UNIVERSITÁ DEGLI STUDI DI PISA
AND
VRIJE UNIVERSITEIT AMSTERDAM

Department of Computer Science
Master in Computer Science

Compositionality from the Operator's Point of View:

from Concrete to Abstract Semantics

Vittorio Massaro

Supervisors

Prof. dr. Roberto Bruni
Department of Computer Science
Universitá degli Studi di Pisa

Prof. dr. Wan Fokkink
Department of Computer Science
Vrije Universiteit Amsterdam

Abstract: Congruence with respect to process algebraic operators is an important characteristic for a behavioural semantics. A widely used method to ensure this property is to single out a semantics, and impose syntactic restrictions on the structural operational semantics definition of operators. In this study, by contrast, we approach this issue from the operator's point of view. Given an operator, we determine constraints on the modal characterization of semantics to guarantee that the semantics is a congruence with respect to that operator. We extend previous results regarding concrete semantics to include the renaming operator, and extend this work to weak semantics.

Contents

Introduction

One of the basic sanity properties of a behavioural semantics is that it constitutes a congruence with respect to standard process operators. This property is particularly important because compositionality of operators is an essential requirement in order to provide sound and complete axiomatizations for collections of process operators, which is one of the fundamental issues of process algebra.

Due to the importance of this characteristic, a lot of studies and researches have been conducted in order to find ways to ensure it. The most commonly used approach in the literature is to select a specific semantics, and provide suitable restrictions on operator definitions. The products of these researches are rule formats, i.e. specifications and limitations on the shape that transition systems specifications can assume, which, if respected during the definition of an operator, guarantee that the semantics is a congruence with respect to that operator. Many rule formats have been provided, for instance, for the bisimulation semantics: from the basic De Simone format [7] to the more expressive *panth* format [22]. With this method, the focus is on the semantics and, since rule formats are often provided taking into account a specific one, they end up being effective only for that semantics.

A different approach that can be used to deal with the compositionality issue is presented in [10]. The main idea is that of facing this quest from the operator's point of view. With this method, the focus is on a single operator, and the aim of the investigation is to provide syntactic restrictions on the modal characterization of a behavioural semantics. If a semantics has a modal characterization that respects the provided constraints, then it is guaranteed to be a congruence with respect to the given operator.

In [10], this investigation is performed for several process algebraic operators, taking into account concrete behavioural semantics. Action prefix, parallel composition and alternative composition operators are treated, as well as two restriction operators, namely projection and encapsulation operators.

In this work, we extend the results concerning concrete semantics with the addition of the Relabelling Operator. Then, we move to abstract semantics and conduct, in these different settings, the same investigation in order to provide sufficient requirement also for what concerns this kind of semantics.

The organization of this work is as follows. In chapter 1, we present some basic knowledge concerning the field of process algebra. The tools we use are introduced as well. The aim of the first chapter is that of providing the knowledge needed to properly understand the remaining part of the work. In chapter 2, the results provided in [10] are briefly presented and extended to include the renaming operator. After providing the constraints that a concrete semantics should respect in order to be a congruence with respect to this operator, we prove that all the concrete semantics presented in [12] have a modal characterization that respects such restrictions. In chapter 3, then, the investigation conducted on weak semantics is presented. Modal requirements for all the investigated operators are provided and, for a subset of the weak semantics presented in [13], we analyse whether those constraints are respected.

Chapter 1

Preliminaries

In this first chapter, we present the basic notions and tools that are used in this work. The aim of the presentation is, if not to provide a complete view of the field in which this work lies, at least to supply the basic knowledge needed to understand and contextualize the results and the properties that are mentioned in the following chapters.

Some basic notions concerning both the behavioural and the operational semantics of processes are presented as well as the Hennessy-Milner logic, that will be crucially important for the enunciation of all the definitions and the theorems illustrated in the paper.

1.1 Process Algebra

In order to give some notions about process algebra, we start analysing the meaning of the word "*process*". In this field, with the word "process" we refer to the *behaviour of a system* [3, 12]. With Process Theory, we refer to the field that studies processes, their representation and their properties.

Process algebra involves an algebraic/axiomatic approach to the studies of processes [3]. The tools provided by the process algebra consist of an algebraic language for the specification of processes and statements about them, and calculi for the proofs of these statements [11].

In this section we start by reviewing the model of *labelled transition systems*, which are used to express the operational semantics of process calculi. Then, we present a higher-level representation of the processes, introducing the notions of behavioural equivalence and pre-order. In the end, we introduce the concept of *transition system specification*, which consists of a set of inductive rules to derive transitions over the set of closed terms.

1.1.1 Labelled Transition Systems

In operational semantics, when describing the behaviour of a system, the focus is on the operations that it can perform. When the focus is mainly on discrete systems, these operations can be modelled as elementary steps which are here called *transitions*. With these transitions, the system goes from a configuration (here called *state*) to another. A transition system [18, 20] is the description of the relation between the states and the transitions.

Definition 1.1. *A transition system is a pair $(\Gamma, \rightarrow)$, where Γ is a set of states and $\rightarrow \subseteq \Gamma \times \Gamma$ is a binary relation on Γ, called transition relation.*
We use the notation $\gamma \rightarrow \gamma'$, with $\gamma, \gamma' \in \Gamma$, to express the fact that $(\gamma, \gamma') \in \rightarrow$.

Transition systems in general, do not give the opportunity to say much about the transitions. It may be the case, in fact, that a transition represents a specific event, with some properties we are interested in. In order to express this information, we can use a name (here called *label*) to indicate the specific event represented by the transition. In order to allow this, we extend definition 1.1 as follows:

Definition 1.2. *A labelled transition system is a triple $(\Gamma, \Sigma, \rightarrow)$, where Γ is a set of states, Σ is a set of labels and $\rightarrow \subseteq \Gamma \times \Sigma \times \Gamma$ is the transition relation. We use the notation $\gamma \xrightarrow{\sigma} \gamma'$, with $\gamma, \gamma' \in \Gamma$ and $\sigma \in \Sigma$, to express the fact that $(\gamma, \sigma, \gamma') \in \rightarrow$.*

In case of concrete processes, we say that the set of label Σ equals the set of concrete actions *Act*. For abstract processes, we say that $\Sigma = Act \cup \{\tau\}$, where τ represents the silent action.

Notation We introduce here some notations that will be used in this paper.

- Given $\gamma \in \Gamma$ and $\sigma \in \Sigma$, we use the notation $\gamma \xrightarrow{\sigma}$ to express that a $\gamma' \in \Gamma$ exists such that $(\gamma, \sigma, \gamma') \in\rightarrow$. Conversely, we use $\gamma \not\xrightarrow{\sigma}$ as an abbreviation for the statement $\forall \gamma' \in \Gamma : (\gamma, \sigma, \gamma') \notin\rightarrow$.

- In case of abstract processes, we use $\stackrel{\epsilon}{\Rightarrow}$ to express the reflexive and transitive closure of $\xrightarrow{\tau}$.

- Given $\overline{\sigma} \in \Sigma^*$, let $\overline{\sigma} = \sigma_1 \dots \sigma_k$, we use $p \xrightarrow{\overline{\sigma}} p'$ to express that $\exists p_0 \dots p_k :$ $\forall i \in [0, k) : p_i \xrightarrow{\sigma_{i+1}} p_{i+1}$ and $p_0 = p \wedge p_k = p'$.

1.1.2 Behavioural Equivalence and Pre-Order

As said before, LTSs describe processes behaviour in great detail, depicting their evolution step by step. Sometimes, one may have the necessity to abstract away from irrelevant information and describe higher-level properties. In order to do so, the notions of behavioural equivalence and pre-order have been introduced.

A deep investigation on these notions is presented in [6, 12] for concrete processes and in [13] for abstract ones. Here we present the semantics that we are going to investigate in the following chapters.

Concrete Semantics

When dealing with concrete processes, as said before, we assume that the set of labels Σ equals the set of concrete performable actions Act.

The linear time/branching time spectrum for concrete semantics is presented in [12] and successively extended to the complete shape, presented in [6]. This lattice (Figure 1.1) contains all the most important behavioural equivalences and pre-orders over LTSs, ordered by inclusion. Here, we report the definitions of the equivalences present in the lattice.

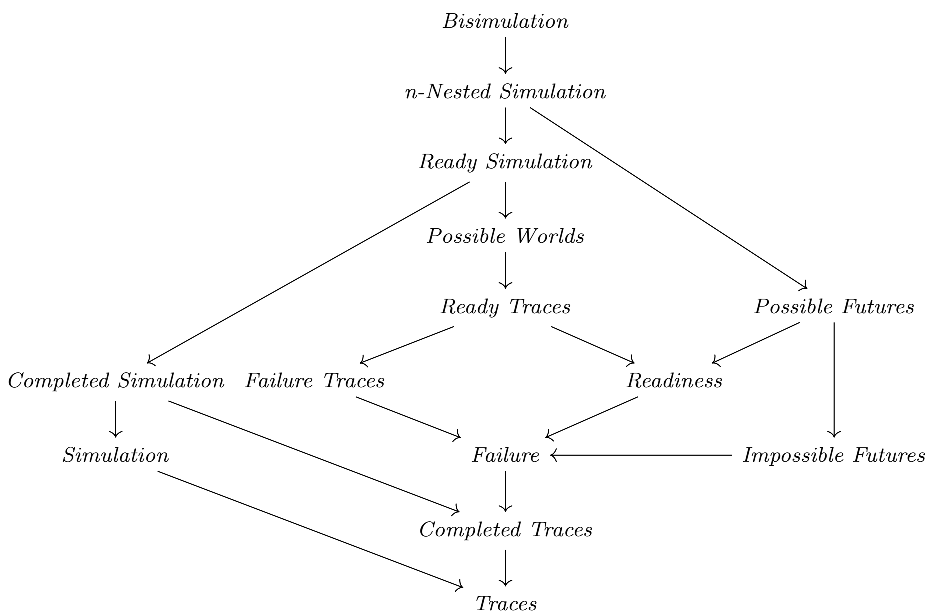

Figure 1.1: Linear Time/Branching Time Spectrum

First, we introduce the notions of trace and completed trace semantics. These semantics take into account the trace that can be observed during executions of a process.

Definition 1.3. *Given $\sigma \in Act^*$ and a process p, we say that σ is a trace of the process p if there is a process p' such that $p \xrightarrow{\sigma} p'$. We denote with $T(p)$ the set of traces of p. Given two processes p and q, we say $p \sqsubseteq_{Trace} q$ if $T(p) \subseteq T(q)$.*

Definition 1.4. *Given $\sigma \in Act^*$ and a process p, we say that σ is a completed trace of the process p if there is a process p' such that $p \xrightarrow{\sigma} p'$ and $\forall a \in Act$:*

$p' \stackrel{a}{\not\rightarrow}$. We denote with $CT(p)$ the set of completed traces of p. Given two processes p and q, we say $p \sqsubseteq_{CompletedTrace} q$ if $CT(p) \subseteq CT(q)$.

Now, we introduce a set of semantics which go under the name of *Decorated Trace Semantics*. These semantics, in fact, are based on the observation of traces, but the possibility of expressing properties between consecutive actions in a trace is given.

Definition 1.5. *Given $\sigma \in Act^*$ and $X \subseteq Act$, we say that (σ, X) is a failure pair for the process p if there is a process p' such that $p \stackrel{\sigma}{\rightarrow} p'$ and $\forall a \in X : p' \stackrel{a}{\not\rightarrow}$. We denote with $F(p)$ the set of failure pairs of p. Given two processes p and q, we say $p \sqsubseteq_{Failure} q$ if $F(p) \subseteq F(q)$.*

Definition 1.6. *Given $\sigma \in Act^*$ and $X \subseteq Act$, we say that (σ, X) is a ready pair for the process p if there is a process p' such that $p \stackrel{\sigma}{\rightarrow} p'$ and $\forall a \in Act : p' \stackrel{a}{\rightarrow} \iff a \in X$. We denote with $R(p)$ the set of ready pairs of p. Given two processes p and q, we say $p \sqsubseteq_{Readiness} q$ if $R(p) \subseteq R(q)$.*

Definition 1.7.

- *Given $X \subseteq Act$, we define the refusal relations $\stackrel{X}{\longrightarrow}$ by: $p \stackrel{X}{\longrightarrow} p'$ if $p = p'$ and $\forall a \in X : p \stackrel{a}{\not\rightarrow}$.*
- *The failure trace relations $\stackrel{\sigma}{\rightarrow}$ for $\sigma \in (Act \cup 2^{Act})^*$ are defined as the reflexive and transitive closure of both actions and refusal relations.*
- *We say that σ is a failure trace for a process p if there is a process p' such that $p \stackrel{\sigma}{\rightarrow} p'$. We denote with $FT(p)$ the set of failure traces of p. Given two processes p and q, we say $p \sqsubseteq_{FailureTrace} q$ if $FT(p) \subseteq FT(q)$.*

Definition 1.8.

- *The ready trace relations $\stackrel{\sigma}{\mapsto}$ for $\sigma \in (Act \cup 2^{Act})^*$ are defined recursively as follows:*
  1. *$p \stackrel{\epsilon}{\mapsto} p$, for any process p*
  2. *$p \stackrel{a}{\rightarrow} p'$, implies $p \stackrel{a}{\mapsto} p'$*
  3. *$p \stackrel{X}{\longmapsto} p'$ with $X \subseteq Act$ if $p = p'$ and $\forall a \in Act : p \stackrel{a}{\rightarrow} \iff a \in X$*
  4. *$p \stackrel{\sigma}{\mapsto} p' \stackrel{\rho}{\mapsto} p''$ implies $p \stackrel{\sigma\rho}{\longmapsto} p''$*
- *$\sigma \in (Act \cup 2^{Act})^*$ is a ready trace for a process if there is a p' such that $p \stackrel{\sigma}{\mapsto} p'$. We denote with $RT(p)$ the set of ready traces of p. Given two processes p and q, we say $p \sqsubseteq_{ReadyTrace} q$ if $RT(p) \subseteq RT(q)$.*

After the decorated trace semantics, we introduce two semantics which are strictly related to each other. With these semantics, we can express what is going to be possible or not in the future of a process, i.e. after it performed a given sequence of actions.

Definition 1.9. *Given $\sigma \in Act^*$ and $X \subseteq Act^*$, we say that (σ, X) is an impossible future for the process p if there is a process p' such that $p \xrightarrow{\sigma} p'$ and $T(p') \cap X = \varnothing$. We denote with $IF(p)$ the set of impossible futures of p. Given two processes p and q, we say $p \sqsubseteq_{ImpossibleFutures} q$ if $IF(p) \subseteq IF(q)$.*

Definition 1.10. *Given $\sigma \in Act^*$ and $X \subseteq Act^*$, we say that (σ, X) is a possible future for the process p if there is a process p' such that $p \xrightarrow{\sigma} p'$ and $T(p') = X$. We denote with $PF(p)$ the set of possible futures of p. Given two processes p and q, we say $p \sqsubseteq_{PossibleFutures} q$ if $PF(p) \subseteq PF(q)$.*

Now, we introduce a new group of semantics. These semantics treat the processes from a different point of view: they are not based on sets of traces, but on the idea of *imitation* between processes.

Definition 1.11. *A simulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall a \in Act : p \xrightarrow{a} p' \implies \exists q' : q \xrightarrow{a} q' \wedge p'Rq'$

Given two processes p and q we say $p \xrightarrow{\subseteq} q$ if there is a simulation relation R such that pRq. We say that the processes are similar $p \xrightarrow{\leftrightarrows} q$ if $p \xrightarrow{\subseteq} q$ and $q \xrightarrow{\subseteq} p$.

Definition 1.12. *A completed simulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall a \in Act : p \xrightarrow{a} p' \implies \exists q' : q \xrightarrow{a} q' \wedge p'Rq'$
- $\forall a \in Act : p \stackrel{a}{\not\rightarrow} \iff \forall a \in Act : q \stackrel{a}{\not\rightarrow}$

Given two processes p and q, they are completed simulation equivalent if there is a completed simulation relation R such that pRq and a completed simulation relation S such that qSp.

Definition 1.13. *A ready simulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall a \in Act : p \xrightarrow{a} p' \implies \exists q' : q \xrightarrow{a} q' \wedge p'Rq'$
- $\forall a \in Act : p \xrightarrow{a} \iff q \xrightarrow{a}$

Given two processes p and q, they are ready simulation equivalent if there is a ready simulation relation R such that pRq and a ready simulation relation S such that qSp.

Definition 1.14. *An n-nested simulation relation is a simulation nested in an (n-1)-nested simulation relation. Given two processes p and q, they are n-nested simulation equivalent if there is an n-nested simulation relation R such that pRq and an n-nested simulation relation S such that qSp.*

Definition 1.15. *A bisimulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall a \in Act : p \xrightarrow{a} p' \implies \exists q' : q \xrightarrow{a} q' \wedge p'Rq'$
- $\forall a \in Act : q \xrightarrow{a} q' \implies \exists p' : p \xrightarrow{a} p' \wedge p'Rq'$

Given two processes p and q, they are bisimilar, notation $p \leftrightarrow q$ if there is a bisimulation relation R such that pRq.

Lastly, we introduce a semantics which is based on a different way of looking at non-deterministic processes. Such a process, in fact, can be described as a set of deterministic processes, its *possible worlds* [21].

Definition 1.16. *A process p' is a possible world of a process p if p' is deterministic and $p' \sqsubseteq_{Readiness} p$. We denote with $PW(p)$ the set of possible worlds of p. Given two processes p and q, we say $p \sqsubseteq_{PossibleWorlds} q$ if $PW(p) \subseteq PW(q)$.*

Abstract Semantics

When discussing about abstract semantics, we assume the existence of a silent action, denoted with the symbol $\tau \notin Act$. We represent with this symbol all the activities performed by the investigated system that are said to be *internal* and not distinguishable.

The linear time/branching time spectrum for abstract semantics is presented in [13]. In this work we investigate a small subset of this spectrum (Figure 1.2). Here, the definitions of the semantics included in the lattice are presented as well as the definition of their *rooted* versions [8, 9].

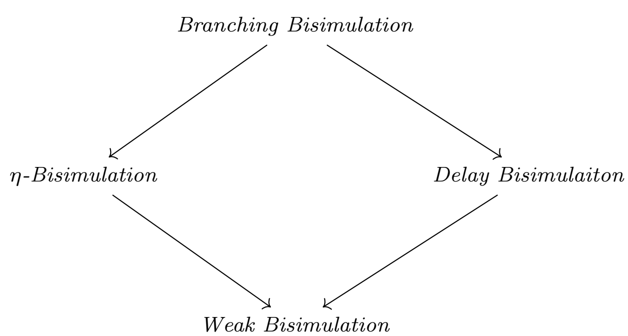

Figure 1.2: The subset of Linear Time/Branching Time Spectrum for Weak Semantics that we investigate in this work.

Definition 1.17. *A weak bisimulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall \alpha \in Act \cup \{\tau\} : p \xrightarrow{\alpha} p'$ *implies that either* $\alpha = \tau$ *and* $p'Rq$ *or a process* q'' *exists such that* $q \stackrel{\epsilon}{\Rightarrow} \xrightarrow{\alpha} \stackrel{\epsilon}{\Rightarrow} q''$ *and* $p'Rq''$.

Given two processes p and q, they are weakly bisimilar, notation $p \leftrightarrows_w q$ *if there is a weak bisimulation relation R such that pRq.*

Definition 1.18. *A delay bisimulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall \alpha \in Act \cup \{\tau\} : p \xrightarrow{\alpha} p'$ *implies that either* $\alpha = \tau$ *and* $p'Rq$ *or a process* q'' *exists such that* $q \stackrel{\epsilon}{\Rightarrow} \xrightarrow{\alpha} q''$ *and* $p'Rq''$.

Given two processes p and q, they are delay bisimilar, notation $p \leftrightarrows_d q$ *if there is a delay bisimulation relation R such that pRq.*

Definition 1.19. *An* η*-bisimulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall \alpha \in Act \cup \{\tau\} : p \xrightarrow{\alpha} p'$ *implies that either* $\alpha = \tau$ *and* $p'Rq$ *or two processes* q' *and* q'' *exist such that* $q \stackrel{\epsilon}{\Rightarrow} q' \xrightarrow{\alpha} \stackrel{\epsilon}{\Rightarrow} q''$ *and* pRq' *and* $p'Rq''$.

Given two processes p and q, they are η*-bisimilar, notation* $p \leftrightarrows_\eta q$ *if there is an* η*-bisimulation relation R such that pRq.*

Definition 1.20. *A branching bisimulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall \alpha \in Act \cup \{\tau\} : p \xrightarrow{\alpha} p'$ *implies that either* $\alpha = \tau$ *and* $p'Rq$ *or two processes* q' *and* q'' *exist such that* $q \stackrel{\epsilon}{\Rightarrow} q' \xrightarrow{\alpha} q''$ *and* pRq' *and* $p'Rq''$.

Given two processes p and q, they are branching bisimilar, notation $p \leftrightarrows_b q$ *if there is a branching bisimulation relation R such that pRq.*

While the bisimulation defined in the previous section for concrete semantics is quite strong and mathematically nice, these abstract bisimulations are less so. In order to fill the gap between these two categories of bisimulations, the notion of *rootedness* is used. Now we present the rooted versions of the previously defined bisimulations.

Definition 1.21. *A rooted weak bisimulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall \alpha \in Act \cup \{\tau\} : p \xrightarrow{\alpha} p'$ *implies that a process* q' *exists such that* $q \stackrel{\epsilon}{\Rightarrow} \xrightarrow{\alpha} \stackrel{\epsilon}{\Rightarrow} q'$ *and* $p' \leftrightarrows_w q'$.

Given two processes p and q, they are rooted weakly bisimilar, notation $p \leftrightarrows_{rw} q$ *if there is a rooted weak bisimulation relation R such that pRq.*

Definition 1.22. *A rooted delay bisimulation is a binary relation R on processes. Given two processes p and q we say that pRq if*

- $\forall \alpha \in Act \cup \{\tau\} : p \xrightarrow{\alpha} p'$ *implies that a process* q' *exists such that* $q \overset{\epsilon}{\Rightarrow} \xrightarrow{\alpha} q'$ *and* $p' \leftrightarroweq_d q'$.

Given two processes p *and* q, *they are rooted delay bisimilar, notation* $p \leftrightarroweq_{rd} q$ *if there is a rooted delay bisimulation relation* R *such that* pRq.

Definition 1.23. *A rooted* η*-bisimulation is a binary relation* R *on processes. Given two processes* p *and* q *we say that* pRq *if*

- $\forall \alpha \in Act \cup \{\tau\} : p \xrightarrow{\alpha} p'$ *implies that a process* q' *exists such that* $q \xrightarrow{\alpha} \overset{\epsilon}{\Rightarrow} q'$ *and* $p' \leftrightarroweq_\eta q'$.

Given two processes p *and* q, *they are rooted* η*-bisimilar, notation* $p \leftrightarroweq_{r\eta} q$ *if there is a rooted* η*-bisimulation relation* R *such that* pRq.

