  $L$ -value under bisimulation (Lem. 2.7) yields

- $L(R_i(T_{i+1} + U_{i+1})) < L(Y)$  for  $i = 1, \dots, k-1$  and
- $L(R_k) < L(Y)$ .

Hence, we may apply induction to equivalences (28) and (29) to infer that:

$$(32) \quad T_i = R_i(T_{i+1} + U_{i+1}) \quad i = 1, \dots, k-1$$

$$(33) \quad T_k = R_k .$$

Reasoning as in the proof of (8), it is not hard to deduce equivalence (31):

$$(34) \quad U_i X' = S_i \quad i = 2, \dots, k .$$

Furthermore, equivalences (29), (30), and Lem. 2.5 imply that  $X' \Leftrightarrow \text{shift}^k(Y)$ . This equivalence can be deduced by case 2.1 of the proof:

$$(35) \quad X' = \text{shift}^k(Y) .$$

We can now use these equalities to derive  $X = Y$  as follows:

$$\begin{aligned} X \equiv P' X' & \stackrel{(22)}{=} (T_1 + U_1) X' \\ & \stackrel{A4, (32), (23)}{=} R_1(T_2 + U_2) X' + S_1 \\ & \stackrel{(32), (34)}{=} R_1(R_2(T_3 + U_3) X' + S_2) + S_1 \\ & \stackrel{(32), (34)}{=} \dots \\ & \stackrel{(33), (34)}{=} R_1(R_2(\dots(R_{k-1}(R_k X' + S_k) + S_{k-1}) \dots) + S_2) + S_1 \\ & \stackrel{(35)}{=} R_1(R_2(\dots(R_{k-1}(R_k \text{shift}^k(Y) + S_k) + S_{k-1}) \dots) + S_2) + S_1 . \end{aligned}$$

Finally, since  $R_i \text{shift}^i(Y) + S_i \stackrel{\text{MEI1}}{=} \text{shift}^{i-1}(Y)$  for  $i = 1, \dots, k$ , we can apply axiom MEI1  $k$  times in order to reduce this last term to  $Y$ . This completes the proof for case 2.2.

- \* CASE 2.3. Assume that  $X \equiv P'(P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$  for some proper derivative  $P'$  of  $P_m$ , and that  $Y \equiv R'(R_1, \dots, R_n)^*(S_1, \dots, S_n)$ , for some proper derivative  $R'$  of  $R_n$ . For convenience, put  $X' \equiv (P_1, \dots, P_m)^*(Q_1, \dots, Q_m)$  and  $Y' \equiv (R_1, \dots, R_n)^*(S_1, \dots, S_n)$ .

We proceed with the proof by analyzing the equivalence  $P'X' \Leftrightarrow R'Y'$  using Lem. 2.5. By symmetry, we may restrict ourselves to considering the following two sub-cases, depending on whether  $|X'| = |Y'|$  or  $|X'| < |Y'|$ , respectively.

- o CASE 2.3.1. Assume that  $|X'| = |Y'|$ . Then an application of Lem. 2.5 yields the equivalences  $P' \Leftrightarrow R'$  and  $X' \Leftrightarrow Y'$ . As  $L(P') < L(X)$  and  $L(R') < L(Y)$  (Lem. 2.7), the inductive hypothesis yields  $P' = R'$ . To conclude the proof for this case, it is therefore sufficient to note that the equality  $X' = Y'$  follows from  $X' \Leftrightarrow Y'$  by case 2.1 of the proof.
- o CASE 2.3.2. Assume that  $|X'| < |Y'|$ . Then an application of Lem. 2.5 yields a proper derivative  $P''$  of  $P'$  such that

$$\begin{array}{ccc} P' & \Leftrightarrow & R'P'' \\ P''X' & \Leftrightarrow & Y' \end{array} .$$

As  $L(P') < L(X)$  (Lem. 2.7), invariance of  $L$ -value under bisimulation (Lem. 2.7) yields  $L(R'P'') < L(Y)$ . Then we may apply the inductive hypothesis to deduce the equality  $P' = R'P''$ . Furthermore,  $P''X' = Y'$  follows from the equivalence  $P''X' \Leftrightarrow Y'$  by case 2.2 of the proof. Hence,  $P'X' = R'P''X' = R'Y'$ .

This completes the proof for case 2.3. As we have examined all the possible forms  $X, Y \in \mathbb{H}$  may take, the proof for case 2 is complete. □

We have therefore shown the completeness theorem. □

In light of Lem. 2.4, Thm. 3.11 has the following corollary.

**Corollary 3.13** *The axiom system  $\mathcal{E}$  is an  $\omega$ -complete axiomatization of the algebra  $\text{T}(\text{BPA}^{\text{mex}}(A))$  modulo bisimulation equivalence.*

## 4 An Expressiveness Hierarchy for Multi-Exit Iteration

As shown in [8], the addition of multi-exit iteration to BPA yields a language that, modulo bisimulation equivalence, is strictly more expressive than that obtained by augmenting BPA with the standard binary Kleene star. More precisely, it is proven *ibidem* that, in the presence of at least two actions, the process  $(a, a)^*(a, b)$  cannot be expressed, modulo bisimulation equivalence, in ACP [7], and *a fortiori* in BPA, enriched with the binary Kleene star (cf. Lem. 3.2.3 in *op. cit.*).

Let us say that a term of the form  $(P_1, \dots, P_m)^*(Q_1, \dots, Q_n)$  has  $n$ -exit iteration. By analogy with the aforementioned result from [8], we shall now argue that, in the presence of a *non-empty* set of actions, the sequence of  $k$ -exit iteration operations induces a hierarchy of super-languages of BPA with a strictly increasing expressive power modulo bisimulation equivalence. To this end, we shall show that, for every positive integer  $k$ , there is a process over a single action that can be specified using  $(k + 1)$ -exit iteration, but not using  $h$ -exit iteration with  $h \leq k$ .

Before embarking in the proof of this fact, we introduce some notions that will be useful in our argument.

**Definition 4.1** *A non-empty sequence of transitions*

$$P_0 \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} P_2 \cdots \xrightarrow{\alpha_{n+1}} P_{n+1}$$

is called a loop from  $P$  iff  $P_0 \equiv P_{n+1} \equiv P$ . The terms  $P_i$  are said to be traversed by the loop.

The set of termination options of a term  $P$ , notation  $\mathcal{O}(P)$ , is the smallest collection of terms satisfying:

- If  $P \xrightarrow{a} \surd$ , then  $a$  is contained in  $\mathcal{O}(P)$ ;
- If  $P \xrightarrow{a} Q$  and  $Q$  does not contain occurrences of multi-exit iteration, then  $aQ$  is contained in  $\mathcal{O}(P)$ .

Finally, for every positive integer  $k$ , we write  $\text{BPA}^{k*}$  for the set of terms in the language  $\mathbb{T}(\text{BPA}^{me*}(A))$  that may use  $h$ -exit iteration with  $h \leq k$ .

Intuitively, the termination options of a term  $P$  are its summands, in the sense of Sect. 2.4, that cannot embark in an infinite computation.

The following simple result, to the effect that bisimilar terms have the same termination options modulo bisimulation, will be useful in later developments. (Cf. the proof of Cor. 4.5.)

**Fact 4.2** *Let  $P, Q \in \mathbb{T}(\text{BPA}^{me*}(A))$ . Assume that  $P$  is bisimilar to  $Q$ . Then, for every  $P' \in \mathcal{O}(P)$ , there exists a term  $Q' \in \mathcal{O}(Q)$  such that  $P' \Leftrightarrow Q'$ .*

The following lemma provides the key to our expressiveness result.

**Lemma 4.3** *Let  $k \geq 1$  and let  $P$  be a term in the language  $\text{BPA}^{k*}$ . Then every loop from  $P$  traverses at most  $k$  terms with distinct, non-empty sets of termination options.*

**Proof:** By structural induction on  $P$ . We proceed by a case analysis on the form  $P$  may take. The case  $P \equiv \alpha$  is obviously vacuous because actions and variables have no loop from them.

- CASE:  $P \equiv Q + R$ . By Lem. 3.5, if  $P'$  is a proper derivative of  $P \equiv Q + R$ , then  $g(P') < g(P)$ . Moreover, again by Lem. 3.5, the function  $g$  is non-increasing with respect to transitions. It follows that a term of the form  $Q + R$  has no loop from it. This case is therefore vacuous.
- CASE:  $P \equiv QR$ . Consider a loop

$$P_0 \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} P_2 \cdots \xrightarrow{\alpha_{n+1}} P_{n+1}$$

from  $P$ . Two possibilities may now arise:

1. there exists a loop

$$Q_0 \xrightarrow{\alpha_1} Q_1 \xrightarrow{\alpha_2} Q_2 \cdots \xrightarrow{\alpha_{n+1}} Q_{n+1}$$

from  $Q$  such that  $P_i \equiv Q_i R$  for every  $i = 0, \dots, n+1$ , or

2. there exist  $l \in \{1, \dots, n\}$  and sequences of transitions

$$Q \equiv Q_0 \xrightarrow{\alpha_1} Q_1 \xrightarrow{\alpha_2} Q_2 \cdots Q_{l-1} \xrightarrow{\alpha_l} \checkmark \quad \text{and} \quad R \equiv P_l \xrightarrow{\alpha_{l+1}} P_{l+1} \cdots \xrightarrow{\alpha_{n+1}} P_{n+1}$$

such that  $P_i \equiv Q_i R$  for  $i = 0, \dots, l-1$ .