Definition 1.24. *A rooted branching bisimulation is a binary relation* R *on processes. Given two processes* p *and* q *we say that* pRq *if*

- $\forall \alpha \in Act \cup \{\tau\} : p \xrightarrow{\alpha} p'$ *implies that a process* q' *exists such that* $q \xrightarrow{\alpha} q'$ *and* $p' \leftrightarroweq_b q'$.

Given two processes p *and* q, *they are rooted branching bisimilar, notation* $p \leftrightarroweq_{rb} q$ *if there is a rooted branching bisimulation relation* R *such that* pRq.

1.1.3 Transition System Specifications

A Transition System Specification (TSS) is defined as a *collection of inductive proof rules to derive the transitions over the set of closed terms* [1]. In order to give a formal definition of this concept, we need to introduce some basic notions of term algebra. We start from a countably infinite set Var of variables.

Definition 1.25. *A signature* S *is a set of function symbols, disjoint from* Var, *together with an* arity *mapping that assigns a natural number* $\mathrm{ar}(f)$ *to each function symbol* f. *A function symbol of arity zero is called a constant, while function symbols of arity one and two are called unary and binary, respectively.*

Definition 1.26. *The set* $\mathbb{T}(\mathrm{S})$ *of (open) terms over a signature* S, *is the least set such that:*

- *each* $x \in \mathrm{Var}$ *is a term*
- $f(t_1, \ldots, t_{\mathrm{ar}(f)})$ *is a term, if* f *is a function symbol and* $t_1, \ldots, t_{\mathrm{ar}(f)}$ *are terms*

$\mathrm{T}(\mathrm{S})$ *denotes the set of closed terms over* S, *i.e., terms that do not contain variables.*

Given these necessary definitions, we proceed to the formalization of the concept of TSS.

Definition 1.27. *Let* S *be a signature and* $t, t' \in \mathbb{T}(\mathrm{S})$. *A transition rule* ρ *is of the shape* H/α, *where* H *is a set of premises and* α *is the conclusion. Each premise can be of the shape* $t \xrightarrow{a} t'$ *or* $t \stackrel{a}{\not\to}$ *or, given a predicate* P, *of the shape* tP, $t\neg P$. *A Transition Specification System is a set of transition rules.*

From now on, transition rules will be displayed with the format $\frac{H}{\alpha}$. In Table 1.1 an example of transition system specification is provided, namely, the specification of the Basic Process Algebra with Empty Process (BPA_ε).

$$\frac{}{a \xrightarrow{a} \varepsilon} \qquad \frac{}{\varepsilon\checkmark}$$

$$\frac{p\checkmark}{p+q\checkmark} \qquad \frac{p \xrightarrow{a} p'}{p+q \xrightarrow{a} p'} \qquad \frac{q\checkmark}{p+q\checkmark} \qquad \frac{q \xrightarrow{a} q'}{p+q \xrightarrow{a} q'}$$

$$\frac{p\checkmark \quad q\checkmark}{p \bullet q\checkmark} \qquad \frac{p\checkmark \quad q \xrightarrow{a} q'}{p \bullet q \xrightarrow{a} q'} \qquad \frac{p \xrightarrow{a} p'}{p \bullet q \xrightarrow{a} p' \bullet q}$$

Table 1.1: Transition Rules for BPA_ε

1.2 Hennessy-Milner Logic

The Hennessy-Milner Logic (HML), is a multi-modal logic that has been presented in [16] as a *language for talking about programs.* Although the expressive power of this logic is fixed, the choice of syntax plays an important role. The requirements established for a given HML version (i.e. for a given syntax) may be insufficient for a different one. Here, we present the syntax we use in this work.

Definition 1.28. *Given a set* Σ *of labels, representing the set of possible experiments and a set* I *of indices. The language* HML *is the least set of formulas such that:*

1. $\phi \in HML \implies \neg\phi \in HML$
2. $\forall i \in I : \phi_i \in HML \implies \bigwedge_{i \in I} \phi_i \in HML$
3. $\phi \in HML, \sigma \in \Sigma \implies \langle\sigma\rangle\phi \in HML$

Furthermore, we use:

- $\top$ *as an abbreviation for* $\bigwedge_{i \in \varnothing} \varphi_i$
- $\bot$ *as an abbreviation for* $\neg\top$
- $\varphi_1 \wedge \varphi_2$ *as an abbreviation for* $\bigwedge_{i \in \{1,2\}} \varphi_i$

Given $\mathscr{P}$ the set of processes, the satisfaction relation $\models \subseteq \mathscr{P} \times HML$ is the least set such that, for all $p \in \mathscr{P}$:

1. $p \models \neg\phi \iff p \not\models \phi$
2. $p \models \bigwedge_{i \in I} \phi_i \iff \forall i \in I : p \models \phi_i$
3. $p \models \langle\sigma\rangle\phi \iff p \xrightarrow{\sigma} p' \wedge p' \models \phi$

Together with the Hennessy-Milner Logic, we present some related concepts that are used, as well as the HML itself, in the following chapters.

Definition 1.29. *Given a set of formulas $\mathscr{O} \subseteq HML$, we define*

$$\mathscr{O}^{\equiv} = \{\phi \mid \exists\phi' : \phi' \in \mathscr{O} \wedge \phi \equiv \phi'\}$$

Definition 1.30. *A context, notation $C[]$, is a HML formula with one occurrence of $[]$.*

Definition 1.31. *A multi-context, notation $C[]_{i \in I}$, is a HML formula with one or more indexed occurrences of $[]$.*

Definition 1.32. *An n-level context, notation $C_n[]$, is a context where $[]$ has n diamond operators above it. It is defined as follows:*

- *$[]$ is a 0-level context.*
- *If $C_n[]$ is an n-level context, then $\neg C_n[]$ and $C_n[] \wedge \bigwedge_{i \in I} \varphi_i$ are n-level contexts.*
- *If $C_n[]$ is an n-level context, then $\langle\sigma\rangle C_n[]$ is an (n+1)-level context.*

An important property of the HML is the result, presented in [17], telling us that two processes are bisimilar if they satisfy the same HML formulas. In order to formalize this result, we need to introduce the notion of equivalent processes with respect to a set of formulas.

Definition 1.33. *Given a set of formulas $\mathscr{O}$, we say that two processes p and q are equivalent with respect to $\mathscr{O}$, notation $p \sim_{\mathscr{O}} q$, if*

$$\forall\phi \in \mathscr{O} : p \models \phi \iff q \models \phi$$

Theorem 1.1. *The equivalence relations $\leftrightarroweq$ and $\sim_{HML}$ coincide. Namely, given two processes p and q:*

$$p \leftrightarroweq q \iff p \sim_{HML} q$$

As we will see in the following section, congruences and pre-orders can be expressed with an opportune subset of HML. This set of formulas is called the modal characterization of the congruence/pre-order.

1.3 Modal Characterizations

In section 1.1.2 we presented the notions of behavioural equivalence and pre-order. Then, in section 1.2, we introduced the HML and enunciated an interesting result, Theorem 1.1, which creates a connection among this logic and one of the behavioural semantics previously presented, namely the bisimulaton.

In this section we provide a modal characterization for all the other equivalences and pre-orders presented. In order to do this, we define a modal characterization as follows:

Definition 1.34. *Given a behavioural pre-order $\sqsubseteq_N$ and a set of formulas $\mathscr{O}_N \subseteq HML$, we say that $\mathscr{O}_N$ is a modal characterization of $\sqsubseteq_N$ if:*

$$(p \sqsubseteq_N q) \iff (\forall \phi \in \mathscr{O}_N : p \models \phi \implies q \models \phi)$$

Given a behavioural equivalence $=_N$ and a set of formulas $\mathscr{O}_N \subseteq HML$, we say that $\mathscr{O}_N$ is a modal characterization of $=_N$ if:

$$(p =_N q) \iff (\forall \phi \in \mathscr{O}_N : p \models \phi \iff q \models \phi)$$

All the modal characterizations that we present, are taken from [5, 6, 8, 9, 12]. It is worth mentioning here that the modal characterizations provided are not necessarily unique. It is possible, for instance, to find different characterizations for the same semantic among [5] and [12]. Different characterizations can specify the same behavioural equivalence, so one could say they are equivalent: they are, indeed, but their syntactic differences can play a role while proving some properties. So, in case of multiple characterizations, we choose the most convenient one for our goals.

1.3.1 Concrete Semantics

The modal characterizations for concrete semantics that we present, can be found in [5, 6, 12].

Definition 1.35. *In the following definitions a ranges over Act. Notice that $\top$ is in all the defined sets, but it is not explicitly present in characterizations where it can be obtained using other production rules.*

- *The modal characterization of the trace semantics is the following:* $\mathscr{O}_T \ \phi ::= \top \mid \langle a \rangle \phi' \ (\phi' \in \mathscr{O}_T)$
- *The modal characterization of the complete trace semantics is the following:*

  $\mathscr{O}_{CT} \ \phi ::= \top \mid \langle a \rangle \phi' \ (\phi' \in \mathscr{O}_{CT}) \mid \bigwedge_{a \in Act} \neg \langle a \rangle \top$
- *The modal characterization of the failure semantics is the following:*

  $\mathscr{O}_F \ \phi ::= \langle a \rangle \phi' \ (\phi' \in \mathscr{O}_F) \mid \bigwedge_{i \in I} \neg \langle a_i \rangle \top$

- *The modal characterization of the readiness semantics is the following:*
  $\mathscr{O}_R\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_R)\ |\ \bigwedge_{i\in I}\neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J}\langle a_j\rangle\top$
- *The modal characterization of the failure trace semantics is the following:*
  $\mathscr{O}_{FT}\ \phi ::= \top\ |\ \langle a\rangle\phi'\ |\ \bigwedge_{i\in I}\neg\langle a_i\rangle\top \wedge \phi'\ (\phi' \in \mathscr{O}_{FT})$
- *The modal characterization of the ready trace semantics is the following:*
  $\mathscr{O}_{RT}\ \phi ::= \top\ |\ \langle a\rangle\phi'\ |\ \bigwedge_{i\in I}\neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J}\langle a_j\rangle\top \wedge \phi'\ (\phi' \in \mathscr{O}_{RT})$
- *The modal characterization of the impossible futures semantics is the following:*
  $\mathscr{O}_{IF}\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_{IF})\ |\ \bigwedge_{i\in I}\neg\phi_i\ (\phi_i \in \mathscr{O}_T)$
- *The modal characterization of the possible futures semantics is the following:*
  $\mathscr{O}_{PF}\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_{PF})\ |\ \bigwedge_{i\in I}\neg\phi_i \wedge \bigwedge_{j\in J}\phi_j\ (\phi_i, \phi_j \in \mathscr{O}_T)$
- *The modal characterization of the simulation is the following:*
  $\mathscr{O}_{1S}\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_{1S})\ |\ \bigwedge_{i\in I}\phi_i\ (\phi_i \in \mathscr{O}_{1S})$
- *The modal characterization of the completed simulation is the following:*
  $\mathscr{O}_{CS}\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_{CS})\ |\ \bigwedge_{i\in I}\phi_i\ (\phi_i \in \mathscr{O}_{CS})\ |\ \bigwedge_{a\in Act}\neg\langle a\rangle\top$
- *The modal characterization of the possible worlds semantics is the following:*
  $\mathscr{O}_{PW}\ \phi ::= \bigwedge_{i\in I}\langle a_i\rangle\phi_i\ (\phi_i \in \mathscr{O}_{PW})\ |\ \bigwedge_{i\in I}\neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J}\langle a_i\rangle\top$
- *The modal characterization of the ready simulation is the following:*
  $\mathscr{O}_{RS}\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_{RS})\ |\ \neg\langle a\rangle\top\ |\ \bigwedge_{i\in I}\phi_i\ (\phi_i \in \mathscr{O}_{RS})$
- *The modal characterization of the n-nested simulation is the following:*
  $\mathscr{O}_{nS}\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_{nS})\ |\ \bigwedge_{i\in I}\phi_i\ (\phi_i \in \mathscr{O}_{nS})\ |\ \neg\phi'\ (\phi' \in \mathscr{O}_{(n-1)S})$

1.3.2 Abstract Semantics

The modal characterizations that we use for the abstract semantics, can be found in [8, 9].

Definition 1.36. *In the following definitions a ranges over Act and α ranges over $Act \cup \{\tau\}$. We use $\varphi\langle\alpha\rangle\varphi'$ to indicate $\varphi \wedge \langle\alpha\rangle\varphi'$ and $\varphi\langle\hat{\tau}\rangle\varphi'$ to indicate $\varphi \wedge \langle\hat{\tau}\rangle\varphi'$, where $\langle\hat{\tau}\rangle\varphi \equiv \varphi \vee \langle\tau\rangle\varphi \equiv \neg(\neg\varphi \wedge \neg\langle\tau\rangle\varphi)$.*

- *The modal characterization of the weak bisimulation is the following:*
  $\mathscr{O}_w\ \phi ::= \bigwedge_{i\in I}\phi_i\ |\ \neg\phi\ |\ \langle\epsilon\rangle\phi\ |\ \langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\phi\ (\phi_i, \phi \in \mathscr{O}_w)$
- *The modal characterization of the delay bisimulation is the following:*
  $\mathscr{O}_d\ \phi ::= \bigwedge_{i\in I}\phi_i\ |\ \neg\phi\ |\ \langle\epsilon\rangle\phi\ |\ \langle\epsilon\rangle\langle a\rangle\phi\ (\phi_i, \phi \in \mathscr{O}_d)$

- *The modal characterization of the $\eta-$bisimulation is the following:*
  $\mathscr{O}_\eta\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle\phi \mid \langle\epsilon\rangle(\phi\langle a\rangle\langle\epsilon\rangle\phi)\ (\phi_i,\phi\in\mathscr{O}_\eta)$
- *The modal characterization of the branching bisimulation is the following:*
  $\mathscr{O}_b\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle(\phi\langle\hat{\tau}\rangle\phi) \mid \langle\epsilon\rangle(\phi\langle a\rangle\phi)\ (\phi_i,\phi\in\mathscr{O}_b)$
- *The modal characterization of the rooted weak bisimulation is the following:*
  $\mathscr{O}_{rw}\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\hat{\phi} \mid \hat{\phi}\ (\phi_i,\phi\in\mathscr{O}_{rw},\hat{\phi}\in\mathscr{O}_w)$
- *The modal characterization of the rooted delay bisimulation is the following:*
  $\mathscr{O}_{rd}\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle\langle\alpha\rangle\hat{\phi} \mid \hat{\phi}\ (\phi_i,\phi\in\mathscr{O}_{rd},\hat{\phi}\in\mathscr{O}_d)$
- *The modal characterization of the rooted $\eta-$bisimulation is the following:*
  $\mathscr{O}_{r\eta}\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\alpha\rangle\langle\epsilon\rangle\hat{\phi} \mid \hat{\phi}\ (\phi_i,\phi\in\mathscr{O}_{r\eta},\hat{\phi}\in\mathscr{O}_\eta)$
- *The modal characterization of the rooted branching bisimulation is the following:*
  $\mathscr{O}_{rb}\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\alpha\rangle\hat{\phi} \mid \hat{\phi}\ (\phi_i,\phi\in\mathscr{O}_{rb},\hat{\phi}\in\mathscr{O}_b)$

1.4 Congruence

In section 1.1.2, we introduced the idea of Behavioural Semantics and some well-known concrete semantics as well as abstract ones. In this section, we introduce one of the main properties of a behavioural semantics: congruence with respect to operators. One of the most important issues of the process algebra is, in fact, providing sound and complete axiomatizations for collections of process operators and this requires that such operators are compositional.

As a first step, we provide a formal characterization of this property[15]:

Definition 1.37. *Given a process operator* O *of arity* n *and a behavioural preorder* $\sqsubseteq_N$*, we say that* $\sqsubseteq_N$ *is a congruence with respect to* O *if:*

$$(\forall i\in[0,n): p_i\sqsubseteq_N q_i) \implies \mathrm{O}(p_0,\ldots,p_{n-1})\sqsubseteq_N \mathrm{O}(q_0,\ldots,q_{n-1})$$

Given a process operator O *of arity* n *and a behavioural equivalence* $=_N$*, we say that* $=_N$ *is a congruence with respect to* O *if:*

$$(\forall i\in[0,n): p_i=_N q_i) \implies \mathrm{O}(p_0,\ldots,p_{n-1})=_N \mathrm{O}(q_0,\ldots,q_{n-1})$$

Given the importance for a semantics to be endowed with this characteristic, it's not surprising that a lot of research to find way to ensure the congruence property has been done.

The most common methodology used to achieve this goal, is that of imposing restrictions on the operators definitions. This is done by providing rule formats for transition system specifications, which are syntactical constraints on the form of the rules. Once a rule format is provided, it is sufficient for an operator to be defined with rules that fit within the specified format, in order to guarantee that a given semantics is a congruence with respect to the operator.

Several rule formats have been provided, for instance, for the strong bisimulation. All these formats can be represented in a lattice (Figure 1.3): an arrow from a rule format to another indicates that the operators definable in the first format are a subset of the operators definable in the second format.

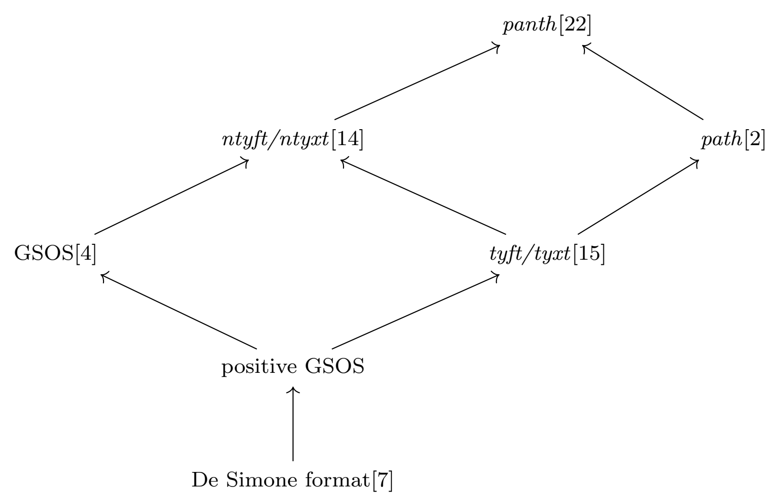

Figure 1.3: Rule Formats Lattice

It is important to notice that most of the time these rule formats are provided with a specific semantics in mind (as said before, the formats presented in figure 1.3 have been developed for strong bisimilarity) and the focus of the process is always on the semantics. What comes in mind is that a different approach could be used when facing the compositionality issue: the focus could be on the operator.

The first attempt of looking at this issue form the operator's point of view is done in [10], where, for some basic operators, constraints on the modal characterization of semantics are found, in order to guarantee that a given semantics is a congruence for those operators.

The focus of that work, as we will see in detail in section 2.1, is on concrete semantics. In this work we extend the investigation conducted in [10] adding another operator and then we move on to abstract semantics, providing new constraints, in order to guarantee the congruence property also for the semantics presented in figure 1.2 and their rooted versions.

Chapter 2

Concrete Semantics

In this chapter we cope with concrete semantics. As said in the first chapter, we want to determine the conditions that a process semantic should satisfy in order to be a congruence with respect to a given operator.

In section 2.1 we briefly state the results presented in [10] regarding some basic operators, such as the alternative and parallel composition, the action prefix and two restriction operators, namely encapsulation and projection. Then, in section 2.2, we conduct the same study on the relabelling operator, providing the properties that a process semantic has to satisfy in order to be a congruence with respect to such an operator. Finally, in section 2.3, we analyse the properties obtained with respect to existing concrete semantics, namely the ones presented in the preliminary chapter.

2.1 Other Operators

In section 1.4, we illustrated the importance for a semantics to be a congruence with respect to operators and briefly presented the main results obtained in order to guarantee this. We also highlighted the fact that these results have been mainly obtained conducting studies whose focus was on the semantics.

The first attempt to treat the compositionality issue from the operator's point of view can be found in [10], where, for some basic operators, syntactic constraints on modal characterization of concrete semantics have been presented, in order to guarantee that the given operators are compositional with respect to this semantics.

In this section, we are going to briefly present these results. In the next chapter, all these operators will be investigated in order to conduct the same study for what concerns abstract semantics.

Action Prefix Operator The crucial observation in order to find suitable constraints for this operator, is that $a.p \models \langle a \rangle \phi \iff p \models \phi$. Intuitively, what may be a problem is the existence of a formula of the shape $\langle a\rangle\phi$ in our set $\mathscr{O}$, with the corresponding $\phi \notin \mathscr{O}$. The condition proposed is then the following:

$$\text{(AP) } C_0[\langle\alpha\rangle\phi] \in \mathscr{O} \implies \phi \in \mathscr{O}^{\equiv}$$

Alternative Composition Operator An important characteristic of this operator is that, after performing the first action, the behaviour of an alternative composition is determined by just one of the two processes. Therefore, the only potential problem can be a conjunction at level 0, where no actions have been performed yet. Thus, the condition proposed for this operator is:

$$\text{(AC) } C_0[\textstyle\bigwedge_{i\in I}\phi_i] \in \mathscr{O} \implies \forall i \in I : \phi_i \in \mathscr{O}^{\equiv}$$

Restriction Operators The restriction operators investigated in [10] are the projection operator and the encapsulation operator. These operators are characterized by the fact that, given a formula $\phi \in HML$, we can deduce in advance which of its sub-formulas $\langle a\rangle\varphi$ will always yield false. The constraint proposed for this kind of operators is:

$$\text{(RES) } C[\neg\phi] \in \mathscr{O} \implies C[\top] \in \mathscr{O}^{\equiv}$$

Parallel Composition Operator The last operator presented is the parallel composition operator. The intuition here, is that if $p||q \models \phi$, there should be a pair of formulas, ϕ_p and ϕ_q, that are in some sort of way sub-formulas of ϕ, such that $p \models \phi_p$ and $q \models \phi_q$: we want them to be part of our set of formulas as well. This notion of sub-formula is defined as follows:

- $\phi \in Sub(\phi)$
- $\phi' \in Sub(\phi) \implies \{D[\psi] \mid \phi' = D[C[\psi]]\} \subseteq Sub(\phi)$

With this definition of sub-formulas, the requirement found for this operator is:

$$\text{(PAR) } \phi \in \mathscr{O} \implies Sub(\phi) \subseteq \mathscr{O}^{\equiv}$$

2.2 Relabelling Operator

Definition 2.1. *The relabelling operator is written $p[f]$ where the function f : $Act \to Act$ is the relabelling function. The effect is that of renaming the actions performed by p according to f. The transition rule for this operator is:*

$$\frac{p \xrightarrow{x} p'}{p[f] \xrightarrow{f(x)} p'[f]} \text{ (REL)}$$

In order to establish which syntactic properties a modal language $\mathscr{O} \subseteq HML$ should satisfy to guarantee that the induced equivalence is a congruence with respect to this operator, we introduce the following function.

Definition 2.2. *The inverse relabelling operator for HML is written* $\phi\{f^{-1}\}$ *where the function* f *is a relabelling function. Its behavior is defined as follows:*

- $(\neg\phi)\{f^{-1}\} = \{\bigwedge_{\phi' \in \phi\{f^{-1}\}} \neg\phi'\}$
- $(\bigwedge_{i \in I} \phi_i)\{f^{-1}\} = \{\bigwedge_{i \in I} \phi'_i | \phi'_i \in \phi_i\{f^{-1}\}\}$
- $(\langle x \rangle \phi)\{f^{-1}\} = \{\langle y \rangle \phi' | f(y) = x \wedge \phi' \in \phi\{f^{-1}\}\}$

Intuitively, we just defined an operator that, given a formula ϕ, produces a set of formulas that *are mapped* in ϕ by f. This intuitive concept is made clear with the following result.

Lemma 2.1. $p[f] \models \phi \iff \exists \phi' \in \phi\{f^{-1}\} : p \models \phi'$

Proof. We apply induction on the structure of ϕ.

- $\phi \equiv \neg\varphi$:

  $p[f] \models \phi$

  $\iff$

  $p[f] \not\models \varphi$

  $\iff$ [*Inductive Hypothesis*]

  $\neg\exists \varphi' \in \varphi\{f^{-1}\} : p \models \varphi'$

  $\iff$

  $\forall \varphi' \in \varphi\{f^{-1}\} : p \not\models \varphi'$

  $\iff$

  $\forall \varphi' \in \varphi\{f^{-1}\} : p \models \neg\varphi'$

  $\iff$

  $p \models \bigwedge_{\varphi' \in \varphi\{f^{-1}\}} \neg\varphi'$

  $\iff$ [*Definition of* $\phi\{f^{-1}\}$]

  $\exists \phi' \in \phi\{f^{-1}\} : p \models \phi'$

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:

  $p[f] \models \phi$

  $\iff$

  $\forall i \in I : p[f] \models \phi_i$

  $\iff$ [*Inductive Hypothesis*]

  $\forall i \in I : \exists \phi'_i \in \phi\{f^{-1}\} : p \models \phi'_i$

  $\iff$ [*Definition of* $\phi\{f^{-1}\}$]

  $\exists \phi' \in \phi\{f^{-1}\} : p \models \phi'$

- $\phi \equiv \langle x \rangle \varphi$ *with* $x \notin f(Act)$:
  $p[f] \models \phi$
  $\iff$ [$x \notin f(Act)$ *and (REL)*]
  $\bot$
  $\iff$ [$\exists$ *in empty set*]
  $\exists \phi' \in \varnothing : p \models \phi'$
  $\iff$ [*Definition of* $\phi\{f^{-1}\}$]
  $\exists \phi' \in \phi\{f^{-1}\} : p \models \phi'$
- $\phi \equiv \langle x \rangle \varphi$ *with* $x \in f(Act)$:
  $p[f] \models \phi$
  $\iff$
  $p[f] \xrightarrow{x} p'[f] \wedge p'[f] \models \varphi$
  $\iff$ [*Inductive Hypothesis*]
  $p[f] \xrightarrow{x} p'[f] \wedge \exists \varphi' \in \varphi\{f^{-1}\} : p' \models \varphi'$
  $\iff$ [*(REL)*]
  $\exists y : f(y) = x : p \xrightarrow{y} p' \wedge \exists \varphi' \in \varphi\{f^{-1}\} : p' \models \varphi'$
  $\iff$
  $\exists y : f(y) = x : \exists \varphi' \in \varphi\{f^{-1}\} : p \models \langle y \rangle \varphi'$
  $\iff$ [*Definition of* $\phi\{f^{-1}\}$]
  $\exists \phi' \in \phi\{f^{-1}\} : p \models \phi'$

□

At this point, we have all the ingredients to provide the constraints we are looking for:

Theorem 2.1. *Let* $\mathscr{O} \subseteq HML$*. If for any* $\phi \in HML$

$$(R)\ \phi \in \mathscr{O} \implies \phi\{f^{-1}\} \subseteq \mathscr{O}$$

then $\sim_{\mathscr{O}}$ *is a congruence with respect to the relabelling operator.*

Proof. Assume a modal language $\mathscr{O}$ with the (R) property. Let $p \sim_{\mathscr{O}} q$. We show that for any $\phi \in \mathscr{O}$

$$p[f] \models \phi \iff q[f] \models \phi$$

$p[f] \models \phi$
$\iff$ [*Lemma 2.1*]
$\exists \phi' \in \phi\{f^{-1}\} : p \models \phi'$
$\iff$ [(R) and $p \sim_{\mathscr{O}} q$]
$\exists \phi' \in \phi\{f^{-1}\} : q \models \phi'$
$\iff$ [*Lemma 2.1*]
$q[f] \models \phi$ □

2.3 Existing Semantics

In this section, we investigate the concrete semantics presented in section 1.1.2. We prove that all of them have a modal characterization which satisfies (R), i.e. they are all congruences with respect to the Relabelling Operator. For the sake of clarity, we report the modal characterization of each semantic.

2.3.1 Trace Semantics

The modal characterization of the trace semantics is the following:
$\mathscr{O}_T \ \phi ::= \top \mid \langle a \rangle \phi' \ (\phi' \in \mathscr{O}_T)$

Theorem 2.2. *The modal characterization of the trace semantics satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_T$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_T$. We apply induction on the structure of ϕ. The base case $\top$ is trivial.

- $\phi \equiv \langle x \rangle \varphi$:
  $\phi \in \mathscr{O}_T$
  $\iff$ [*Modal Characterization of* $\mathscr{O}_T$]
  $\varphi \in \mathscr{O}_T$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_T$
  $\iff$
  $\forall \varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_T$
  $\iff$ [*Modal Characterization of* $\mathscr{O}_T$]
  $\forall a \in Act : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a \rangle \varphi' \in \mathscr{O}_T$
  $\implies$
  $\forall a \ s.t. \ f(a) = x : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a \rangle \varphi' \in \mathscr{O}_T$
  $\iff$
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_T$

□

2.3.2 Complete Trace Semantics

The modal characterization of the complete trace semantics is the following:
$\mathscr{O}_{CT} \ \phi ::= \top \mid \langle a \rangle \phi' \ (\phi' \in \mathscr{O}_{CT}) \mid \bigwedge_{a \in Act} \neg \langle a \rangle \top$

Theorem 2.3. *The modal characterization of the complete trace semantics satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{CT}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{CT}$. We apply induction on the structure of ϕ. The base case $\top$ is trivial.

- $\phi \equiv \langle x\rangle\varphi$:
  $\phi \in \mathscr{O}_{CT}$
  $\iff$ [*Modal Characterization of* $\mathscr{O}_{CT}$]
  $\varphi \in \mathscr{O}_{CT}$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{CT}$
  $\iff$
  $\forall \varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_{CT}$
  $\iff$ [*Modal Characterization of* $\mathscr{O}_{CT}$]
  $\forall a \in Act : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{CT}$
  $\implies$
  $\forall a \ s.t. \ f(a) = x : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{CT}$
  $\iff$
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{CT}$
- $\phi \equiv \bigwedge_{a\in Act} \neg\langle a\rangle\top$:
  We assume [1]$\phi \in \mathscr{O}_{CT}$.
  We will prove [2]$\phi\{f^{-1}\} = \{\phi\}$.

  $\phi\{f^{-1}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{a\in Act} \phi'_a \mid \phi'_a \in (\neg\langle a\rangle\top)\{f^{-1}\}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{a\in Act} \bigwedge_{\phi'\in(\langle a\rangle\top)\{f^{-1}\}} \neg\phi'\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{a\in Act} \bigwedge_{\phi'\in\{\langle y\rangle\phi'' \mid f(y)=a \ \wedge \ \phi''\in\top\{f^{-1}\}\}} \neg\phi'\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{a\in Act} \bigwedge_{\phi'\in\{\langle y\rangle\top \mid f(y)=a\}} \neg\phi'\}$
  $=$
  $\{\bigwedge_{a\in Act} \bigwedge_{x\in\{y \mid f(y)=a\}} \neg\langle x\rangle\top\}$
  $=$
  $\{\bigwedge_{x\in\{y \mid f(y)\in Act\}} \neg\langle x\rangle\top\}$
  $=$ [$f(Act) \subseteq Act$]
  $\{\bigwedge_{a\in Act} \neg\langle a\rangle\top\}$
  $=$ [*Definition of* ϕ]

$\{\phi\}$

Using [1] and [2] we can conclude that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{CT}$.

□

2.3.3 Failure Semantics

The modal characterization of the failure semantics is the following:
$\mathscr{O}_F\ \phi ::= \langle a \rangle \phi'\ (\phi' \in \mathscr{O}_F)\ |\ \bigwedge_{i \in I} \neg \langle a_i \rangle \top$

Theorem 2.4. *The modal characterization of the failure semantics satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_F$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_F$. We apply induction on the structure of ϕ.

- $\phi \equiv \langle x \rangle \varphi$:

  $\phi \in \mathscr{O}_F$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_F$]

  $\varphi \in \mathscr{O}_F$

  $\implies$ [*Inductive Hypothesis*]

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_F$

  $\iff$

  $\forall \varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_F$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_F$]

  $\forall a \in Act : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a \rangle \varphi' \in \mathscr{O}_F$

  $\implies$

  $\forall a\ s.t.\ f(a) = x : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a \rangle \varphi' \in \mathscr{O}_F$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_F$

- $\phi \equiv \bigwedge_{i \in I} \neg \langle a_i \rangle \top$:
  We will prove [1]$\phi\{f^{-1}\} = \{\bigwedge_{i \in I'} \neg \langle a_i \rangle \top\}$ for some I'.

  $\phi\{f^{-1}\}$

  $=$ [*Definition of* $\{f^{-1}\}$]

  $\{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in (\neg \langle a_i \rangle \top)\{f^{-1}\}\}$

  $=$ [*Definition of* $\{f^{-1}\}$]

  $\{\bigwedge_{i \in I} \bigwedge_{\phi' \in (\langle a_i \rangle \top)\{f^{-1}\}} \neg \phi'\}$

$= [\textit{Definition of } \{f^{-1}\}]$
$\{\bigwedge_{i\in I} \bigwedge_{\phi' \in \{\langle y\rangle\phi'' \mid f(y)=a_i \wedge \phi'' \in \top\{f^{-1}\}\}} \neg\phi'\}$
$= [\textit{Definition of } \{f^{-1}\}]$
$\{\bigwedge_{i\in I} \bigwedge_{\phi' \in \{\langle y\rangle\top \mid f(y)=a_i\}} \neg\phi'\}$
$=$
$\{\bigwedge_{i\in I} \bigwedge_{x \in \{y \mid f(y)=a_i\}} \neg\langle x\rangle\top\}$
$=$
$\{\bigwedge_{x \in \{y \mid \exists i\in I:\ f(y)=a_i\}} \neg\langle x\rangle\top\}$
$= [\textit{with } I' = \{i' \mid \exists i \in I :\ f(a_{i'}) = a_i\}]$
$\{\bigwedge_{i\in I'} \neg\langle a_i\rangle\top\}$

Using [1] and the Modal Characterization of $\mathscr{O}_F$ we can conclude that $\phi\{f^{-1}\} \subseteq \mathscr{O}_F$.

□

2.3.4 Readiness Semantics

The modal characterization of the readiness semantics is the following:
$\mathscr{O}_R\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_R) \mid \bigwedge_{i\in I} \neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J}\langle a_j\rangle\top$

Theorem 2.5. *The modal characterization of the readiness semantics satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_R$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_R$. We apply induction on the structure of ϕ.

- $\phi \equiv \langle x\rangle\varphi$:
  $\phi \in \mathscr{O}_R$
  $\iff [\textit{Modal Characterization of } \mathscr{O}_R]$
  $\varphi \in \mathscr{O}_R$
  $\implies [\textit{Inductive Hypothesis}]$
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_R$
  $\iff$
  $\forall\varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_R$
  $\iff [\textit{Modal Characterization of } \mathscr{O}_R]$
  $\forall a \in \mathit{Act} : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_R$
  $\implies$
  $\forall a \textit{ s.t. } f(a) = x : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_R$
  $\iff$
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_R$

- $\phi \equiv \bigwedge_{i\in I} \neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J} \langle a_j\rangle\top$:

  $\phi\{f^{-1}\}$

  $= [\textit{Definition of } \{f^{-1}\}]$

  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in (\bigwedge_{i\in I} \neg\langle a_i\rangle\top)\{f^{-1}\} \wedge \varphi_j \in (\bigwedge_{j\in J} \langle a_j\rangle\top)\{f^{-1}\}\}$

  $= [\textit{Definition of } \{f^{-1}\}]$

  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in (\neg\langle a_i\rangle\top)\{f^{-1}\}\} \ \wedge$
  $\quad \varphi_j \in \{\bigwedge_{j\in J} \phi'_j \mid \phi'_j \in (\langle a_j\rangle\top)\{f^{-1}\}\}\}$

  $= [\textit{Definition of } \{f^{-1}\}]$

  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{\phi' \in (\langle a_i\rangle\top)\{f^{-1}\}} \neg\phi'\} \ \wedge$
  $\quad \varphi_j \in \{\bigwedge_{j\in J} \langle x\rangle\phi' \mid f(x) = a_j \ \wedge \ \phi' \in (\top)\{f^{-1}\}\}\}$

  $= [\textit{Definition of } \{f^{-1}\}]$

  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{\{f(x)=a_i \ \wedge \ \phi' \in (\top)\{f^{-1}\}\}} \neg\langle x\rangle\phi'\} \ \wedge$
  $\quad \varphi_j \in \{\bigwedge_{j\in J} \langle x\rangle\top \mid f(x) = a_j\}\}$

  $= [\textit{Definition of } \{f^{-1}\}]$

  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{f(x)=a_i} \neg\langle x\rangle\top\} \ \wedge$
  $\quad \varphi_j \in \{\bigwedge_{j\in J} \langle x\rangle\top \mid f(x) = a_j\}\}$

  $= [\textit{with } I' = \{i' \mid \exists i \in I : f(a_{i'}) = a_i\}]$

  $\{\bigwedge_{i\in I'} \neg\langle a_i\rangle\top \wedge \varphi_j \mid \varphi_j \in \{\bigwedge_{j\in J} \langle x\rangle\top \mid f(x) = a_j\}\}$

  We now have two cases:

  - $\exists j \in J : a_j \notin f(Act)$:

    $\phi\{f^{-1}\} = \varnothing$

    $\Longrightarrow$

    $\phi\{f^{-1}\} \subseteq \mathscr{O}_R$

  - $\forall j \in J : a_j \in f(Act)$:

    φ_j is of the shape $\bigwedge_{j\in J'} \langle a_j\rangle\top$ for some J'

    $\Longrightarrow$

    $\forall \varphi \in \phi\{f^{-1}\} : \varphi$ is of the shape $\bigwedge_{i\in I'} \neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J'} \langle a_j\rangle\top$ for some J'

    $\Longleftrightarrow$ [*Modal Characterization of* $\mathscr{O}_R$]

    $\forall \varphi \in \phi\{f^{-1}\} : \varphi \in \mathscr{O}_R$

    $\Longleftrightarrow$

    $\phi\{f^{-1}\} \subseteq \mathscr{O}_R$

$\square$

2.3.5 Failure Trace Semantics

The modal characterization of the failure trace semantics is the following:
$\mathscr{O}_{FT}$ $\phi ::= \top \mid \langle a\rangle\phi'$ $(\phi' \in \mathscr{O}_{FT}) \mid \bigwedge_{i\in I} \neg\langle a_i\rangle\top \wedge \phi'$ $(\phi' \in \mathscr{O}_{FT})$

Theorem 2.6. *The modal characterization of the failure trace semantics satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{FT}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{FT}$. We apply induction on the structure of ϕ. The base case $\top$ is trivial.

- $\phi \equiv \langle x\rangle\varphi$:

  $\phi \in \mathscr{O}_{FT}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{FT}$]

  $\varphi \in \mathscr{O}_{FT}$

  $\implies$ [*Inductive Hypothesis*]

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{FT}$

  $\iff$

  $\forall\varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_{FT}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{FT}$]

  $\forall a \in Act : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{FT}$

  $\implies$

  $\forall a$ *s.t.* $f(a) = x : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{FT}$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{FT}$

- $\phi \equiv \bigwedge_{i\in I} \neg\langle a_i\rangle\top \wedge \phi'$:

  We will prove [1]$\phi\{f^{-1}\} = \{\bigwedge_{i\in I'} \neg\langle a_i\rangle\top \wedge \phi' \mid \phi' \in \mathscr{O}' \ \wedge \ \mathscr{O}' \subseteq \mathscr{O}_{FT}\}$ for some I'.

  $\phi\{f^{-1}\}$

  $=$ [*Definition of* $\{f^{-1}\}$]

  $\{\varphi_i \wedge \varphi' \mid \varphi_i \in (\bigwedge_{i\in I} \neg\langle a_i\rangle\top)\{f^{-1}\} \wedge \varphi' \in (\phi')\{f^{-1}\}\}$

  $=$ [*Definition of* $\{f^{-1}\}$]

  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in (\neg\langle a_i\rangle\top)\{f^{-1}\}\} \ \wedge \varphi' \in (\phi')\{f^{-1}\}\}$

  $=$ [*Definition of* $\{f^{-1}\}$]

  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{\phi' \in (\langle a_i\rangle\top)\{f^{-1}\}} \neg\phi'\} \ \wedge \ \varphi' \in (\phi')\{f^{-1}\}\}$

  $=$ [*Definition of* $\{f^{-1}\}$]

  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{f(x)=a_i \ \wedge \ \phi' \in (\top)\{f^{-1}\}\}} \neg\langle x\rangle\phi'\} \ \wedge \ \varphi' \in (\phi')\{f^{-1}\}\}$

$= [\textit{Definition of } \{f^{-1}\}]$

$\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i \in I} \bigwedge_{f(x)=a_i} \neg\langle x\rangle\top\} \;\wedge\; \varphi' \in (\phi')\{f^{-1}\}\}$

$= [\textit{with } I' = \{i' \mid \exists i \in I : f(a_{i'}) = a_i\} \text{ and } \mathscr{O}' = (\phi')\{f^{-1}\}]$

$\{\bigwedge_{i \in I'} \neg\langle a_i\rangle\top \wedge \varphi' \mid \varphi' \in \mathscr{O}'\}$

$= [\textit{Inductive Hypothesis}]$

$\{\bigwedge_{i \in I'} \neg\langle a_i\rangle\top \wedge \varphi' \mid \varphi' \in \mathscr{O}' \;\wedge\; \mathscr{O}' \subseteq \mathscr{O}_{FT}\}$

Using [1] and the Modal Characterization of $\mathscr{O}_{FT}$ we can conclude that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{FT}$.

□

2.3.6 Ready Trace Semantics

The modal characterization of the ready trace semantics is the following:
$\mathscr{O}_{RT}$ $\phi ::= \top \mid \langle a\rangle\phi' \; (\phi' \in \mathscr{O}_{RT}) \mid \bigwedge_{i \in I} \neg\langle a_i\rangle\top \wedge \bigwedge_{j \in J}\langle a_j\rangle\top \wedge \phi' \; (\phi' \in \mathscr{O}_{RT})$

Theorem 2.7. *The modal characterization of the ready trace semantics satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{RT}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{RT}$. We apply induction on the structure of ϕ. The base case $\top$ is trivial.

- $\phi \equiv \langle x\rangle\varphi$:

  $\phi \in \mathscr{O}_{RT}$

  $\iff [\textit{Modal Characterization of } \mathscr{O}_{RT}]$

  $\varphi \in \mathscr{O}_{RT}$

  $\implies [\textit{Inductive Hypothesis}]$

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{RT}$

  $\iff$

  $\forall \varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_{RT}$

  $\iff [\textit{Modal Characterization of } \mathscr{O}_{RT}]$

  $\forall a \in \mathit{Act} : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{RT}$

  $\implies$

  $\forall a \textit{ s.t. } f(a) = x : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{RT}$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{RT}$

- $\phi \equiv \bigwedge_{i\in I} \neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J}\langle a_j\rangle\top \wedge \phi'$:

  $\phi\{f^{-1}\}$
  $= [\textit{Definition of } \{f^{-1}\}]$
  $\{\varphi_i \wedge \varphi_j \wedge \varphi' |\ \varphi_i \in (\bigwedge_{i\in I} \neg\langle a_i\rangle\top)\{f^{-1}\} \wedge \varphi_j \in (\bigwedge_{j\in J}\langle a_j\rangle\top)\{f^{-1}\} \wedge \varphi' \in (\phi')\{f^{-1}\}\}$
  $= [\textit{Definition of } \{f^{-1}\}]$
  $\{\varphi_i \wedge \varphi_j \wedge \varphi' |\ \varphi_i \in \{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in (\neg\langle a_i\rangle\top)\{f^{-1}\}\} \ \wedge \varphi_j \in \{\bigwedge_{j\in J} \phi'_j \mid \phi'_j \in (\langle a_j\rangle\top)\{f^{-1}\}\} \wedge \varphi' \in (\phi')\{f^{-1}\}\}$
  $= [\textit{Definition of } \{f^{-1}\}]$
  $\{\varphi_i \wedge \varphi_j \wedge \varphi' |\ \varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{\phi'\in(\langle a_i\rangle\top)\{f^{-1}\}} \neg\phi'\} \ \wedge \varphi_j \in \{\bigwedge_{j\in J}\langle x\rangle\phi' \mid f(x) = a_j \ \wedge \ \phi' \in (\top)\{f^{-1}\}\} \wedge \varphi' \in (\phi')\{f^{-1}\}\}$
  $= [\textit{Definition of } \{f^{-1}\}]$
  $\{\varphi_i \wedge \varphi_j \wedge \varphi' |\ \varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{f(x)=a_i \ \wedge \ \phi'\in(\top)\{f^{-1}\}\}} \neg\langle x\rangle\phi'\} \ \wedge \varphi_j \in \{\bigwedge_{j\in J}\langle x\rangle\top \mid f(x) = a_j\} \wedge \varphi' \in (\phi')\{f^{-1}\}\}$
  $= [\textit{Definition of } \{f^{-1}\}]$
  $\{\varphi_i \wedge \varphi_j \wedge \varphi' |\ \varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{f(x)=a_i} \neg\langle x\rangle\top\} \ \wedge \varphi_j \in \{\bigwedge_{j\in J}\langle x\rangle\top \mid f(x) = a_j\} \wedge \varphi' \in (\phi')\{f^{-1}\}\}$
  $= [\textit{with } I' = \{i' |\ \exists i \in I : f(a_{i'}) = a_i\} \ \wedge \ \mathscr{O}' = (\phi')\{f^{-1}\}]$
  $\{\bigwedge_{i\in I'} \neg\langle a_i\rangle\top \wedge \varphi_j \wedge \varphi' \mid \varphi_j \in \{\bigwedge_{j\in J}\langle x\rangle\top \mid f(x) = a_j\} \wedge \varphi' \in \mathscr{O}'\}$
  $= [\textit{Inductive Hypothesis}]$
  $\{\bigwedge_{i\in I'} \neg\langle a_i\rangle\top \wedge \varphi_j \wedge \varphi' \mid \varphi_j \in \{\bigwedge_{j\in J}\langle x\rangle\top \mid f(x) = a_j\} \wedge \varphi' \in \mathscr{O}' \wedge \mathscr{O}' \subseteq \mathscr{O}_{RT}\}$

  We now have two cases:

  - $\exists j \in J : a_j \notin f(Act)$:
    $\phi\{f^{-1}\} = \varnothing$
    $\Longrightarrow$
    $\phi\{f^{-1}\} \subseteq \mathscr{O}_{RT}$
  - $\forall j \in J : a_j \in f(Act)$:
    φ_j is of the shape $\bigwedge_{j\in J'}\langle a_j\rangle\top$ for some J'
    $\Longrightarrow$
    $\forall\varphi \in \phi\{f^{-1}\} : \varphi$ is of the shape $\bigwedge_{i\in I'} \neg\langle a_i\rangle\top \ \wedge \ \bigwedge_{j\in J'}\langle a_j\rangle\top \ \wedge \ \varphi'$ with $\varphi' \in \mathscr{O}_{RT}$ for some J'
    $\Longleftrightarrow$ [*Modal Characterization of* $\mathscr{O}_{RT}$]
    $\forall\varphi \in \phi\{f^{-1}\} : \varphi \in \mathscr{O}_{RT}$
    $\Longleftrightarrow$
    $\phi\{f^{-1}\} \subseteq \mathscr{O}_{RT}$

□

2.3.7 Impossible Futures Semantics

The modal characterization of the impossible futures semantics is the following:
$\mathscr{O}_{IF}\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_{IF}) \mid \bigwedge_{i\in I} \neg\phi_i\ (\phi_i \in \mathscr{O}_T)$

Theorem 2.8. *The modal characterization of the impossible futures semantics satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{IF}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{IF}$. We apply induction on the structure of ϕ.