Assume that the first possibility applies. We proceed with the proof by considering two cases, depending on whether the term  $R$  contains occurrences of multi-exit iteration, or not.

- If the term  $R$  contains occurrences of multi-exit iteration, then all the terms  $P_i \equiv Q_i R$  traversed by the loop from  $P$  have an empty set of termination options.
- Assume that the term  $R$  does *not* contain occurrences of multi-exit iteration. Under this assumption, the set of termination options of each of the terms  $P_i \equiv Q_i R$  is

$$\mathcal{O}(P_i) = \{Q'R \mid Q' \in \mathcal{O}(Q_i)\} .$$

The inductive hypothesis yields that there are at most  $k$  terms  $Q_i$  with distinct, non-empty sets of termination options. The claim now follows immediately.

If the second possibility applies, then the loop from  $P$  traverses exactly the same terms visited by the following loop from  $R$ :

$$R \equiv P_l \xrightarrow{\alpha_{l+1}} P_{l+1} \cdots \xrightarrow{\alpha_{n+1}} P_{n+1} \equiv P \xrightarrow{\alpha_1} Q_1 R \xrightarrow{\alpha_2} Q_2 R \cdots Q_{l-1} R \xrightarrow{\alpha_l} R$$

and the claim follows immediately by induction.

- CASE:  $P \equiv (Q_1, \dots, Q_m)^*(R_1, \dots, R_h)$ , where  $h \leq k$ . Note, first of all, that no proper derivative of any of the terms  $R_i$  can be traversed by a loop from  $P$  (Lem. 3.5). Thus the only terms with possibly non-empty sets of termination options in loops from  $P$  are those of the form  $\text{shift}^i(P)$  for some non-negative integer  $i$ . There are at most  $h$  terms with this form that have distinct, non-empty sets of termination options.

□

**Notation 4.4** For a positive integer  $i$  and action  $a$ , we write  $a^i$  for the term obtained by concatenating  $i$  copies of action  $a$ .

As an immediate corollary of the above lemma, we now obtain the following expressiveness result.

**Corollary 4.5** Let  $k$  be a positive integer. If the set of actions  $A$  contains at least the action  $a$ , then the process

$$a^*(a, a^2, \dots, a^{k+1})$$

cannot be specified in the language  $\text{BPA}^{k*}$  modulo bisimulation equivalence.

**Proof:** Assume, towards a contradiction, that there is a term  $P$  in the language  $\text{BPA}^{k*}$  that is bisimilar to  $a^*(a, a^2, \dots, a^{k+1})$ . Then there exists an infinite sequence of terms  $P_0, P_1, \dots$  such that  $P \equiv P_0$ , and, for every non-negative integer  $i$ ,

1.  $P_i \xrightarrow{a} P_{i+1}$ , and
2.  $P_i \xleftrightarrow{\text{shift}^i} a^*(a, a^2, \dots, a^{k+1})$ .

As the term  $P$  has only finitely many derivatives (Fact 2.2), there exists a derivative  $Q$  of  $P$  that occurs infinitely often in the sequence  $P_0, P_1, \dots$ . Thus we may choose positive integers  $i$  and  $j$ , with  $j - i$  greater than  $k + 1$ , such that  $Q \equiv P_i \equiv P_j$ . The term  $Q$  is in the language  $\text{BPA}^{k*}$ , and, by Fact 4.2 and our choice of indices  $i$  and  $j$ , has a loop that traverses  $k + 1$  terms with distinct, non-empty sets of termination options. However, this contradicts Lem. 4.3.  $\square$

By Cor. 4.5, it follows that, for every  $k \geq 1$ , the process

$$a^*(a, a^2, \dots, a^{k+1})$$

cannot be specified in the language  $\text{BPA}^{k*}$  modulo bisimulation equivalence. Thus, for every  $k \geq 1$ , the language  $\text{BPA}^{(k+1)*}$  is strictly more expressive than  $\text{BPA}^{k*}$  modulo bisimulation equivalence. This establishes the promised expressiveness hierarchy for the collection of languages  $\text{BPA}^{k*}$  ( $k \geq 1$ ).

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