- $\phi \equiv \langle x\rangle\varphi$:
  $\phi \in \mathscr{O}_{IF}$
  $\iff$ [*Modal Characterization of* $\mathscr{O}_{IF}$]
  $\varphi \in \mathscr{O}_{IF}$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{IF}$
  $\iff$
  $\forall\varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_{IF}$
  $\iff$ [*Modal Characterization of* $\mathscr{O}_{IF}$]
  $\forall a \in Act : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{IF}$
  $\implies$
  $\forall a\ s.t.\ f(a) = x : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{IF}$
  $\iff$
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{IF}$

- $\bigwedge_{i\in I} \neg\phi_i$:
  $\phi\{f^{-1}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in (\neg\phi_i)\{f^{-1}\}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{i\in I} \bigwedge_{\phi'_i\in(\phi_i)\{f^{-1}\}} \neg\phi'_i\}$
  $=$ [*with* $I' = \{i' \mid \phi_{i'} \in \bigcup_{i\in I}(\phi_i)\{f^{-1}\}\}$]
  $\{\bigwedge_{i\in I'} \neg\phi_i\}$
  $=$ [$Theorem$ 2.2]
  $\{\bigwedge_{i\in I'} \neg\phi_i \mid \forall i \in I' : \phi_i \in \mathscr{O}_T\}$

Using the modal characterization of $\mathscr{O}_{IF}$, we conclude

$$\phi\{f^{-1}\} \subseteq \mathscr{O}_{IF}$$

□

2.3.8 Possible Futures Semantics

The modal characterization of the possible futures semantics is the following: $\mathscr{O}_{PF}$ $\phi ::= \langle a\rangle\phi'$ $(\phi' \in \mathscr{O}_{PF})$ | $\bigwedge_{i\in I} \neg\phi_i \wedge \bigwedge_{j\in J} \phi_j$ $(\phi_i, \phi_j \in \mathscr{O}_T)$

Theorem 2.9. *The modal characterization of the possible futures semantics satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{PF}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{PF}$. We apply induction on the structure of ϕ.

- $\phi \equiv \langle x\rangle\varphi$:

  $\phi \in \mathscr{O}_{PF}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{PF}$]

  $\varphi \in \mathscr{O}_{PF}$

  $\implies$ [*Inductive Hypothesis*]

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{PF}$

  $\iff$

  $\forall\varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_{PF}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{PF}$]

  $\forall a \in Act : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{PF}$

  $\implies$

  $\forall a$ *s.t.* $f(a) = x : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{PF}$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{PF}$

- $\bigwedge_{i\in I} \neg\phi_i \wedge \bigwedge_{j\in J} \phi_j$:

  $\phi\{f^{-1}\}$

  $=$ [*Definition of* $\{f^{-1}\}$]

  $\{\varphi_i \wedge \varphi_j |\ \varphi_i \in (\bigwedge_{i\in I} \neg\phi_i)\{f^{-1}\} \wedge \varphi_j \in (\bigwedge_{j\in J} \phi_j)\{f^{-1}\}\}$

  $=$ [*Definition of* $\{f^{-1}\}$]

  $\{\varphi_i \wedge \varphi_j |\ \varphi_i \in \{\bigwedge_{i\in I} \phi'_i |\ \phi'_i \in (\neg\phi_i)\{f^{-1}\} \wedge$
  $\varphi_j \in \{\bigwedge_{j\in J} \phi'_j |\ \phi'_j \in (\phi_j)\{f^{-1}\}\}\}$

  $=$ [*Definition of* $\{f^{-1}\}$]

$\{\varphi_i \wedge \varphi_j |\ \varphi_i \in \{\bigwedge_{i\in I}\bigwedge_{\phi'_i \in(\phi_i)\{f^{-1}\}} \neg\phi'_i\}$
$\quad\quad \varphi_j \in \{\bigwedge_{j\in J}\phi'_j |\ \phi'_j \in (\phi_j)\{f^{-1}\}\}\}$
$= [with\ I' = \{i' \mid \phi_{i'} \in \bigcup_{i\in I}(\phi_i)\{f^{-1}\}\}]$
$\{\bigwedge_{i\in I'}\neg\phi_i \wedge \varphi_j \mid \varphi_j \in \{\bigwedge_{j\in J}\phi'_j \mid \phi'_j \in (\phi_j)\{f^{-1}\}\}\}$
$= [Theorem\ 2.2]$
$\{\bigwedge_{i\in I'}\neg\phi_i \wedge \varphi_j \mid \phi_i \in \mathscr{O}_T \wedge$
$\quad\quad \varphi_j \in \{\bigwedge_{j\in J}\phi'_j \mid \phi'_j \in (\phi_j)\{f^{-1}\} \wedge (\phi_j)\{f^{-1}\} \subseteq \mathscr{O}_T\}\}$

Using the modal characterization of $\mathscr{O}_{PF}$, we conclude

$$\phi\{f^{-1}\} \subseteq \mathscr{O}_{PF}$$

□

2.3.9 Simulation

The modal characterization of the simulation is the following:
$\mathscr{O}_{1S}\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathscr{O}_{1S}) \mid\ \bigwedge_{i\in I}\phi_i\ (\phi_i \in \mathscr{O}_{1S})$

Theorem 2.10. *The modal characterization of the simulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{1S}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{1S}$. We apply induction on the structure of ϕ.

- $\phi \equiv \langle x\rangle\varphi$:

  $\phi \in \mathscr{O}_{1S}$

  $\iff [Modal\ Characterization\ of\ \mathscr{O}_{1S}]$

  $\varphi \in \mathscr{O}_{1S}$

  $\implies [Inductive\ Hypothesis]$

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{1S}$

  $\iff$

  $\forall\varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_{1S}$

  $\iff [Modal\ Characterization\ of\ \mathscr{O}_{1S}]$

  $\forall a \in Act : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{1S}$

  $\implies$

  $\forall a\ s.t.\ f(a) = x : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathscr{O}_{1S}$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{1S}$

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:

  $\phi \in \mathscr{O}_{1S}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{1S}$]

  $\forall i \in I : \phi_i \in \mathscr{O}_{1S}$

  $\implies$ [*Inductive Hypothesis*]

  $\forall i \in I : \phi_i\{f^{-1}\} \subseteq \mathscr{O}_{1S}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{1S}$]

  $\forall \phi' \in \{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} : \phi' \in \mathscr{O}_{1S}$

  $\iff$ [*Definition of* $\{f^{-1}\}$]

  $\forall \phi' \in \phi\{f^{-1}\} : \phi' \in \mathscr{O}_{1S}$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{1S}$

□

2.3.10 Completed Simulation

The modal characterization of the completed simulation is the following:
$\mathscr{O}_{CS}$ $\phi ::= \langle a \rangle \phi'$ $(\phi' \in \mathscr{O}_{CS})$ $\mid$ $\bigwedge_{i \in I} \phi_i$ $(\phi_i \in \mathscr{O}_{CS})$ $\mid$ $\bigwedge_{a \in Act} \neg\langle a \rangle \top$

Theorem 2.11. *The modal characterization of the completed simulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{CS}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{CS}$. We apply induction on the structure of ϕ.

- $\phi \equiv \langle x \rangle \varphi$:

  $\phi \in \mathscr{O}_{CS}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{CS}$]

  $\varphi \in \mathscr{O}_{CS}$

  $\implies$ [*Inductive Hypothesis*]

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{CS}$

  $\iff$

  $\forall \varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_{CS}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{CS}$]

  $\forall a \in Act : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a \rangle \varphi' \in \mathscr{O}_{CS}$

  $\implies$

  $\forall a$ *s.t.* $f(a) = x : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a \rangle \varphi' \in \mathscr{O}_{CS}$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{CS}$

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:
  $\phi \in \mathscr{O}_{CS}$
  $\iff$ [*Modal Characterization of* $\mathscr{O}_{CS}$]
  $\forall i \in I : \phi_i \in \mathscr{O}_{CS}$
  $\implies$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi_i\{f^{-1}\} \subseteq \mathscr{O}_{CS}$
  $\iff$ [*Modal Characterization of* $\mathscr{O}_{CS}$]
  $\forall \phi' \in \{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} : \phi' \in \mathscr{O}_{CS}$
  $\iff$ [*Definition of* $\{f^{-1}\}$]
  $\forall \phi' \in \phi\{f^{-1}\} : \phi' \in \mathscr{O}_{CS}$
  $\iff$
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{CS}$
- $\phi \equiv \bigwedge_{a\in Act} \neg\langle a\rangle\top$:
  We assume [1]$\phi \in \mathscr{O}_{CS}$.
  We will prove [2]$\phi\{f^{-1}\} = \{\phi\}$.

  $\phi\{f^{-1}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{a\in Act} \phi'_a \mid \phi'_a \in (\neg\langle a\rangle\top)\{f^{-1}\}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{a\in Act} \bigwedge_{\phi'\in(\langle a\rangle\top)\{f^{-1}\}} \neg\phi'\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{a\in Act} \bigwedge_{\phi'\in\{\langle y\rangle\phi'' \mid f(y)=a \,\wedge\, \phi''\in\top\{f^{-1}\}\}} \neg\phi'\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\bigwedge_{a\in Act} \bigwedge_{\phi'\in\{\langle y\rangle\top \mid f(y)=a\}} \neg\phi'\}$
  $=$
  $\{\bigwedge_{a\in Act} \bigwedge_{x\in\{y \mid f(y)=a\}} \neg\langle x\rangle\top\}$
  $=$
  $\{\bigwedge_{x\in\{y \mid f(y)\in Act\}} \neg\langle x\rangle\top\}$
  $=$ [$f(Act) \subseteq Act$]
  $\{\bigwedge_{a\in Act} \neg\langle a\rangle\top\}$
  $=$ [*Definition of* ϕ]
  $\{\phi\}$

  Using [1] and [2] we can conclude that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{CS}$.

□

2.3.11 Possible Worlds Semantics

The modal characterization of the possible worlds semantics is the following: $\mathscr{O}_{PW}\ \phi ::= \bigwedge_{i\in I}\langle a_i\rangle\phi_i\ (\phi_i \in \mathscr{O}_{PW})\ |\ \bigwedge_{i\in I}\neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J}\langle a_i\rangle\top$

Theorem 2.12. *The modal characterization of the possible worlds semantics satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{PW}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{PW}$. We apply induction on the structure of ϕ.

- $\phi \equiv \bigwedge_{i\in I}\langle a_i\rangle\phi_i$:

  We will prove that, for some I',
  $[1]\phi\{f^{-1}\} = \{\bigwedge_{i\in I'}\langle a_i\rangle\varphi_i \mid \forall i \in I' : \varphi_i \in \mathscr{O}_i \wedge \mathscr{O}_i \subseteq \mathscr{O}_{PW}\}$.

  $\phi\{f^{-1}\}$
  $= [Definition\ of\ \phi\{f^{-1}\}]$
  $\{\bigwedge_{i\in I}\varphi_i \mid \varphi_i \in (\langle a_i\rangle\phi_i)\{f^{-1}\}\}$
  $= [Definition\ of\ \phi\{f^{-1}\}]$
  $\{\bigwedge_{i\in I}\langle x\rangle\varphi_i \mid f(x) = a_i \wedge \varphi_i \in (\phi_i)\{f^{-1}\}\}$
  $= [with\ I' = \{i' \mid \exists i \in I : f(a_{i'}) = a_i\}\ and$
  $\forall i \in I' : \mathscr{O}_i = (\phi_i)\ \{f^{-1}\}\ s.t.\ f(a_{i'}) = a_i]$
  $\{\bigwedge_{i\in I'}\langle a_i\rangle\varphi_i \mid \forall i \in I' : \varphi_i \in \mathscr{O}_i\}$
  $= [Modal\ Characterization\ of\ \mathscr{O}_{PW}\ and\ Inductive\ Hypothesis]$
  $\{\bigwedge_{i\in I'}\langle a_i\rangle\varphi_i \mid \forall i \in I' : \varphi_i \in \mathscr{O}_i \wedge \mathscr{O}_i \subseteq \mathscr{O}_{PW}\}$.

  Using [1] and the Modal Characterization of $\mathscr{O}_{PW}$ we can conclude that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{PW}$.

- $\phi \equiv \bigwedge_{i\in I}\neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J}\langle a_j\rangle\top$:

  $\phi\{f^{-1}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in (\bigwedge_{i\in I}\neg\langle a_i\rangle\top)\{f^{-1}\} \wedge \varphi_j \in (\bigwedge_{j\in J}\langle a_j\rangle\top)\{f^{-1}\}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i\in I}\phi'_i \mid \phi'_i \in (\neg\langle a_i\rangle\top)\{f^{-1}\}\} \wedge$
  $\varphi_j \in \{\bigwedge_{j\in J}\phi'_j \mid \phi'_j \in (\langle a_j\rangle\top)\{f^{-1}\}\}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]
  $\{\varphi_i \wedge \varphi_j \mid \varphi_i \in \{\bigwedge_{i\in I}\bigwedge_{\phi'\in(\langle a_i\rangle\top)\{f^{-1}\}}\neg\phi'\} \wedge$
  $\varphi_j \in \{\bigwedge_{j\in J}\langle x\rangle\phi' \mid f(x) = a_j\ \wedge\ \phi' \in (\top)\{f^{-1}\}\}\}$
  $=$ [*Definition of* $\{f^{-1}\}$]

$\{\varphi_i \wedge \varphi_j \ |\varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{f(x)=a_i \ \wedge \ \phi' \in (\top)\{f^{-1}\}} \neg\langle x\rangle\phi'\} \ \wedge$
$\quad \varphi_j \in \{\bigwedge_{j\in J}\langle x\rangle\top \mid f(x) = a_j\}\}$
$= [\textit{Definition of } \{f^{-1}\}]$
$\{\varphi_i \wedge \varphi_j \ |\varphi_i \in \{\bigwedge_{i\in I} \bigwedge_{f(x)=a_i} \neg\langle x\rangle\top\} \ \wedge$
$\quad \varphi_j \in \{\bigwedge_{j\in J}\langle x\rangle\top \mid f(x) = a_j\}\}$
$= [\textit{with } I' = \{i' |\ \exists i \in I : f(a_{i'}) = a_i\}]$
$\{\bigwedge i \in I' \neg\langle a_i\rangle\top \wedge \varphi_j \mid \varphi_j \in \{\bigwedge_{j\in J}\langle x\rangle\top \mid f(x) = a_j\}\}$

We now have two cases:

- $\exists j \in J : a_j \notin f(Act)$:
  $\phi\{f^{-1}\} = \varnothing$
  $\Longrightarrow$
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{PW}$
- $\forall j \in J : a_j \in f(Act)$:
  φ_j is of the shape $\bigwedge_{j\in J'}\langle a_j\rangle\top$ for some J'
  $\Longrightarrow$
  $\forall\varphi \in \phi\{f^{-1}\} : \varphi$ is of the shape $\bigwedge_{i\in I'} \neg\langle a_i\rangle\top \wedge \bigwedge_{j\in J'}\langle a_j\rangle\top$ for some J'
  $\Longleftrightarrow$ [*Modal Characterization of* $\mathscr{O}_{PW}$]
  $\forall\varphi \in \phi\{f^{-1}\} : \varphi \in \mathscr{O}_{PW}$
  $\Longleftrightarrow$
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{PW}$

□

2.3.12 Ready Simulation

The modal characterization of the ready simulation is the following:
$\mathscr{O}_{RS}\ \phi ::= \langle a\rangle\phi' \ (\phi' \in \mathscr{O}_{RS}) \mid \neg\langle a\rangle\top \mid \ \bigwedge_{i\in I} \phi_i \ (\phi_i \in \mathscr{O}_{RS})$

Theorem 2.13. *The modal characterization of the ready simulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{RS}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{RS}$. We apply induction on the structure of ϕ.

- $\phi \equiv \langle x\rangle\varphi$:
  $\phi \in \mathscr{O}_{RS}$
  $\Longleftrightarrow$ [*Modal Characterization of* $\mathscr{O}_{RS}$]
  $\varphi \in \mathscr{O}_{RS}$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{RS}$

$\iff$

$\forall \varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathscr{O}_{RS}$

$\iff$ [*Modal Characterization of* $\mathscr{O}_{RS}$]

$\forall a \in Act : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a \rangle \varphi' \in \mathscr{O}_{RS}$

$\implies$

$\forall a \ s.t. \ f(a) = x : \forall \varphi' \in \varphi\{f^{-1}\} : \langle a \rangle \varphi' \in \mathscr{O}_{RS}$

$\iff$

$\phi\{f^{-1}\} \subseteq \mathscr{O}_{RS}$

- $\phi \equiv \neg\langle a \rangle \top$:

  $\phi \in \mathscr{O}_{RS}$

  $\implies$

  $\forall x \in Act : \neg\langle x \rangle \top \in \mathscr{O}_{RS}$

  $\implies$

  $\forall x : \ f(x) = a \implies \neg\langle x \rangle \top \in \mathscr{O}_{RS}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{RS}$]

  $\bigwedge\limits_{f(x)=a} \neg\langle x \rangle \top \in \mathscr{O}_{RS}$

  $\iff$ $[\phi\{f^{-1}\} = \{\bigwedge_{f(x)=a} \neg\langle x \rangle \top\}]$

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{RS}$

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:

  $\phi \in \mathscr{O}_{RS}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{RS}$]

  $\forall i \in I : \phi_i \in \mathscr{O}_{RS}$

  $\implies$ [*Inductive Hypothesis*]

  $\forall i \in I : \phi_i\{f^{-1}\} \subseteq \mathscr{O}_{RS}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{RS}$]

  $\forall \phi' \in \{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} : \phi' \in \mathscr{O}_{RS}$

  $\iff$ [*Definition of* $\{f^{-1}\}$]

  $\forall \phi' \in \phi\{f^{-1}\} : \phi' \in \mathscr{O}_{RS}$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{RS}$

□

2.3.13 n-Nested Simulation

The modal characterization of the n-nested simulation is the following:
$\mathcal{O}_{nS}\ \phi ::= \langle a\rangle\phi'\ (\phi' \in \mathcal{O}_{nS}) \mid \bigwedge_{i\in I}\phi_i\ (\phi_i \in \mathcal{O}_{nS}) \mid \neg\phi'\ (\phi' \in \mathcal{O}_{(n-1)S})$

Theorem 2.14. *The modal characterization of the n-nested simulation satisfies (R).*

Proof. Given $\phi \in \mathcal{O}_{nS}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathcal{O}_{nS}$. We apply induction on the structure of ϕ.

- $\phi \equiv \langle x\rangle\varphi$:

  $\phi \in \mathcal{O}_{nS}$

  $\iff$ [*Modal Characterization of* $\mathcal{O}_{nS}$]

  $\varphi \in \mathcal{O}_{nS}$

  $\implies$ [*Inductive Hypothesis*]

  $\varphi\{f^{-1}\} \subseteq \mathcal{O}_{nS}$

  $\iff$

  $\forall\varphi' \in \varphi\{f^{-1}\} : \varphi' \in \mathcal{O}_{nS}$

  $\iff$ [*Modal Characterization of* $\mathcal{O}_{nS}$]

  $\forall a \in Act : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathcal{O}_{nS}$

  $\implies$

  $\forall a\ s.t.\ f(a) = x : \forall\varphi' \in \varphi\{f^{-1}\} : \langle a\rangle\varphi' \in \mathcal{O}_{nS}$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathcal{O}_{nS}$

- $\phi \equiv \bigwedge_{i\in I}\phi_i$:

  $\phi \in \mathcal{O}_{nS}$

  $\iff$ [*Modal Characterization of* $\mathcal{O}_{nS}$]

  $\forall i \in I : \phi_i \in \mathcal{O}_{nS}$

  $\implies$ [*Inductive Hypothesis*]

  $\forall i \in I : \phi_i\{f^{-1}\} \subseteq \mathcal{O}_{nS}$

  $\iff$ [*Modal Characterization of* $\mathcal{O}_{nS}$]

  $\forall\phi' \in \{\bigwedge_{i\in I}\phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} : \phi' \in \mathcal{O}_{nS}$

  $\iff$ [*Definition of* $\{f^{-1}\}$]

  $\forall\phi' \in \phi\{f^{-1}\} : \phi' \in \mathcal{O}_{nS}$

  $\iff$

  $\phi\{f^{-1}\} \subseteq \mathcal{O}_{nS}$

- $\phi \equiv \neg\phi'$:

  $\phi \in \mathscr{O}_{nS}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{nS}$]

  $\phi' \in \mathscr{O}_{(n-1)S}$

  $\implies$ [*Inductive Hypothesis*]

  $\phi'\{f^{-1}\} \subseteq \mathscr{O}_{(n-1)S}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{nS}$]

  $\forall\varphi \in \{\neg\varphi' \mid \varphi' \in \phi'\{f^{-1}\}\} : \varphi \in \mathscr{O}_{nS}$

  $\iff$ [*Modal Characterization of* $\mathscr{O}_{nS}$]

  $\{\bigwedge_{\varphi' \in \phi'\{f^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_{nS}$

  $\iff$ [*Definition of* $\{f^{-1}\}$]

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{nS}$

$\square$

Chapter 3

Weak Semantics

In this chapter we cope with abstract semantics. As said in the first chapter, in abstract semantics we take into account the existence of the *silent move* τ.

All the operators presented both in section 2.1 and in section 2.2 are investigated in sections 3.1-3.5. Moreover, the abstraction operator is added to the collection (section 3.6). For all of them, we provide suitable constraints. Then, in section 3.7, we conduct a study, in a similar way to what we have done in the concrete case, on the existing semantics.

As in the concrete case, in order to achieve reliable and clear results, we present the HML syntax we use. Here, we need a more powerful syntax, which allows us to express the act of performing the τ-transition.

Definition 3.1. *In this section the Hennessy-Milner logic is extended with the modal connective $\langle\epsilon\rangle\varphi$. We say that $p \models \langle\epsilon\rangle\varphi$ if $p \stackrel{\epsilon}{\Rightarrow} p'$, i.e. it can perform zero or more τ-transitions to the state p', for some p' and $p' \models \varphi$. Therefore, the extended syntax is as follows:*

$$\varphi ::= \neg\varphi \mid \bigwedge_{i\in I}\varphi_i \mid \langle\alpha\rangle\varphi \mid \langle\epsilon\rangle\varphi$$

where I is an arbitrary index set and α ranges over the set $Act \cup \{\tau\}$ of actions. Again, we use $\top$, $\bot$ and $\varphi_1 \wedge \varphi_2$ with the same meaning as we did in the previous chapter.

With the introduction of the prefix $\langle\epsilon\rangle$, we have now two diamond operators. Exactly as the $\langle\alpha\rangle$ prefix, also this one contributes to increase the level of a context. In order to formalize this, we redefine the notion of n-level context:

Definition 3.2. *A n-level context, is defined as follows:*

- *$[]$ is a 0-level context.*
- *If $C_n[]$ is a n-level context, then $\neg C_n[]$ and $C_n[] \wedge \bigwedge_{i\in I}\varphi_i$ are n-level contexts.*
- *If $C_n[]$ is a n-level context, then $\langle\alpha\rangle C_n[]$ and $\langle\epsilon\rangle C_n[]$ are (n+1)-level contexts.*

3.1 Action Prefix Operator

Definition 3.3. *The action-prefix operator is written $\alpha.p$, where $\alpha \in Act \cup \{\tau\}$. The transition rule for this operator is:*

$$\frac{}{\alpha.p \xrightarrow{\alpha} p} \text{ (PREF)}$$

In order to obtain a sufficient congruence requirement, we use the following observation and define the *depth* function:

Lemma 3.1. *Given $\alpha, \alpha' \in Act \cup \{\tau\}$ and $a \in Act$ we have:*

1. $\alpha.p \models \langle\alpha'\rangle\varphi \iff p \models \varphi \wedge \alpha' = \alpha$
2. $\tau.p \models \langle\epsilon\rangle\varphi \iff p \models \langle\epsilon\rangle\varphi \ \vee \ \tau.p \models \varphi$
3. $a.p \models \langle\epsilon\rangle\varphi \iff a.p \models \varphi$

Definition 3.4. *Given a HML formula ϕ we define its depth, notation $depth(\phi)$, as follows:*

- $depth(\top) = 0$
- $depth(\neg\phi) = depth(\phi)$
- $depth(\bigwedge_{i\in I} \phi_i) = sup_{i\in I}(depth(\phi_i))$
- $depth(\langle\alpha\rangle\phi) = depth(\phi) + 1$
- $depth(\langle\epsilon\rangle\phi) = depth(\phi) + 1$

The result presented in Lemma 3.1 is a description of how a process $\alpha.p$ interacts with a formula starting with a diamond. This result is used, as well as the *depth* function, while stating the congruence requirements for the action prefix operator.

Theorem 3.1. *Let $\mathscr{O} \subseteq HML$. If for any $\phi \in HML$, for any $\alpha \in Act \cup \{\tau\}$:*

(AP1) $C[\langle\alpha\rangle\phi] \in \mathscr{O} \implies \phi \in \mathscr{O}^{\equiv}$
(AP2) $C[\langle\epsilon\rangle\phi] \in \mathscr{O} \implies \langle\epsilon\rangle\phi \in \mathscr{O}^{\equiv}$

then $\sim_{\mathscr{O}}$ is a congruence with respect to the action-prefix operator.

Proof. Assume a modal language $\mathscr{O}$ with the (AP1) and (AP2) properties. Let $p \sim_{\mathscr{O}} q$. We show that for any $\phi \in \mathscr{O}$

$$\alpha.p \models \phi \iff \alpha.q \models \phi$$

First of all, we prove that, given $\varphi \in HML$, if there is a context $C[]$ such that $C[\varphi] = \phi$, then $\alpha.p \models \varphi \iff \alpha.q \models \varphi$. We prove it using transfinite induction on the depth of the formula.

Basis Step: If $depth(\varphi) = 0$, it's sufficient to notice that either $\varphi \equiv \top$ or $\varphi \equiv \bot$. In both cases, we know that $\alpha.p \models \varphi \iff \alpha.q \models \varphi$.

Inductive Step: We can express φ as $C[\langle x_i \rangle \varphi_i]_{i \in I}$, with $x_i \in Act \cup \{\tau, \epsilon\}$, such that the $\langle x_i \rangle \phi_i$ are all action prefix sub-formulas that appear at level zero. Since $C[]_{i \in I}$ is built only by conjunction and negation, we know that $\forall i \in I : p \models \langle x_i \rangle \phi_i \iff q \models \langle x_i \rangle \phi_i$ implies $p \models \phi \iff q \models \phi$
So, we need to prove that $\forall i \in I : \alpha.p \models \langle x_i \rangle \phi_i \iff \alpha.q \models \langle x_i \rangle \phi_i$. We split the proof in two sub-cases:

- $x_i \in Act \cup \{\tau\}$:

  $\alpha.p \models \langle x_i \rangle \phi_i$

  $\iff$ [*Lemma 3.1.1*]

  $\alpha = x_i \wedge p \models \phi_i$

  $\iff$ [*(AP1): $\phi_i \in \mathscr{O}^{\equiv}$, $p \sim_{\mathscr{O}} q$*]

  $\alpha = x_i \wedge q \models \phi_i$

  $\iff$ [*Lemma 3.1.1*]

  $\alpha.q \models \langle x_i \rangle \phi_i$

- $x_i = \epsilon$:

  Here, we split in two sub-cases:

  $\alpha = \tau$**:**

  $\tau.p \models \langle \epsilon \rangle \phi_i$
  $\iff$ [*Lemma 3.1.2*]
  $p \models \langle \epsilon \rangle \phi_i \vee \tau.p \models \phi_i$
  $\iff$ [*(AP2): $\langle \epsilon \rangle \phi_i \in \mathscr{O}^{\equiv}$, $p \sim_{\mathscr{O}} q$*]
  $q \models \langle \epsilon \rangle \phi_i \vee \tau.p \models \phi_i$
  $\iff$ [*Inductive Hypothesis*]
  $q \models \langle \epsilon \rangle \phi_i \vee \tau.q \models \phi_i$
  $\iff$ [*Lemma 3.1.2*]
  $\tau.q \models \langle \epsilon \rangle \phi_i$

  $\alpha \in Act$**:**

  $\alpha.p \models \langle \epsilon \rangle \phi_i$
  $\iff$ [*Lemma 3.1.3*]
  $\alpha.p \models \phi_i$
  $\iff$ [*Inductive Hypothesis*]
  $\alpha.q \models \phi_i$
  $\iff$ [*Lemma 3.1.3*]
  $\alpha.q \models \langle \epsilon \rangle \phi_i$

It is easy to see that there is a context such that $C[\phi] = \phi$, namely $C[] = []$. So we derive that $\alpha.p \models \phi \iff \alpha.q \models \phi$. □

3.2 Alternative Composition Operator

Definition 3.5. *The alternative composition operator is written $p + q$. It expresses the non-deterministic choice between two processes. The transition rules for this operator are:*

$$\frac{p \xrightarrow{\alpha} p'}{p + q \xrightarrow{\alpha} p'}\ \text{(ALT1)} \qquad\qquad \frac{q \xrightarrow{\alpha} q'}{p + q \xrightarrow{\alpha} q'}\ \text{(ALT2)}$$

In order to obtain a sufficient congruence requirement for this operator, we use the following observations:

Lemma 3.2. $p + q \models \langle\alpha\rangle\varphi \iff p \models \langle\alpha\rangle\varphi \vee q \models \langle\alpha\rangle\varphi$*, with* $\alpha \in Act \cup \{\tau\}$.

Lemma 3.3. $p \models \langle\epsilon\rangle\varphi \iff p \models \varphi \vee p \models \langle\tau\rangle\langle\epsilon\rangle\varphi$.

Lemma 3.4. $p + q \models \langle\epsilon\rangle\langle\alpha\rangle\varphi \iff p \models \langle\epsilon\rangle\langle\alpha\rangle\varphi \vee q \models \langle\epsilon\rangle\langle\alpha\rangle\varphi$*, with* $\alpha \in Act \cup \{\tau\}$.

With these simple equivalences in mind, we proceed to present the congruence requirements for the alternative composition operator.

Theorem 3.2. *Let* $\mathscr{O} \subseteq HML$*. If for any* $\phi \in HML$

(AC1) $\bigwedge_{i\in I} \varphi_i \in \mathscr{O} \implies \forall i \in I : \varphi_i \in \mathscr{O}^{\equiv}$
(AC2) $\neg\varphi \in \mathscr{O} \implies \varphi \in \mathscr{O}^{\equiv}$
(AC3) $\langle\epsilon\rangle\varphi \in \mathscr{O} \implies \varphi \equiv \langle\alpha\rangle\varphi' \vee (\langle\tau\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}^{\equiv} \wedge \varphi \in \mathscr{O}^{\equiv})$

then $\sim_{\mathscr{O}}$ *is a congruence with respect to the alternative composition operator.*

Proof. Assume a modal language $\mathscr{O}$ with the (AC1), (AC2) and (AC3) properties. Let $p \sim_{\mathscr{O}} p'$ and $q \sim_{\mathscr{O}} q'$. We show that for any $\phi \in \mathscr{O}$

$$p + q \models \phi \iff p' + q' \models \phi$$

We use induction on the structure of ϕ.

- $\phi = \bigwedge_{i\in I} \phi_i$:

  $p + q \models \phi$

  $\iff$

  $\forall i \in I : p + q \models \phi_i$

  $\iff$ [*(AC1):* $\phi_i \in \mathscr{O}^{\equiv}$*, Inductive Hypothesis*]

  $\forall i \in I : p' + q' \models \phi_i$

  $\iff$

  $p' + q' \models \phi$

- $\phi = \neg\phi'$:
  $p + q \models \phi$
  $\iff$
  $p + q \not\models \phi'$
  $\iff$ [*(AC2): $\phi' \in \mathscr{O}^{\equiv}$, Inductive Hypothesis*]
  $p' + q' \not\models \phi'$
  $\iff$
  $p' + q' \models \phi$
- $\phi = \langle\alpha\rangle\phi'$:
  $p + q \models \phi$
  $\iff$ [*Lemma 3.2*]
  $p \models \phi \vee q \models \phi$
  $\iff$ [$p \sim_{\mathscr{O}} p'$ and $q \sim_{\mathscr{O}} q'$]
  $p' \models \phi \vee q' \models \phi$
  $\iff$ [*Lemma 3.2*]
  $p' + q' \models \phi$
- $\phi = \langle\epsilon\rangle\phi'$ *with* $\phi' = \langle\alpha\rangle\varphi'$:
  $p + q \models \phi$
  $\iff$
  $p + q \models \langle\epsilon\rangle\langle\alpha\rangle\varphi'$
  $\iff$ [*Lemma 3.4*]
  $p \models \langle\epsilon\rangle\langle\alpha\rangle\varphi' \vee q \models \langle\epsilon\rangle\langle\alpha\rangle\varphi'$
  $\iff$ [$p \sim_{\mathscr{O}} p'$ and $q \sim_{\mathscr{O}} q'$]
  $p' \models \langle\epsilon\rangle\langle\alpha\rangle\varphi' \vee q' \models \langle\epsilon\rangle\langle\alpha\rangle\varphi'$
  $\iff$ [*Lemma 3.4*]
  $p' + q' \models \langle\epsilon\rangle\langle\alpha\rangle\varphi'$
  $\iff$
  $p' + q' \models \phi$
- $\phi = \langle\epsilon\rangle\phi'$ *with* $\phi' \neq \langle\alpha\rangle\varphi'$:
  $p + q \models \phi$
  $\iff$
  $p + q \models \langle\epsilon\rangle\phi'$
  $\iff$ [*Lemma 3.3*]
  $p + q \models \phi' \vee p + q \models \langle\tau\rangle\langle\epsilon\rangle\phi'$
  Now, we split the proof in two sub-cases:

$p + q \models \phi'$**:**

$\iff$ [*(AC3), Inductive Hypothesis*]

$p' + q' \models \phi'$

$p + q \models \langle\tau\rangle\langle\epsilon\rangle\phi'$**:**

$\iff$ [*Lemma 3.2*]

$p \models \langle\tau\rangle\langle\epsilon\rangle\phi' \vee q \models \langle\tau\rangle\langle\epsilon\rangle\phi'$

$\iff$ [*(AC3),* $p \sim_{\mathscr{O}} p' \wedge q \sim_{\mathscr{O}} q'$]

$p' \models \langle\tau\rangle\langle\epsilon\rangle\phi' \vee q' \models \langle\tau\rangle\langle\epsilon\rangle\phi'$

$\iff$ [*Lemma 3.2*]

$p' + q' \models \langle\tau\rangle\langle\epsilon\rangle\phi'$

So, we have:

$\iff$

$p' + q' \models \phi' \vee p' + q' \models \langle\tau\rangle\langle\epsilon\rangle\phi'$

$\iff$ [*Lemma 3.3*]

$p' + q' \models \langle\epsilon\rangle\phi'$

$\iff$

$p' + q' \models \phi$

□

3.3 Restriction Operators

Now we consider two restriction operators, namely projection and encapsulation.

Definition 3.6. *The nth projection operator is written $\pi_n(p)$. It expresses a boundary on the number of transitions that can be performed by the process p. The transition rules for this operator are:*

$$\frac{p \xrightarrow{a} p' \;\; (a \in Act)}{\pi_{n+1}(p) \xrightarrow{a} \pi_n(p')} \text{ (PROJ1)} \qquad\qquad \frac{p \xrightarrow{\tau} p'}{\pi_n(p) \xrightarrow{\tau} \pi_n(p')} \text{ (PROJ2)}$$

Definition 3.7. *The encapsulation operator is written $\partial_B(p)$, where $B \subseteq Act$. It expresses a limitation on the transitions that can be performed by the process p. The transition rule for this operator is:*

$$\frac{p \xrightarrow{a} p' \;\; (a \notin B)}{\partial_B(p) \xrightarrow{a} \partial_B(p')} \text{ (ENC)}$$

In order to find sufficient congruence requirements for these operators, we use the following two lemmas from [10].

Lemma 3.5. *Let f be a unary process operator. Suppose there exists a cutting function $cut_f : HML \to HML$ such that for any process p and any $\phi \in HML$:*

$$(CUT) \quad f(p) \models \phi \iff p \models cut_f(\phi)$$

Then, for any language $\mathscr{O}$ satisfying

$$\phi \in \mathscr{O} \implies (cut_f(\phi) \in \mathscr{O}^{\equiv} \vee cut_f(\phi) \equiv \bot)$$

the corresponding equivalence $\sim_{\mathscr{O}}$ is a congruence with respect to f.

Proof. Suppose (CUT) holds on $\mathscr{O}$ and $p \sim_{\mathscr{O}} q$. Given $\phi \in \mathscr{O}$, we have:
$f(p) \models \phi$
$\iff$ [(CUT)]
$p \models cut_f(\phi)$
$\iff$ [*Either* $cut_f(\phi) \equiv \bot$ *or* $cut_f(\phi) \in \mathscr{O} \wedge p \sim_{\mathscr{O}} q$]
$q \models cut_f(\phi)$
$\iff$ [(CUT)]
$f(q) \models \phi$ □

Lemma 3.6. *Let f and cut_f be as in Lemma 3.5, satisfying (CUT). Suppose that for each $\phi \in HML$, there exists a multi-context $C[]_{i\in I}$ such that $\phi = C[\langle a_i\rangle\phi_i]_{i\in I}$ and $cut_f(\phi) \equiv cut_f(C[\bot]_{i\in I})$. Then, for each language $\mathscr{O} \subseteq HML$, that satisfies for any context $C'[]$ and for any $\phi \in HML$,*

$$\text{(RES) } C'[\neg\phi] \in \mathscr{O} \implies C'[\top] \in O^{\equiv},$$

the corresponding equivalence $\sim_{\mathscr{O}}$ is a congruence with respect to f.

Proof. By Lemma 3.5, it's sufficient to prove that for all $\phi \in \mathscr{O}$, either $cut_f(\phi) \equiv \bot$ or $cut_f(\phi) \in \mathscr{O}^{\equiv}$. Take any $\phi \in \mathscr{O}$ such that $cut_f(\phi) \not\equiv \bot$. By assumption $cut_f(\phi) \equiv C[\bot]_{i\in I}$ for some $C[]_{i\in I}$. Since $cut_f(\phi) \not\equiv \bot$, each occurrence of $\bot$ must be in the scope of a negation symbol. Hence $cut_f(\phi) \equiv C'[\neg D^i[\bot]]_{i\in I}$ where we can choose $D^i[]$ for all $i \in I$ such that $[]$ is not within the scope of a negation. Then $C'[\neg D^i[\bot]]_{i\in I} \equiv C'[\top]_{i\in I}$. Since $\mathscr{O}$ satisfies (RES), $C'[\top]_{i\in I} \in \mathscr{O}^{\equiv}$. Hence $cut_f(\phi) \in \mathscr{O}^{\equiv}$. □

A natural step in our process, now, is to present suitable cutting functions (i.e. cutting functions that respect (CUT)) for the projection and encapsulation operators.

Definition 3.8. *The cutting function cut_n for the projection operator π_n is defined as follows:*

- $cut_n(\bigwedge_{i\in I} \phi_i) = \bigwedge_{i\in I} cut_n(\phi_i)$
- $cut_n(\neg\phi) = \neg cut_n(\phi)$
- $cut_{n+1}(\langle a\rangle\phi) = \langle a\rangle cut_n(\phi)$ $(a \in Act)$

- $cut_0(\langle a \rangle \phi) = \bot$
- $cut_n(\langle \tau \rangle \phi) = \langle \tau \rangle cut_n(\phi)$
- $cut_n(\langle \epsilon \rangle \phi) = \langle \epsilon \rangle cut_n(\phi)$

The cutting function cut_B for the projection operator ∂_B is defined as follows:

- $cut_B(\bigwedge_{i \in I} \phi_i) = \bigwedge_{i \in I} cut_B(\phi_i)$
- $cut_B(\neg \phi) = \neg cut_B(\phi)$
- $cut_B(\langle \alpha \rangle \phi) = \langle \alpha \rangle cut_B(\phi)$ $(\alpha \notin B)$
- $cut_B(\langle \alpha \rangle \phi) = \bot$ $(\alpha \in B)$
- $cut_B(\langle \epsilon \rangle \phi) = \langle \epsilon \rangle cut_B(\phi)$

Lemma 3.7. *The cutting functions cut_n respects (CUT)*

Proof. Given a process p and a formula $\phi \in HML$, we want to prove that

$$\pi_n(p) \models \phi \iff p \models cut_n(\phi)$$

We apply induction on the structure of ϕ.

- $\phi = \bigwedge_{i \in I} \phi_i$:
  $\pi_n(p) \models \phi$
  $\iff$
  $\forall i \in I : \pi_n(p) \models \phi_i$
  $\iff$ [*Inductive Hypothesis*]
  $\forall i \in I : p \models cut_n(\phi_i)$
  $\iff$
  $p \models \bigwedge_{i \in I} cut_n(\phi_i)$
  $\iff$ [*Definition of cut_n*]
  $p \models cut_n(\phi)$
- $\phi = \neg \phi'$:
  $\pi_n(p) \models \phi$
  $\iff$
  $\pi_n(p) \not\models \phi'$
  $\iff$ [*Inductive Hypothesis*]
  $p \not\models cut_n(\phi')$
  $\iff$
  $p \models \neg cut_n(\phi')$
  $\iff$ [*Definition of cut_n*]
  $p \models cut_n(\phi)$

- $\phi = \langle\tau\rangle\phi'$:

  $\pi_n(p) \models \phi$

  $\iff$

  $\pi_n(p) \xrightarrow{\tau} \pi_n(p') \wedge \pi_n(p') \models \phi'$

  $\iff$ [*Inductive Hypothesis,* (PROJ2)]

  $p \xrightarrow{\tau} p' \wedge p' \models cut_n(\phi')$

  $\iff$

  $p \models \langle\tau\rangle cut_n(\phi')$

  $\iff$ [*Definition of* cut_n]

  $p \models cut_n(\phi)$

- $\phi = \langle a\rangle\phi'$ $(a \in Act)$

  We want to prove $\pi_n(p) \models \phi \iff p \models cut_n(\phi)$. We apply mathematical induction over n:

  n = 0:

  $\pi_0(p) \models \phi$

  $\iff$

  $\bot$

  $\iff$

  $p \models \bot$

  $\iff$ [*Definition of* cut_n]

  $p \models cut_0(\phi)$

  n = k + 1

  $\pi_{k+1}(p) \models \phi$

  $\iff$

  $\pi_{k+1}(p) \xrightarrow{a} \pi_k(p') \wedge \pi_k(p') \models \phi'$

  $\iff$ [*Inductive Hypothesis,* (PROJ1)]

  $p \xrightarrow{a} p' \wedge p' \models cut_k(\phi')$

  $\iff$

  $p \models \langle a\rangle cut_k(\phi')$

  $\iff$ [*Definition of* cut_n]

  $p \models cut_{k+1}(\phi)$

- $\phi = \langle\epsilon\rangle\phi'$:

  $\pi_n(p) \models \phi$

  $\iff$

  $\pi_n(p) \stackrel{\epsilon}{\Rightarrow} \pi_n(p') \wedge \pi_n(p') \models \phi'$

  $\iff$ [*Definition of* $\stackrel{\epsilon}{\Rightarrow}$]

$\exists m \in \mathbb{N} : \forall i \in [0, m) : \pi_n(p_i) \xrightarrow{\tau} \pi_n(p_{i+1})$
$\land\, p_0 = p \land p_m = p' \land \pi_n(p') \models \phi'$

$\iff$ [*Inductive Hypothesis,* (PROJ2)]

$\exists m \in \mathbb{N} : \forall i \in [0, m) : p_i \xrightarrow{\tau} p_{i+1}$
$\land\, p_0 = p \land p_m = p' \land p' \models cut_n(\phi')$

$\iff$ [*Definition of* $\overset{\epsilon}{\Rightarrow}$]

$p \overset{\epsilon}{\Rightarrow} p' \land p' \models cut_n(\phi')$

$\iff$

$p \models \langle\epsilon\rangle cut_n(\phi')$

$\iff$ [*Definition of* cut_n]

$p \models cut_n(\phi)$

□

Lemma 3.8. *The cutting functions* cut_B *respects (CUT)*

Proof. Given a process p and a formula $\phi \in HML$, we want to prove that

$$\partial_B(p) \models \phi \iff p \models cut_B(\phi)$$

We apply induction on the structure of ϕ.

- $\phi = \bigwedge_{i \in I} \phi_i$:

  $\partial_B(p) \models \phi$

  $\iff$

  $\forall i \in I : \partial_B(p) \models \phi_i$

  $\iff$ [*Inductive Hypothesis*]

  $\forall i \in I : p \models cut_B(\phi_i)$

  $\iff$ [*Definition of* cut_B]

  $p \models cut_B(\phi)$

- $\phi = \neg\phi'$:

  $\partial_B(p) \models \phi$

  $\iff$

  $\partial_B(p) \not\models \phi'$

  $\iff$ [*Inductive Hypothesis*]

  $p \not\models cut_B(\phi')$

  $\iff$

  $p \models \neg cut_B(\phi')$

  $\iff$ [*Definition of* cut_B]

  $p \models cut_B(\phi)$

- $\phi = \langle\alpha\rangle\phi'$ $(\alpha \in B)$:

  $\partial_B(p) \models \phi$

  $\iff$ [ENC]

  $\bot$

  $\iff$

  $p \models \bot$

  $\iff$ [*Definition of* cut_b]

  $p \models cut_B(\phi)$

- $\phi = \langle\alpha\rangle\phi'$ $(\alpha \notin B)$:

  $\partial_B(p) \models \phi$

  $\iff$

  $\partial_B(p) \xrightarrow{\alpha} \partial_B(p') \wedge \partial_b(p') \models \phi'$

  $\iff$ [*Inductive Hypothesis,* (ENC)]

  $p \xrightarrow{\alpha} p' \wedge p' \models cut_B(\phi')$

  $\iff$

  $p \models \langle\alpha\rangle cut_B(\phi')$

  $\iff$ [*Definition of* cut_B]

  $p \models cut_B(\phi)$

- $\phi = \langle\epsilon\rangle\phi'$:

  $\partial_B(p) \models \phi$

  $\iff$

  $\partial_B(p) \overset{\epsilon}{\Rightarrow} \partial_B(p') \wedge \partial_B(p') \models \phi'$

  $\iff$ [*Definition of* $\overset{\epsilon}{\Rightarrow}$]

  $\exists n \in \mathbb{N} : \forall i \in [0, n) : \partial_B(p_i) \xrightarrow{\tau} \partial_B(p_{i+1})$
  $\wedge\, p_0 = p \wedge p_m = p' \wedge d_B(p') \models \phi'$

  $\iff$ [*Inductive Hypothesis,* (ENC)]

  $\exists n \in \mathbb{N} : \forall i \in [0, n) : p_i \xrightarrow{\tau} p_{i+1}$
  $\wedge\, p_0 = p \wedge p_m = p' \wedge p' \models cut_B(\phi')$

  $\iff$ [*Definition of* $\overset{\epsilon}{\Rightarrow}$]

  $p \overset{\epsilon}{\Rightarrow} p' \wedge p' \models cut_B(\phi')$

  $\iff$

  $p \models \langle\epsilon\rangle cut_B(\phi')$

  $\iff$ [*Definition of* cut_B]

  $p \models cut_B(\phi)$

□

All we need to do, at this point, is to combine the results we obtained. In the following theorem, we do this, presenting the resulting congruence requirements.

Theorem 3.3. *Let $\mathscr{O} \subseteq HML$. If $\mathscr{O}$ satisfies (RES), then $\sim_{\mathscr{O}}$ is a congruence with respect to the projection operators π_n and the encapsulation operator ∂_B.*

Proof. From Lemma 3.7 and Lemma 3.8, cut_n and cut_B respect (CUT). Moreover, in the way they are defined, they only replace certain sub-formulas of the shape $\langle a \rangle \phi$ of the original formula with $\bot$, so they meet the requirements of Lemma 3.6. Congruence is thus an immediate consequence of Lemma 3.6. □

3.4 Parallel Composition Operator

Definition 3.9. *The parallel composition operator (without communication) is written $p||q$. Its behaviour consists of all possible interleavings of the component processes. This operator behaves as $p||\_q + q||\_p$, where the left-merge operator is defined as follows:*

$$\frac{p \xrightarrow{\alpha} p'}{p||\_q \xrightarrow{\alpha} p'||q} \text{ (LMRG)}$$

Since the behaviour of this operator consists of all possible interleavings of two processes, the first thing that we can observe is that if $p||q \models \phi$, then probably there should be two formulas ϕ_p, ϕ_q such that $p \models \phi_p$ and $q \models \phi_q$ and that the formula ϕ can be, in some way, obtained combining ϕ_p and ϕ_q. A way to formalize this intuition, is to think about two operators, the first one that allows us to go from a formula to its sub-formulas, and a second one that allows us to build a formula starting from two sub-formulas. These ideas are formalized in an effective way in [10] for concrete semantics. Here, we formalize these concepts for the abstract semantics.

Definition 3.10. *We define the set of sub-formulas of a given formula ϕ, notation $Sub(\phi)$, as follows:*

- $Sub(\neg\phi) = \{\neg\phi' \mid \phi' \in Sub(\phi)\}$
- $Sub(\bigwedge_{i\in I} \phi_i) = \bigcup_{i\in I} Sub(\phi_i) \cup \{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in Sub(\phi_i)\}$
- $Sub(\langle\alpha\rangle\phi) = Sub(\phi) \cup \{\langle\alpha\rangle\phi' \mid \phi' \in Sub(\phi)\}$
- $Sub(\langle\epsilon\rangle\phi) = \{\langle\epsilon\rangle\phi' \mid \phi' \in Sub(\phi)\}$

Definition 3.11. *Given $A, B \subseteq HML$, we now define*

$$Par : 2^{HML} \times 2^{HML} \to 2^{HML}$$

using induction as follows:

- $\neg\phi \in Par(A,B) \overset{def}{\iff} \forall C,D \subseteq Sub(\phi) : \phi \in Par(C,D) \implies (\exists\varphi_c \in C : \neg\varphi_c \in A^{\Rightarrow}) \vee (\exists\varphi_d \in D : \neg\varphi_d \in B^{\Rightarrow})$
- $\bigwedge_{i\in I}\phi_i \in Par(A,B) \overset{def}{\iff} \forall i \in I : \phi_i \in Par(A,B)$
- $\langle\alpha\rangle\phi \in Par(A,B) \overset{def}{\iff}$
  $\exists A' : (\forall\phi_{A'} \in A' : \langle\alpha\rangle\phi_{A'} \in A^{\Rightarrow}) \wedge (\phi \in Par(A',B)) \vee$
  $\exists B' : (\forall\phi_{B'} \in B' : \langle\alpha\rangle\phi_{B'} \in B^{\Rightarrow}) \wedge (\phi \in Par(A,B'))$
- $\langle\epsilon\rangle\phi \in Par(A,B) \overset{def}{\iff} \exists A' : (\forall\phi_{A'} \in A' : \langle\epsilon\rangle\phi_{A'} \in A^{\Rightarrow}) \wedge$
  $\exists B' : (\forall\phi_{B'} \in B' : \langle\epsilon\rangle\phi_{B'} \in B^{\Rightarrow}) \wedge \phi \in Par(A',B')$

where, given $\mathscr{O} \subseteq HML$*, we use the notation* $\mathscr{O}^{\Rightarrow}$ *to denote the set* $\{\phi \mid \forall p : ((\forall\varphi \in \mathscr{O} : p \models \varphi) \implies p \models \phi)$.

In the following results, by abuse of notation, we will use A to denote either the set $A \subseteq HML$ or the formula $\bigwedge_{\phi_A\in A}\phi_A \in HML$. Namely, given $A \subseteq HML$, we will use the notation $p \models A$ to express $p \models \bigwedge_{\phi_A\in A}\phi_A$.

Lemma 3.9. *Given two processes* p *and* q *and a formula* $\phi \in HML$*, we have*

$$p||q \models \phi \implies \exists A,B \subseteq Sub(\phi) : (p \models A \wedge q \models B \wedge \phi \in Par(A,B))$$

Proof. We apply induction on the structure of the formula.

- $\phi \equiv \neg\varphi$:
  $p||q \models \phi$
  $\iff$
  $\neg(p||q \models \varphi)$
  $\implies$ [*Inductive Hypothesis*]
  $\neg(\exists C,D \subseteq Sub(\varphi) : (p \models C \wedge q \models D \wedge \varphi \in Par(C,D)))$
  $\iff$
  $\forall C,D \subseteq Sub(\varphi) : \varphi \in Par(C,D) \implies (\exists\phi_C \in C : p \not\models \phi_C) \vee (\exists\phi_D \in D : q \not\models \phi_D)$
  We take:
  $A = \bigcup_{C,D\subseteq Sub(\varphi):\varphi\in Par(C,D)}\{\neg\varphi_C | \varphi_C \in C \wedge p \not\models \varphi_C\}$
  $B = \bigcup_{C,D\subseteq Sub(\varphi):\varphi\in Par(C,D)}\{\neg\varphi_D | \varphi_D \in D \wedge q \not\models \varphi_D\}$
- $\phi \equiv \bigwedge_{i\in I}\phi_i$:
  $p||q \models \phi$
  $\iff$
  $\forall i \in I : p||q \models \phi_i$
  $\implies$ [*Inductive Hypothesis*]

$\forall i \in I : \exists A_i, B_i \subseteq Sub(\phi_i) : (p \models A_i \wedge q \models B_i \wedge \phi_i \in Par(A_i, B_i))$
We take:
$A = \bigcup_{i \in I} A_i$
$B = \bigcup_{i \in I} B_i$

- $\phi \equiv \langle\alpha\rangle\varphi$
  $p||q \models \phi$
  $\iff [p||q = p||\_q + q||\_p]$
  $(p \xrightarrow{\alpha} p' \wedge p'||q \models \varphi) \vee (q \xrightarrow{\alpha} q' \wedge p||q' \models \varphi)$
  Without loss of generality, we assume $p \xrightarrow{\alpha} p' \wedge p'||q \models \varphi$. The case $q \xrightarrow{\alpha} q' \wedge p||q' \models \varphi$ is symmetric.
  $p \xrightarrow{\alpha} p' \wedge p'||q \models \varphi$
  $\implies$ [*Inductive Hypothesis*]
  $p \xrightarrow{\alpha} p' \wedge \exists A', B' \subseteq Sub(\varphi) : (p' \models A' \wedge q \models B' \wedge \varphi \in Par(A', B'))$
  We take:
  $A = \{\langle\alpha\rangle\varphi \mid \varphi \in A'\}$
  $B = B'$

- $\phi \equiv \langle\epsilon\rangle\varphi$
  $p||q \models \phi$
  $\iff$
  $p \overset{\epsilon}{\Rightarrow} p' \wedge q \overset{\epsilon}{\Rightarrow} q' \wedge p'||q' \models \varphi$
  $\implies$ [*Inductive Hypothesis*]
  $p \overset{\epsilon}{\Rightarrow} p' \wedge q \overset{\epsilon}{\Rightarrow} q' \wedge \exists A', B' \subseteq Sub(\varphi) : (p' \models A' \wedge q' \models B' \wedge \varphi \in Par(A', B'))$
  We take:
  $A = \{\langle\epsilon\rangle\varphi \mid \varphi \in A'\}$
  $B = \{\langle\epsilon\rangle\varphi \mid \varphi \in B'\}$

□

Lemma 3.10. *Given two processes p and q and a formula $\phi \in HML$, we have*

$$\exists A, B : (p \models A \wedge q \models B \wedge \phi \in Par(A, B)) \implies p||q \models \phi$$

Proof. We apply induction on the structure of the formula.

- $\phi \equiv \neg\varphi$:
  From $\neg\varphi \in Par(A, B)$ we have:
  $\forall C, D \subseteq Sub(\varphi) : \varphi \in Par(C, D) \implies$
  $(\exists \varphi_c \in C : \neg\varphi_c \in A^{\Rightarrow}) \vee (\exists \varphi_d \in D : \neg\varphi_d \in B^{\Rightarrow})$

Let's suppose that $p||q \models \varphi$. From Lemma 3.9, there exist $C, D \subseteq Sub(\varphi)$ such that $p \models C$, $q \models D$ and $\varphi \in Par(C, D)$. But, from the earlier remark, we have either $\exists \varphi_c \in C : \neg\varphi_c \in A^{\Rightarrow}$ or $\exists \varphi_d \in D : \neg\varphi_d \in B^{\Rightarrow}$. This contradicts the hypothesis $p \models A \wedge q \models B$.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:

  From $\bigwedge_{i \in I} \phi_i \in Par(A, B)$ we have that $\forall i \in I : \phi_i \in Par(A, B)$.

  Using the Inductive Hypothesis, we know $\forall i \in I : p||q \models \phi_i$. This implies $p||q \models \bigwedge_{i \in I} \phi_i$.

- $\phi \equiv \langle\alpha\rangle\varphi$:

  From $\langle\alpha\rangle\varphi \in Par(A, B)$ we have that either
  $\exists A' : (\forall \varphi_{A'} \in A' : \langle\alpha\rangle\varphi_{A'} \in A^{\Rightarrow}) \wedge (\varphi \in Par(A', B))$ or
  $\exists B' : (\forall \varphi_{B'} \in B' : \langle\alpha\rangle\varphi_{B'} \in B^{\Rightarrow}) \wedge (\varphi \in Par(A, B'))$.

  Without loss of generality, we assume
  $\exists A' : (\forall \varphi_{A'} \in A' : \langle\alpha\rangle\varphi_{A'} \in A^{\Rightarrow}) \wedge (\varphi \in Par(A', B))$. The case
  $\exists B' : (\forall \varphi_{B'} \in B' : \langle\alpha\rangle\varphi_{B'} \in B^{\Rightarrow}) \wedge (\varphi \in Par(A, B'))$ is symmetric.

  From $p \models A$, we have $p \xrightarrow{\alpha} p' \wedge p' \models A'$. Using this and $\varphi \in Par(A', B)$, we apply the Inductive Hypothesis obtaining $p'||q \models \varphi$. Since $p||q \xrightarrow{\alpha} p'||q$, we have $p||q \models \langle\alpha\rangle\varphi$.

- $\phi \equiv \langle\epsilon\rangle\varphi$:

  From $\langle\epsilon\rangle\varphi \in Par(A, B)$ we have: $\exists A' : (\forall \varphi_{A'} \in A' : \langle\epsilon\rangle\varphi_{A'} \in A^{\Rightarrow}) \wedge$
  $\exists B' : (\forall \varphi_{B'} \in B' : \langle\epsilon\rangle\varphi_{B'} \in B^{\Rightarrow}) \wedge \varphi \in Par(A', B')$

  From $p \models A$ and $q \models B$, we have $p \overset{\epsilon}{\Rightarrow} p' \wedge p' \models A'$ and $q \overset{\epsilon}{\Rightarrow} q' \wedge q' \models B'$. Using this and $\varphi \in Par(A', B')$, we apply the Inductive Hypothesis obtaining $p'||q' \models \varphi$. Since $p||q \overset{\epsilon}{\Rightarrow} p'||q'$, we have $p||q \models \langle\epsilon\rangle\varphi$.

□

Using the results we just proved, we can now prove that a sufficient congruence requirement is that of containing all the sub-formulas of the contained formulas.

Theorem 3.4. *Let $\mathscr{O} \subseteq HML$. If for any $\phi \in HML$*

$$\text{(P) } \phi \in \mathscr{O} \implies Sub(\phi) \subseteq \mathscr{O}^{\equiv}$$

then $\sim_{\mathscr{O}}$ is a congruence with respect to the abstraction operator.

Proof. Assume a modal language $\mathscr{O}$ with the (P) property. Let $p \sim_{\mathscr{O}} p'$ and $q \sim_{\mathscr{O}} q'$. We show that for any $\phi \in \mathscr{O}$

$$p||q \models \phi \iff p'||q' \models \phi$$

We will prove $p||q \models \phi \implies p'||q' \models \phi$, the converse implication is symmetric.

$p||q \models \phi$
$\implies$ [*Lemma 3.9*]
$\exists A, B \subseteq Sub(\phi) : (p \models A \wedge q \models B \wedge \phi \in Par(A, B))$
$\implies$ [(P) *and* $p \sim_{\mathscr{O}} p' \wedge q \sim_{\mathscr{O}} q'$]
$\exists A, B : (p' \models A \wedge q' \models B \wedge \phi \in Par(A, B))$
$\implies$ [*Lemma 3.10*]
$p'||q' \models \phi$ □

3.5 Relabelling Operator

The relabelling operator for abstract processes, is defined in a similar way as its concrete version. Again, it's written $p[f]$, but in this case the relabelling function $f : Act \cup \{\tau\} \to Act \cup \{\tau\}$ is made up from two sub-functions: its concrete component, let's call it $\overline{f} : Act \to Act$, and its abstract part, which is basically an identity function $Id : \{\tau\} \to \{\tau\}$. We have then:

$$f(\alpha) = \begin{cases} \overline{f}(\alpha), & \text{if } \alpha \in Act \\ \tau, & \text{if } \alpha = \tau \end{cases}$$

Using this new definition of the relabelling function, the transition rule for the operator stays the same as the concrete case. We need to update some results in order to cope with the modal connective $\langle\epsilon\rangle$ that we introduced in our HML.

Definition 3.12. *The inverse relabelling operator for HML is written* $\phi\{f^{-1}\}$ *where the function* f *is a relabelling function. We keep the behaviour defined as in Definition 2.2 for* $(\neg\phi)\{f^{-1}\}$, $(\bigwedge_{i\in I}\phi_i)\{f^{-1}\}$ *and* $(\langle x\rangle\phi)\{f^{-1}\}$ *and we extend it as follows:*

- $(\langle\epsilon\rangle\phi)\{f^{-1}\} = \{\langle\epsilon\rangle\phi' | \phi' \in \phi\{f^{-1}\}\}$

Lemma 3.11. $p[f] \models \phi \iff \exists\phi' \in \phi\{f^{-1}\} : p \models \phi'$

Proof. We apply induction on the structure of ϕ. The cases $\phi \equiv \neg\varphi$, $\phi \equiv \bigwedge_{i\in I}\phi_i$, $\phi \equiv \langle x\rangle\varphi$ are already covered in the proof of Lemma 2.1.

- $\phi \equiv \langle\epsilon\rangle\varphi$:

  $p[f] \models \phi$

  $\iff$

  $p[f] \stackrel{\epsilon}{\Rightarrow} p'[f] \wedge p'[f] \models \varphi$

  $\iff$ [*Inductive Hypothesis*]

  $p[f] \stackrel{\epsilon}{\Rightarrow} p'[f] \wedge \exists\varphi' \in \varphi\{f^{-1}\} : p' \models \varphi'$

  $\iff$ [*Definition of* $\stackrel{\epsilon}{\Rightarrow}$]

  $\exists n \in \mathbb{N} : \forall i \in [0, n) : p_i[f] \xrightarrow{\tau} p[f]_{i+1} \wedge p_0 = p \wedge p_n = p'$
  $\wedge\ \exists\varphi' \in \varphi\{f^{-1}\} : p' \models \varphi'$

$\iff$ [(REL) *and Definiton of* f]

$\exists n \in \mathbb{N} : \forall i \in [0, n) : p_i \xrightarrow{\tau} p_{i+1} \wedge p_0 = p \wedge p_n = p'$
$\wedge \exists \varphi' \in \varphi\{f^{-1}\} : p' \models \varphi'$

$\iff$ [*Definition of* $\stackrel{\epsilon}{\Rightarrow}$]

$p \stackrel{\epsilon}{\Rightarrow} p' \wedge \exists \varphi' \in \varphi\{f^{-1}\} : p' \models \varphi'$

$\iff$

$\exists \varphi' \in \varphi\{f^{-1}\} : p \models \langle\epsilon\rangle\varphi'$

$\iff$ [*Definition of* $\phi\{f^{-1}\}$]

$\exists \phi' \in \phi\{f^{-1}\} : p \models \phi'$

□

With these small modifications, we can safely say that the congruence requirement we found for the concrete case, is sufficient also for the abstract semantics.

Theorem 3.5. *Let* $\mathscr{O} \subseteq HML$*. If for any* $\phi \in HML$

$$\text{(R) } \phi \in \mathscr{O} \implies \phi\{f^{-1}\} \subseteq \mathscr{O}$$

then $\sim_{\mathscr{O}}$ *is a congruence with respect to the relabelling operator.*

Proof. The proof can be obtained in the same way as in Theorem 2.1, using Lemma 3.11 instead of Lemma 2.1. □

3.6 Abstraction Operator

Definition 3.13. *The abstraction operator is written* $\mathcal{T}_H(p)$*, where* $H \subseteq Act$*. The transition rules for this operator are:*

$$\frac{p \xrightarrow{a} p' \wedge a \notin H}{\mathcal{T}_H(p) \xrightarrow{a} \mathcal{T}_H(p')} \text{ (ABS1)} \qquad \frac{p \xrightarrow{a} p' \wedge a \in H}{\mathcal{T}_H(p) \xrightarrow{\tau} \mathcal{T}_H(p')} \text{ (ABS2)}$$

$$\frac{p \xrightarrow{\tau} p'}{\mathcal{T}_H(p) \xrightarrow{\tau} \mathcal{T}_H(p')} \text{ (ABS3)}$$

Analogously as we did for the relabelling operator, we introduce an abstraction reverse operator for HML. Again we will try to obtain a relationship between the abstraction operator for processes and the abstraction operator for HML.

Definition 3.14. *The inverse abstraction operator for HML is written* $\phi\{\mathcal{T}_H^{-1}\}$*, where* $H \subseteq Act$*. Its behaviour is defined as follows:*

- $(\neg\phi)\{\mathcal{T}_H^{-1}\} = \{\bigwedge_{\phi' \in \phi\{\mathcal{T}_H^{-1}\}} \neg\phi'\}$

- $(\bigwedge_{i\in I}\phi_i)\{\mathcal{T}_H^{-1}\} = \{\bigwedge_{i\in I}\phi_i' \mid \phi_i' \in \phi_i\{\mathcal{T}_H^{-1}\}\}$
- $(\langle\tau\rangle\phi)\{\mathcal{T}_H^{-1}\} = \{\langle\alpha\rangle\phi' \mid \phi' \in \phi\{\mathcal{T}_H^{-1}\} \wedge \alpha \in H \cup \{\tau\}\}$
- $(\langle a\rangle\phi)\{\mathcal{T}_H^{-1}\} = \begin{cases}\{\langle a\rangle\phi' \mid \phi' \in \phi\{\mathcal{T}_H^{-1}\}\}, & \textit{if } a \notin H\\ \varnothing, & \textit{if } a \in H\end{cases}$
- $(\langle\epsilon\rangle\langle\tau\rangle\phi)\{\mathcal{T}_H^{-1}\} = S_\tau \cup S_{\not\tau}, where:$
  $S_\tau = \{\langle\epsilon\rangle\langle\tau\rangle\phi' \mid \phi' \in \phi\{\mathcal{T}_H^{-1}\}\}$
  $S_{\not\tau} = \{\langle\epsilon\rangle\langle h\rangle\langle\epsilon\rangle\phi' \mid \phi' \in \phi\{\mathcal{T}_H^{-1}\} \wedge h \in H\} \cup \{\langle\epsilon\rangle\langle h\rangle\phi' \mid \phi' \in S_{\not\tau} \wedge h \in H\}$
- $(\langle\epsilon\rangle\phi)\{\mathcal{T}_H^{-1}\} = \{\langle\epsilon\rangle\phi' \mid \phi' \in \phi\{\mathcal{T}_H^{-1}\}\} \cup$
  $\{\langle\epsilon\rangle\langle h\rangle\phi' \mid \phi' \in (\langle\epsilon\rangle\phi)\{\mathcal{T}_H^{-1}\} \wedge h \in H\}$

As we can see in Definition 3.14, the set $(\langle\epsilon\rangle\phi)\{\mathcal{T}_H^{-1}\}$ is defined in a recursive fashion. There are two cases: either ϕ is of the shape $\langle\tau\rangle\varphi$, or it isn't. Intuitively, in the first case, the process $\mathcal{T}_H(p)$ has to perform a number of τ-transitions greater than zero, and reach a state which satisfies φ. This can happen in two ways: either the process p can perform at least one h-transition (with $h \in H$), or not. In the first case the process will satisfy a formula of the shape $\langle\epsilon\rangle\langle h_1\rangle \dots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi'$, such that $m > 0$, $h_i \in H$ and $\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}$, while in the second case, p can perform at least one τ-transition as well, so it satisfies a formula of the shape $\langle\epsilon\rangle\langle\tau\rangle\varphi'$. In the latter case, the one where ϕ is not of the shape $\langle\tau\rangle\varphi$, the number of τ-transitions must be zero, so we only need formulas of the shape $\langle\epsilon\rangle\langle h_1\rangle \dots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\phi'$ such that $h_i \in H$ and $\phi' \in \phi\{\mathcal{T}_H^{-1}\}$.

Lemma 3.12. $\mathcal{T}_H(p) \models \phi \iff \exists\phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$

Proof. We apply induction on the structure of ϕ.

- $\phi \equiv \neg\varphi$:

  $\mathcal{T}_H(p) \models \phi$

  $\iff$

  $\mathcal{T}_H(p) \not\models \varphi$

  $\iff$ [*Inductive Hypothesis*]

  $\forall\varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \not\models \varphi'$

  $\iff$

  $\forall\varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \models \neg\varphi'$

  $\iff$

  $p \models \bigwedge_{\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}} \neg\varphi'$

  $\iff$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]

  $\exists\phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\mathcal{T}_H(p) \models \phi$
  $\iff$
  $\forall i \in I : \mathcal{T}_H(p) \models \phi_i$
  $\iff$ [*Inductive Hypothesis*]
  $\forall i \in I : \exists \phi'_i \in \phi_i\{\mathcal{T}_H^{-1}\} : p \models \phi'_i$
  $\iff$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\exists \phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$
- $\phi \equiv \langle \tau \rangle \varphi$
  $\mathcal{T}_H \models \phi$
  $\iff$
  $\mathcal{T}_H(p) \xrightarrow{\tau} \mathcal{T}_H(p') \wedge \mathcal{T}_H(p') \models \varphi$
  $\iff$ [*Inductive Hypothesis*]
  $\mathcal{T}_H(p) \xrightarrow{\tau} \mathcal{T}_H(p') \wedge \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi'$
  $\iff$ [(ABS2) *and* (ABS3)]
  $\exists a \in H \cup \{\tau\} : p \xrightarrow{a} p' \wedge \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi'$
  $\iff$
  $\exists a \in H \cup \{\tau\}, \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \models \langle a \rangle \varphi'$
  $\iff$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\exists \phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$
- $\phi \equiv \langle a \rangle \varphi$ *with* $a \in H$:
  $\mathcal{T}_H(p) \models \phi'$
  $\iff$
  $\mathcal{T}_H(p) \xrightarrow{a} \mathcal{T}_H(p') \wedge \mathcal{T}_H(p') \models \varphi$
  $\iff$ [$a \in H$ *and* (ABS1), (ABS2), (ABS3)]
  $\perp$
  $\iff$ [$\exists$ *on empty set*]
  $\exists \phi' \in \varnothing : p \models \phi'$
  $\iff$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\exists \phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$
- $\phi \equiv \langle a \rangle \varphi$ *with* $a \notin H$:
  $\mathcal{T}_H(p) \models \phi'$
  $\iff$
  $\mathcal{T}_H(p) \xrightarrow{a} \mathcal{T}_H(p') \wedge \mathcal{T}_H(p') \models \varphi$

$\iff$ [*Inductive Hypothesis*]

$\mathcal{T}_H(p) \xrightarrow{a} \mathcal{T}_H(p') \wedge \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi'$

$\iff$ [(ABS1)]

$p \xrightarrow{a} p' \wedge \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi'$

$\iff$

$\exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \models \langle a \rangle \varphi'$

$\iff$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]

$\exists \phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$

- $\phi \equiv \langle \epsilon \rangle \langle \tau \rangle \varphi$

  $\mathcal{T}_H(p) \models \phi$

  $\iff$

  $\mathcal{T}_H(p) \stackrel{\epsilon}{\Rightarrow} \xrightarrow{\tau} \mathcal{T}_H(p') \wedge \mathcal{T}_H(p') \models \varphi$

  $\iff$ [*Inductive Hypothesis*]

  $\mathcal{T}_H(p) \stackrel{\epsilon}{\Rightarrow} \xrightarrow{\tau} \mathcal{T}_H(p') \wedge \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi'$

  $\iff$ [*Definition of* $\stackrel{\epsilon}{\Rightarrow}$]

  $\exists n > 0 : \forall i \in [0, n) : \mathcal{T}_H(p_i) \xrightarrow{\tau} \mathcal{T}_H(p_{i+1}) \wedge p_0 = p \wedge p_n = p'$
  $\wedge \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi'$

  $\iff$ [(ABS2), (ABS3)]

  $(\exists m > 0 : \forall i \in [0, m) : [p_i \stackrel{\epsilon}{\Rightarrow} \xrightarrow{h_i} \stackrel{\epsilon}{\Rightarrow} p_{i+1} \wedge h_i \in H] \wedge p_0 = p \wedge p_n = p'$
  $\wedge \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi')$
  $\vee$
  $(\exists n > 0 : \forall i \in [0, n) : p_i \xrightarrow{\tau} p_{i+1} \wedge p_0 = p \wedge p_n = p' \wedge \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi')$

  $\iff$

  $\exists m > 0 : h_0, \ldots, h_{m-1} \in H, \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \models \langle \epsilon \rangle \langle h_0 \rangle \ldots \langle \epsilon \rangle \langle h_{m-1} \rangle \langle \epsilon \rangle \varphi'$
  $\vee \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \models \langle \epsilon \rangle \langle \tau \rangle \varphi'$

  $\iff$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]

  $\exists \phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$

- $\phi \equiv \langle \epsilon \rangle \varphi$

  $\mathcal{T}_H(p) \models \phi$

  $\iff$

  $\mathcal{T}_H(p) \stackrel{\epsilon}{\Rightarrow} \mathcal{T}_H(p') \wedge \mathcal{T}_H(p') \models \varphi$

  $\iff$ [*Inductive Hypothesis*]

  $\mathcal{T}_H(p) \stackrel{\epsilon}{\Rightarrow} \mathcal{T}_H(p') \wedge \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi'$

  $\iff$ [*Definition of* $\stackrel{\epsilon}{\Rightarrow}$]

$\exists n \in \mathbb{N} : \forall i \in [0, n) : \mathcal{T}_H(p_i) \xrightarrow{\tau} \mathcal{T}_H(p_{i+1}) \wedge p_0 = p \wedge p_n = p'$
$\wedge\, \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi'$
$\iff$ [(ABS2)]
$\exists m \in \mathbb{N} : \forall i \in [0, m) : [p_i \overset{\epsilon}{\Rightarrow}\xrightarrow{h_i}\overset{\epsilon}{\Rightarrow} p_{i+1} \wedge h_i \in H] \wedge p_0 = p \wedge p_n = p'$
$\wedge\, \exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p' \models \varphi'$
$\iff$
$\exists h_0, \ldots, h_{m-1} \in H, \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \models \langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_{m-1}\rangle\langle\epsilon\rangle\varphi'$
$\iff$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
$\exists \phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$

□

All the definitions and results we introduced for this operator are, in some sense, similar to the ones we used for the relabelling operator. Therefore, it is not surprising that the congruence requirement for this operator is analogous to the constraint we provided for the previous one.

Theorem 3.6. *Let $\mathscr{O} \subseteq HML$. If for any $\phi \in HML$*

$$\text{(AB) } \phi \in \mathscr{O} \implies \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}^{\equiv}$$

then $\sim_{\mathscr{O}}$ is a congruence with respect to the abstraction operator.

Proof. Assume a modal language $\mathscr{O}$ with the (AB) property. Let $p \sim_{\mathscr{O}} q$. We show that for any $\phi \in \mathscr{O}$

$$\mathcal{T}_H(p) \models \phi \iff \mathcal{T}_H(q) \models \phi$$

$\mathcal{T}_H(p) \models \phi$
$\iff$ [*Lemma 3.12*]
$\exists \phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$
$\iff$ [(AB) and $p \sim_{\mathscr{O}} q$]
$\exists \phi' \in \phi\{f^{-1}\} : q \models \phi'$
$\iff$ [*Lemma 3.12*]
$\mathcal{T}_H(q) \models \phi$ □

3.7 Existing Semantics

As was done in section 2.3 for concrete semantics, here we investigate the abstract semantics presented in section 1.1.2. We prove whether the modal characterizations of (rooted and unrooted) weak, delay, η and branching bisimulations satisfy the different requirements that we found for the different operators, i.e. we investigate whether they are congruences with respect to the presented operators.

For the sake of clarity, we report here the modal characterizations of these congruences.

- The modal characterization of the weak bisimulation is the following:
  $\mathscr{O}_w\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle\phi \mid \langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\phi\ (\phi\in\mathscr{O}_w)$
- The modal characterization of the delay bisimulation is the following:
  $\mathscr{O}_d\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle\phi \mid \langle\epsilon\rangle\langle a\rangle\phi\ (\phi\in\mathscr{O}_d)$
- The modal characterization of the $\eta-$bisimulation is the following:
  $\mathscr{O}_\eta\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle\phi \mid \langle\epsilon\rangle(\phi\langle a\rangle\langle\epsilon\rangle\phi)\ (\phi\in\mathscr{O}_\eta)$
- The modal characterization of the branching bisimulation is the following:
  $\mathscr{O}_b\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle(\phi\langle\hat{\tau}\rangle\phi) \mid \langle\epsilon\rangle(\phi\langle a\rangle\phi)\ (\phi\in\mathscr{O}_b)$
- The modal characterization of the rooted weak bisimulation is the following:
  $\mathscr{O}_{rw}\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\hat{\phi} \mid \hat{\phi}\ (\phi\in\mathscr{O}_{rw}, \hat{\phi}\in\mathscr{O}_w)$
- The modal characterization of the rooted delay bisimulation is the following:
  $\mathscr{O}_{rd}\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\epsilon\rangle\langle\alpha\rangle\hat{\phi} \mid \hat{\phi}\ (\phi\in\mathscr{O}_{rd}, \hat{\phi}\in\mathscr{O}_d)$
- The modal characterization of the rooted $\eta-$bisimulation is the following:
  $\mathscr{O}_{r\eta}\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\alpha\rangle\langle\epsilon\rangle\hat{\phi} \mid \hat{\phi}\ (\phi\in\mathscr{O}_{r\eta}, \hat{\phi}\in\mathscr{O}_\eta)$
- The modal characterization of the rooted branching bisimulation is the following:
  $\mathscr{O}_{rb}\ \phi ::= \bigwedge_{i\in I}\phi_i \mid \neg\phi \mid \langle\alpha\rangle\hat{\phi} \mid \hat{\phi}\ (\phi\in\mathscr{O}_{rb}, \hat{\phi}\in\mathscr{O}_b)$

3.7.1 Action Prefix Operator

The requirements found for the action prefix operator are the following:
(AP1) $C[\langle\alpha\rangle\phi]\in\mathscr{O} \implies \phi\in\mathscr{O}^{\equiv}$
(AP2) $C[\langle\epsilon\rangle\phi]\in\mathscr{O} \implies \langle\epsilon\rangle\phi\in\mathscr{O}^{\equiv}$

Theorem 3.7. *The modal characterization of the weak bisimulation satisfies (AP1) and (AP2).*

Proof. We prove the two requirements separately:

(AP1) :

$C[\langle\alpha\rangle\phi]\in\mathscr{O}_w$

$\implies$ [*Modal Characterization of* $\mathscr{O}_w$]

$\phi = \langle\epsilon\rangle\varphi$ with $\varphi\in\mathscr{O}_w$

$\implies$ [*Modal Characterization of* $\mathscr{O}_w$]

$\phi\in\mathscr{O}_w$.

(AP2) :

$C[\langle\epsilon\rangle\phi] \in \mathscr{O}_w$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]

$\phi = \varphi \vee \phi = \langle a\rangle\langle\epsilon\rangle\varphi$ with $\varphi \in \mathscr{O}_w$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]

$\langle\epsilon\rangle\phi \in \mathscr{O}_w$.

□

Theorem 3.8. *The modal characterization of the delay bisimulation satisfies (AP1) and (AP2).*

Proof. We prove the two requirements separately:

(AP1) :

$C[\langle\alpha\rangle\phi] \in \mathscr{O}_d$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_d$]

$\phi = \varphi$ with $\varphi \in \mathscr{O}_d$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_d$]

$\phi \in \mathscr{O}_d$.

(AP2) :

$C[\langle\epsilon\rangle\phi] \in \mathscr{O}_d$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_d$]

$\phi = \varphi \vee \phi = \langle a\rangle\varphi$ with $\varphi \in \mathscr{O}_d$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_d$]

$\langle\epsilon\rangle\phi \in \mathscr{O}_d$.

□

Theorem 3.9. *The modal characterization of the* η*-bisimulation satisfies (AP1) and (AP2).*

Proof. We prove the two requirements separately:

(AP1) :

$C[\langle\alpha\rangle\phi] \in \mathscr{O}_\eta$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]

$\phi = \langle\epsilon\rangle\varphi$ with $\varphi \in \mathscr{O}_\eta$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]

$\phi \in \mathscr{O}_\eta$.

(AP2) :

$C[\langle\epsilon\rangle\phi] \in \mathscr{O}_\eta$
$\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
$\phi = \varphi \vee \phi = \varphi\langle a\rangle\langle\epsilon\rangle\varphi'$ with $\varphi, \varphi' \in \mathscr{O}_\eta$
$\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
$\langle\epsilon\rangle\phi \in \mathscr{O}_\eta$.

□

Theorem 3.10. *The modal characterization of the branching bisimulation satisfies (AP1) and (AP2).*

Proof. We prove the two requirements separately:

(AP1) :

$C[\langle\alpha\rangle\phi] \in \mathscr{O}_b$
$\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
$\phi = \varphi$ with $\varphi \in \mathscr{O}_b$
$\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
$\phi \in \mathscr{O}_b$.

(AP2) :

$C[\langle\epsilon\rangle\phi] \in \mathscr{O}_b$
$\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
$\phi = \varphi\langle a\rangle\varphi' \vee \phi = \varphi\langle\hat{\tau}\rangle\varphi'$ with $\varphi, \varphi' \in \mathscr{O}_b$
$\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
$\langle\epsilon\rangle\phi \in \mathscr{O}_b$.

□

Theorem 3.11. *The modal characterization of the rooted weak bisimulation satisfies (AP1) and (AP2).*

Proof. We prove the two requirements separately:

(AP1) :

$C[\langle\alpha\rangle\phi] \in \mathscr{O}_{rw}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$\phi = \langle\epsilon\rangle\varphi$ with $\varphi \in \mathscr{O}_w \vee C[\langle\alpha\rangle\phi] \in \mathscr{O}_w$
$\implies$ [*Modal Characterization of* $\mathscr{O}_w$*, Theorem 3.7*]
$\phi \in \mathscr{O}_w$.
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$\phi \in \mathscr{O}_{rw}$.

(AP2) :

$C[\langle\epsilon\rangle\phi] \in \mathscr{O}_{rw}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

$\phi = \langle\alpha\rangle\langle\epsilon\rangle\varphi$ with $\varphi \in \mathscr{O}_w \vee C[\langle\epsilon\rangle\phi] \in \mathscr{O}_w$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$, *Theorem 3.7*]

$\langle\epsilon\rangle\phi \in \mathscr{O}_{rw} \vee \langle\epsilon\rangle\phi \in \mathscr{O}_w$.

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

$\phi \in \mathscr{O}_{rw}$.

□

Theorem 3.12. *The modal characterization of the rooted delay bisimulation satisfies (AP1) and (AP2).*

Proof. We prove the two requirements separately:

(AP1) :

$C[\langle\alpha\rangle\phi] \in \mathscr{O}_{rd}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

$\phi = \varphi$ with $\varphi \in \mathscr{O}_d \vee C[\langle\alpha\rangle\phi] \in \mathscr{O}_d$

$\Longrightarrow$ [*Theorem 3.8*]

$\phi \in \mathscr{O}_d$.

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

$\phi \in \mathscr{O}_{rd}$.

(AP2) :

$C[\langle\epsilon\rangle\phi] \in \mathscr{O}_{rd}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

$\phi = \langle\alpha\rangle\varphi$ with $\varphi \in \mathscr{O}_d \vee C[\langle\epsilon\rangle\phi] \in \mathscr{O}_d$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$, *Theorem 3.8*]

$\langle\epsilon\rangle\phi \in \mathscr{O}_{rd} \vee \langle\epsilon\rangle\phi \in \mathscr{O}_d$.

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

$\phi \in \mathscr{O}_{rd}$.

□

Theorem 3.13. *The modal characterization of the rooted η-bisimulation satisfies (AP1) and (AP2).*

Proof. We prove the two requirements separately:

(AP1) :

$C[\langle\alpha\rangle\phi] \in \mathscr{O}_{r\eta}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

$\phi = \langle\epsilon\rangle\varphi$ with $\varphi \in \mathscr{O}_{\eta} \vee C[\langle\alpha\rangle\phi] \in \mathscr{O}_{\eta}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{\eta}$*, Theorem 3.9*]

$\phi \in \mathscr{O}_{\eta}$.

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

$\phi \in \mathscr{O}_{r\eta}$.

(AP2) :

$C[\langle\epsilon\rangle\phi] \in \mathscr{O}_{r\eta}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

$\phi = \varphi$ with $\varphi \in \mathscr{O}_{\eta} \vee C[\langle\epsilon\rangle\phi] \in \mathscr{O}_{\eta}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{\eta}$*, Theorem 3.9*]

$\langle\epsilon\rangle\phi \in \mathscr{O}_{\eta}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

$\phi \in \mathscr{O}_{r\eta}$.

□

Theorem 3.14. *The modal characterization of the rooted branching bisimulation satisfies (AP1) and (AP2).*

Proof. We prove the two requirements separately:

(AP1) :

$C[\langle\alpha\rangle\phi] \in \mathscr{O}_{rb}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

$\phi = \varphi$ with $\varphi \in \mathscr{O}_{b} \vee C[\langle\alpha\rangle\phi] \in \mathscr{O}_{b}$

$\Longrightarrow$ [*Theorem 3.10*]

$\phi \in \mathscr{O}_{b}$.

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

$\phi \in \mathscr{O}_{rb}$.

(AP2) :

$C[\langle\epsilon\rangle\phi] \in \mathscr{O}_{rb}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

$C[\langle\epsilon\rangle\phi] \in \mathscr{O}_{b}$

$\Longrightarrow$ [*Theorem 3.10*]

$\langle\epsilon\rangle\phi \in \mathscr{O}_b$
$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
$\phi \in \mathscr{O}_{rb}$.

□

3.7.2 Alternative Composition Operator

The requirements found for the alternative composition operator are the following:
(AC1) $\bigwedge_{i\in I}\varphi_i \in \mathscr{O} \implies \forall i \in I : \varphi_i \in \mathscr{O}^{\equiv}$
(AC2) $\neg\varphi \in \mathscr{O} \implies \varphi \in \mathscr{O}^{\equiv}$
(AC3) $\langle\epsilon\rangle\varphi \in \mathscr{O} \implies \varphi \equiv \langle\alpha\rangle\varphi' \vee (\langle\tau\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}^{\equiv} \wedge \varphi \in \mathscr{O}^{\equiv})$

Theorem 3.15. *For* $c \in \{w, d, \eta, b\}$, $\mathscr{O}_c$ *and* $\mathscr{O}_{rc}$ *satisfy (AC1) and (AC2).*

Proof. We prove the two requirements separately:

(AC1) :

- $\mathscr{O}_c$:
  $\bigwedge_{i\in I}\varphi_i \in \mathscr{O}_c$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_c$]
  $\forall i \in I : \varphi_i \in \mathscr{O}_c$
- $\mathscr{O}_{rc}$:
  $\bigwedge_{i\in I}\varphi_i \in \mathscr{O}_{rc}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rc}$]
  $\forall i \in I : \varphi_i \in \mathscr{O}_{rc} \vee \bigwedge_{i\in I}\varphi_i \in \mathscr{O}_c$
  $\Longrightarrow$ [$\mathscr{O}_c$ *satisfies (AC1)*]
  $\forall i \in I : \varphi_i \in \mathscr{O}_{rc} \vee \forall i \in I : \varphi_i \in \mathscr{O}_c$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rc}$]
  $\forall i \in I : \varphi_i \in \mathscr{O}_{rc}$

(AC2) :

- $\mathscr{O}_c$:
  $\neg\varphi \in \mathscr{O}_c$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_c$]
  $\varphi \in \mathscr{O}_c$
- $\mathscr{O}_{rc}$:
  $\neg\varphi \in \mathscr{O}_{rc}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rc}$]
  $\varphi \in \mathscr{O}_{rc} \vee \neg\varphi \in \mathscr{O}_c$
  $\Longrightarrow$ [$\mathscr{O}_c$ *satisfies (AC2)*]

$\varphi \in \mathscr{O}_{rc} \vee \varphi \in \mathscr{O}_c$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rc}$]
$\varphi \in \mathscr{O}_{rc}$

□

Theorem 3.16. *For* $c \in \{w, d, \eta, b\}$, $\mathscr{O}_c$ *does not satisfy (AC3).*

Proof. Given $a \in Act$, we define:

- $\phi_w = \langle\epsilon\rangle\neg\langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\top$
- $\phi_d = \langle\epsilon\rangle\neg\langle\epsilon\rangle\langle a\rangle\top$
- $\phi_\eta = \langle\epsilon\rangle\neg\langle\epsilon\rangle((\top)\langle a\rangle\langle\epsilon\rangle(\top))$
- $\phi_b = \langle\epsilon\rangle((\neg\langle\epsilon\rangle((\top)\langle a\rangle(\top)))\langle\hat{\tau}\rangle(\top))$

We observe that $\forall c \in \{w, d, \eta, b\} : \phi_c \in \mathscr{O}_c, \phi_c = \langle\epsilon\rangle\phi'_c, \langle\tau\rangle\langle\epsilon\rangle\phi'_c \notin \mathscr{O}_c^{\equiv}$ □

It is important to convince ourselves that the result stated in Theorem 3.16 it's not sufficient to say that weak, delay, η and branching bisimulations are not congruence with respect to the alternative composition operator. The requirements we found, in fact, are sufficient and not necessary. It may be the case, that they are just too strong.

If we really want to prove that this is actually the case, we need to find a counterexample.

Theorem 3.17. *For* $c \in \{w, d, \eta, b\}$, $\leftrightarroweq_c$ *it's not a congruence with respect to the alternative composition operator.*

Proof. It's sufficient to observe that the processes $b0$ and $\tau b0$ are bisimilar for all the given bisimulations. The same thing, obviously, applies to $a0$ with itself. But when we apply the operator, we observe that $a0 + b0$ and $a0 + \tau b0$ are no longer bisimilar, since

$$\forall c \in \{w, d, \eta, b\} : a0 + \tau b0 \models \phi_c \wedge a0 + b0 \not\models \phi_c$$

where ϕ_c are the formulas defined in the previous theorem. □

Theorem 3.18. *The modal characterization of the rooted weak bisimulation satisfies (AC3)*

Proof.
$\langle\epsilon\rangle\phi \in \mathscr{O}_{rw}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$(\phi = \langle\alpha\rangle\langle\epsilon\rangle\varphi$ with $\varphi \in \mathscr{O}_w) \vee \langle\epsilon\rangle\phi \in \mathscr{O}_w$
$\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
$(\phi = \langle\alpha\rangle\langle\epsilon\rangle\varphi \vee \phi = \langle a\rangle\langle\epsilon\rangle\varphi$ with $\varphi \in \mathscr{O}_w) \vee \phi \in \mathscr{O}_w$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$(\phi = \langle\alpha\rangle\langle\epsilon\rangle\varphi \vee \phi = \langle a\rangle\langle\epsilon\rangle\varphi$ with $\varphi \in \mathscr{O}_w) \vee (\phi \in \mathscr{O}_{rw} \wedge \langle\epsilon\rangle\langle\tau\rangle\langle\epsilon\rangle\phi \in \mathscr{O}_{rw})$
$\implies$
$\phi \equiv \langle\alpha\rangle\varphi \vee (\phi \in \mathscr{O}_{rw}^{\equiv} \wedge \langle\tau\rangle\langle\epsilon\rangle\phi \in \mathscr{O}_{rw}^{\equiv})$ □

Theorem 3.19. *The modal characterization of the rooted delay bisimulation satisfies (AC3)*

Proof.
$\langle\epsilon\rangle\phi \in \mathscr{O}_{rd}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
$(\phi = \langle\alpha\rangle\varphi$ with $\varphi \in \mathscr{O}_d) \vee \langle\epsilon\rangle\phi \in \mathscr{O}_d$
$\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
$(\phi = \langle\alpha\rangle\varphi \vee \phi = \langle a\rangle\varphi$ with $\varphi \in \mathscr{O}_d) \vee \phi \in \mathscr{O}_d$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
$(\phi = \langle\alpha\rangle\varphi \vee \phi = \langle a\rangle\varphi$ with $\varphi \in \mathscr{O}_d) \vee (\phi \in \mathscr{O}_{rd} \wedge \langle\epsilon\rangle\langle\tau\rangle\phi \in \mathscr{O}_{rd})$
$\implies$
$\phi \equiv \langle\alpha\rangle\varphi \vee (\phi \in \mathscr{O}_{rd}^{\equiv} \wedge \langle\tau\rangle\langle\epsilon\rangle\phi \in \mathscr{O}_{rd}^{\equiv})$ □

Theorem 3.20. *The modal characterization of the rooted* η*-bisimulation satisfies (AC3)*

Proof.
$\langle\epsilon\rangle\phi \in \mathscr{O}_{r\eta}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
$\langle\epsilon\rangle\phi \in \mathscr{O}_{\eta}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{\eta}$]
$\phi = \varphi \vee \phi = \varphi\langle a\rangle\langle\epsilon\rangle\varphi$ with $\varphi \in \mathscr{O}_{\eta}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
$\phi \in \mathscr{O}_{r\eta} \wedge \langle\tau\rangle\langle\epsilon\rangle\phi \in \mathscr{O}_{r\eta}$
$\implies$
$\phi \in \mathscr{O}_{r\eta}^{\equiv} \wedge \langle\tau\rangle\langle\epsilon\rangle\phi \in \mathscr{O}_{r\eta}^{\equiv}$ □

Theorem 3.21. *The modal characterization of the rooted branching bisimulation satisfies (AC3)*

Proof.
$\langle\epsilon\rangle\phi \in \mathscr{O}_{rb}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
$\langle\epsilon\rangle\phi \in \mathscr{O}_{b}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{b}$]
$\phi = \varphi\langle\hat{\tau}\rangle\varphi \vee \phi = \varphi\langle a\rangle\varphi$ with $\varphi \in \mathscr{O}_b$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
$\phi \in \mathscr{O}_{rb}^{\equiv} \wedge \langle\tau\rangle\langle\epsilon\rangle\phi \in \mathscr{O}_{rb}$
$\implies$
$\phi \in \mathscr{O}_{rb}^{\equiv} \wedge \langle\tau\rangle\langle\epsilon\rangle\phi \in \mathscr{O}_{rb}^{\equiv}$ □

3.7.3 Restriction Operators

The requirement found for the restriction operators is the following:
(RES) $C'[\neg\phi] \in \mathscr{O} \implies C'[\top] \in O^{\equiv}$

Theorem 3.22. *For* $c \in \{w, d, \eta, b\}$, $\mathscr{O}_c$ *satisfies (RES).*

Proof.
$C'[\neg\phi] \in \mathscr{O}_c$
$\implies$ [*Modal Characterization of* $\mathscr{O}_c$]
$\forall\varphi \in \mathscr{O}_c^{\equiv} : C'[\neg\varphi] \in \mathscr{O}_c^{\equiv}$
$\implies$ [$\bot \in \mathscr{O}_c^{\equiv}$]
$C'[\top] \in \mathscr{O}_c$ □

Theorem 3.23. *For* $c \in \{w, d, \eta, b\}$, $\mathscr{O}_{rc}$ *satisfies (RES).*

Proof.
$C'[\neg\phi] \in \mathscr{O}_{rc}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rc}$]
$(\forall\varphi \in \mathscr{O}_{rc}^{\equiv} : C'[\neg\varphi] \in \mathscr{O}_{rc}^{\equiv}) \vee (\forall\varphi' \in \mathscr{O}_c^{\equiv} : C'[\neg\varphi'] \in \mathscr{O}_{rc}^{\equiv})$
$\implies$ [$\bot \in \mathscr{O}_c^{\equiv}, \bot \in \mathscr{O}_{rc}^{\equiv}$]
$C'[\top] \in \mathscr{O}_{rc}$ □

3.7.4 Parallel Composition Operator

The requirement found for the parallel composition operator is the following:
(P) $\phi \in \mathscr{O} \implies Sub(\phi) \subseteq \mathscr{O}^{\equiv}$

Theorem 3.24. *The modal characterization of the weak bisimulation satisfies (P).*

Proof. Given $\phi \in \mathscr{O}_w$, we want to prove that $Sub(\phi) \subseteq \mathscr{O}_w^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:

  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_w$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]

  $\forall i \in I : \phi_i \in \mathscr{O}_w$

  $\implies$ [*Inductive Hypothesis*]

  $\forall i \in I : Sub(\phi_i) \subseteq \mathscr{O}_w^{\equiv}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]

  $\bigcup_{i\in I} Sub(\phi_i) \subseteq \mathscr{O}_w^{\equiv} \wedge \{\bigwedge_{i\in I} \phi_i' \mid \phi_i' \in Sub(\phi_i)\} \subseteq \mathscr{O}_w^{\equiv}$

  $\implies$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_w^{\equiv}$

- $\phi \equiv \neg\varphi$:

  $\neg\varphi \in \mathscr{O}_w$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]

  $\varphi \in \mathscr{O}_w$

  $\implies$ [*Inductive Hypothesis*]

$Sub(\varphi) \subseteq \mathscr{O}_w^{\equiv}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
$\{\neg\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_w^{\equiv}$
$\implies$ [*Definition of* $Sub(\phi)$]
$Sub(\phi) \subseteq \mathscr{O}_w^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\varphi \in \mathscr{O}_w$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\varphi \in \mathscr{O}_w$
  $\implies$ [*Inductive Hypothesis*]
  $Sub(\varphi) \subseteq \mathscr{O}_w^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_w^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_w^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_w$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\varphi \in \mathscr{O}_w$
  $\implies$ [*Inductive Hypothesis*]
  $Sub(\varphi) \subseteq \mathscr{O}_w^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\{\langle\epsilon\rangle\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_w^{\equiv} \wedge \{\langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_w^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_w^{\equiv}$

□

Theorem 3.25. *The modal characterization of the delay bisimulation satisfies (P).*

Proof. Given $\phi \in \mathscr{O}_d$, we want to prove that $Sub(\phi) \subseteq \mathscr{O}_d^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\forall i \in I : \phi_i \in \mathscr{O}_d$

$\implies$ [*Inductive Hypothesis*]
$\forall i \in I : Sub(\phi_i) \subseteq \mathscr{O}_d^{\equiv}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
$\bigcup_{i \in I} Sub(\phi_i) \subseteq \mathscr{O}_d^{\equiv} \wedge \{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in Sub(\phi_i)\} \subseteq \mathscr{O}_d^{\equiv}$
$\implies$ [*Definition of* $Sub(\phi)$]
$Sub(\phi) \subseteq \mathscr{O}_d^{\equiv}$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $Sub(\varphi) \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\neg\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_d^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\varphi \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $Sub(\varphi) \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_d^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\langle a\rangle\varphi$:
  $\langle\epsilon\rangle\langle a\rangle\varphi \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $Sub(\varphi) \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_d^{\equiv} \wedge \{\langle\epsilon\rangle\langle a\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_d^{\equiv}$

$\Longrightarrow$ [*Definition of* $Sub(\phi)$]

$Sub(\phi) \subseteq \mathscr{O}_d^{\equiv}$

□

Theorem 3.26. *The modal characterization of the* η*-bisimulation satisfies (P).*

Proof. Given $\phi \in \mathscr{O}_\eta$, we want to prove that $Sub(\phi) \subseteq \mathscr{O}_\eta^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:

  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_\eta$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]

  $\forall i \in I : \phi_i \in \mathscr{O}_\eta$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $\forall i \in I : Sub(\phi_i) \subseteq \mathscr{O}_\eta^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]

  $\bigcup_{i\in I} Sub(\phi_i) \subseteq \mathscr{O}_\eta^{\equiv} \wedge \{\bigwedge_{i\in I} \phi_i' \mid \phi_i' \in Sub(\phi_i)\} \subseteq \mathscr{O}_\eta^{\equiv}$

  $\Longrightarrow$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_\eta^{\equiv}$

- $\phi \equiv \neg\varphi$:

  $\neg\varphi \in \mathscr{O}_\eta$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]

  $\varphi \in \mathscr{O}_\eta$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $Sub(\varphi) \subseteq \mathscr{O}_\eta^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]

  $\{\neg\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_\eta^{\equiv}$

  $\Longrightarrow$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_\eta^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\varphi$:

  $\langle\epsilon\rangle\varphi \in \mathscr{O}_\eta$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]

  $\varphi \in \mathscr{O}_\eta$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $Sub(\varphi) \subseteq \mathscr{O}_\eta^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]

$\{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_\eta^\equiv$
$\implies$ [*Definition of* $Sub(\phi)$]
$Sub(\phi) \subseteq \mathscr{O}_\eta^\equiv$

- $\phi \equiv \langle\epsilon\rangle(\varphi\langle a\rangle\langle\epsilon\rangle\psi)$:
  $\langle\epsilon\rangle(\varphi\langle a\rangle\langle\epsilon\rangle\psi) \in \mathscr{O}_\eta$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\varphi, \psi \in \mathscr{O}_\eta$
  $\implies$ [*Inductive Hypothesis*]
  $Sub(\varphi) \subseteq \mathscr{O}_\eta^\equiv \wedge Sub(\psi) \subseteq \mathscr{O}_\eta^\equiv$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $S_1 = \{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_\eta^\equiv \wedge$
  $S_2 = \{\langle\epsilon\rangle\langle\epsilon\rangle\psi' \mid \psi' \in Sub(\psi)\} \cup \{\langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\psi' \mid \psi' \in Sub(\psi)\} \subseteq \mathscr{O}_\eta^\equiv \wedge$
  $S_3 = \{\langle\epsilon\rangle(\phi' \wedge \psi') \mid \phi' \in S_1, \psi' \in S_2\} \subseteq \mathscr{O}_\eta^\equiv$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_\eta^\equiv$

□

Looking at the modal characterization of the branching bisimulation, we can see that the prefix $\langle\hat{\tau}\rangle$ is used. As said before, the formula $\langle\hat{\tau}\rangle\phi$, can be expressed using the HML grammar that we used until now. On the other hand, the operator $Sub()$ is dependent by the structure of a formula. In order to avoid problems given by this, we extend the definitions of $Sub()$ and $Par(,)$:

Definition 3.15. *We define:*

- $Sub(\langle\hat{\tau}\rangle\phi) = Sub(\phi) \cup \{\langle\hat{\tau}\rangle \bigwedge_{\phi' \in S} \phi' \mid S \subseteq Sub(\phi)\}$

And, given $A, B \subseteq HML$*, we say*

- $\langle\hat{\tau}\rangle\phi \in Par(A,B) \overset{def}{\iff} \phi \in Par(A,B) \vee$
  $\exists\langle\hat{\tau}\rangle\phi_A \in A^{\Rightarrow} : \phi \in Par(\phi_A, B) \vee$
  $\exists\langle\hat{\tau}\rangle\phi_B \in B^{\Rightarrow} : \phi \in Par(A, \phi_B)$

Now that we extended the definitions, we need to prove that the Lemmas 3.9 and 3.10 are still valid.

Lemma 3.13. *Given two processes* p *and* q *and a formula* $\phi \in HML$*, we have:*

1. $p||q \models \phi \implies \exists A, B \subseteq Sub(\phi) : (p \models A \wedge q \models B \wedge \phi \in Par(A,B))$

   Proof. We will prove the case $\phi = \langle\hat{\tau}\rangle\varphi$, since the remaining cases have been proved already.

   $p||q \models \phi \iff p||q \models \varphi \vee p||q \models \langle\tau\rangle\varphi$, so we split in two cases:

- $p||q \models \varphi \implies$ [*Inductive Hypothesis*] $\exists A', B' \subseteq Sub(\varphi) : (p \models A' \wedge q \models B' \wedge \varphi \in Par(A, B))$
  We take $A = A'$ and $B = B'$.
- $p||q \models \langle\tau\rangle\varphi \implies (p \xrightarrow{\tau} p' \wedge p'||q \models \varphi) \vee (q \xrightarrow{\tau} q' \wedge p||q' \models \varphi)$
  Without loss of generality, we assume $p \xrightarrow{\tau} p' \wedge p'||q \models \varphi$. The case $q \xrightarrow{\tau} q' \wedge p||q' \models \varphi$ is symmetric.
  $p \xrightarrow{\tau} p' \wedge p'||q \models \varphi \implies$ [*Inductive Hypothesis*] $p \xrightarrow{\tau} p' \wedge \exists A', B' \subseteq Sub(\varphi) : (p' \models A' \wedge q \models B' \wedge \varphi \in Par(A', B'))$
  We take $A = \{\langle\hat{\tau}\rangle \bigwedge_{\phi_{A'} \in A'} \phi_{A'}\}$ and $B = B'$

□

2. $\exists A, B : (p \models A \wedge q \models B \wedge \phi \in Par(A, B)) \implies p||q \models \phi$

Proof. We will prove the case $\phi = \langle\hat{\tau}\rangle\varphi$, since the remaining cases have been proved already.

$$\begin{aligned}\exists A, B : (p \models A \wedge q \models B \wedge \phi \in Par(A, B)) \iff& \\ &\varphi \in Par(A, B) \vee \\ &\exists\langle\hat{\tau}\rangle\phi_A \in A^{\Rightarrow} : \varphi \in Par(\phi_A, B) \vee \\ &\exists\langle\hat{\tau}\rangle\phi_B \in B^{\Rightarrow} : \varphi \in Par(A, \phi_B).\end{aligned}$$

We face the first and the second case. The third case is symmetric to the second.

- $\varphi \in Par(A, B) \implies$ [*Inductive Hypothesis*] $p||q \models \varphi \implies p||q \models \phi$
- $\exists\langle\hat{\tau}\rangle\phi_A \in A^{\Rightarrow} : \varphi \in Par(\phi_A, B)$
  From $p \models A$, we have that $p \models \langle\hat{\tau}\rangle\phi_A$, which means that either $p \models \phi_A$, or $p \xrightarrow{\tau} p' \wedge p' \models \phi_A$. In the first case, by the Inductive Hypothesis, we have $p||q \models \varphi$, which implies $p||q \models \phi$. In the second case, we have $p \xrightarrow{\tau} p' \wedge p'||q \models \varphi$, which implies $p||q \models \langle\tau\rangle\varphi$, which implies $p||q \models \phi$.

□

Theorem 3.27. *The modal characterization of the branching bisimulation satisfies (P).*

Proof. Given $\phi \in \mathcal{O}_b$, we want to prove that $Sub(\phi) \subseteq \mathcal{O}_b^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\bigwedge_{i \in I} \phi_i \in \mathcal{O}_b$
  $\implies$ [*Modal Characterization of* $\mathcal{O}_b$]
  $\forall i \in I : \phi_i \in \mathcal{O}_b$
  $\implies$ [*Inductive Hypothesis*]

$\forall i \in I : Sub(\phi_i) \subseteq \mathscr{O}_b^{\equiv}$

$\implies$ [*Modal Characterization of* $\mathscr{O}_b$]

$\bigcup_{i\in I} Sub(\phi_i) \subseteq \mathscr{O}_b^{\equiv} \wedge \{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in Sub(\phi_i)\} \subseteq \mathscr{O}_b^{\equiv}$

$\implies$ [*Definition of* $Sub(\phi)$]

$Sub(\phi) \subseteq \mathscr{O}_b^{\equiv}$

- $\phi \equiv \neg\varphi$:

  $\neg\varphi \in \mathscr{O}_b$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]

  $\varphi \in \mathscr{O}_b$

  $\implies$ [*Inductive Hypothesis*]

  $Sub(\varphi) \subseteq \mathscr{O}_b^{\equiv}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]

  $\{\neg\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_b^{\equiv}$

  $\implies$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_b^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle(\varphi\langle\hat{\tau}\rangle\psi)$:

  $\langle\epsilon\rangle(\varphi\langle\hat{\tau}\rangle\psi) \in \mathscr{O}_b$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]

  $\varphi, \psi \in \mathscr{O}_b$

  $\implies$ [*Inductive Hypothesis*]

  $Sub(\varphi) \subseteq \mathscr{O}_b^{\equiv} \wedge Sub(\psi) \subseteq \mathscr{O}_b^{\equiv}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]

  $S_1 = \{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_b^{\equiv} \wedge$
  $S_2 = \{\langle\epsilon\rangle\psi' \mid \psi' \in Sub(\psi)\} \cup \{\langle\epsilon\rangle\langle\hat{\tau}\rangle \bigwedge_{\psi'\in S} \psi' \mid S \subseteq Sub(\psi)\} \subseteq \mathscr{O}_b^{\equiv} \wedge$
  $S_3 = \{\langle\epsilon\rangle(\phi' \wedge \psi') \mid \phi' \in S_1, \psi' \in S_2\} \subseteq \mathscr{O}_b^{\equiv}$

  $\implies$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_b^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle(\varphi\langle a\rangle\psi)$:

  $\langle\epsilon\rangle(\varphi\langle a\rangle\psi) \in \mathscr{O}_b$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]

  $\varphi, \psi \in \mathscr{O}_b$

  $\implies$ [*Inductive Hypothesis*]

  $Sub(\varphi) \subseteq \mathscr{O}_b^{\equiv} \wedge Sub(\psi) \subseteq \mathscr{O}_b^{\equiv}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]

$S_1 = \{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_b^{\equiv} \wedge$
$S_2 = \{\langle\epsilon\rangle\psi' \mid \psi' \in Sub(\psi)\} \cup \{\langle\epsilon\rangle\langle a\rangle\psi' \mid \psi' \in Sub(\psi)\} \subseteq \mathscr{O}_b^{\equiv} \wedge$
$S_3 = \{\langle\epsilon\rangle(\phi' \wedge \psi') \mid \phi' \in S_1, \psi' \in S_2\} \subseteq \mathscr{O}_b^{\equiv}$

$\Longrightarrow$ [*Definition of* $Sub(\phi)$]

$Sub(\phi) \subseteq \mathscr{O}_b^{\equiv}$

□

Theorem 3.28. *The modal characterization of the rooted weak bisimulation satisfies (P).*

Proof. Given $\phi \in \mathscr{O}_{rw}$, we want to prove that $Sub(\phi) \subseteq \mathscr{O}_{rw}^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:

  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_{rw}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

  $\forall i \in I : \phi_i \in \mathscr{O}_{rw}$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $\forall i \in I : Sub(\phi_i) \subseteq \mathscr{O}_{rw}^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

  $\bigcup_{i \in I} Sub(\phi_i) \subseteq \mathscr{O}_{rw}^{\equiv} \wedge \{\bigwedge_{i \in I} \phi_i' \mid \phi_i' \in Sub(\phi_i)\} \subseteq \mathscr{O}_{rw}^{\equiv}$

  $\Longrightarrow$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_{rw}^{\equiv}$

- $\phi \equiv \neg\varphi$:

  $\neg\varphi \in \mathscr{O}_{rw}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

  $\varphi \in \mathscr{O}_{rw}$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $Sub(\varphi) \subseteq \mathscr{O}_{rw}^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

  $\{\neg\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rw}^{\equiv}$

  $\Longrightarrow$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_{rw}^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\varphi,$ *with* $\varphi \in \mathscr{O}_w$:

  $\langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_{rw}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

  $\varphi \in \mathscr{O}_w$

$\Longrightarrow$ [*Theorem 3.24*]

$Sub(\varphi) \subseteq \mathscr{O}_w^{\equiv}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$*, Modal Characterization of* $\mathscr{O}_w$]

$\{\langle\epsilon\rangle\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_w^{\equiv} \wedge \{\langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rw}^{\equiv}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

$\{\langle\epsilon\rangle\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rw}^{\equiv} \wedge \{\langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rw}^{\equiv}$

$\Longrightarrow$ [*Definition of* $Sub(\phi)$]

$Sub(\phi) \subseteq \mathscr{O}_{rw}^{\equiv}$

- $\phi \equiv \varphi$*, with* $\varphi \in \mathscr{O}_w$:

  $\varphi \in \mathscr{O}_{rw}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

  $\varphi \in \mathscr{O}_w$

  $\Longrightarrow$ [*Theorem 3.24*]

  $Sub(\varphi) \subseteq \mathscr{O}_w^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

  $Sub(\phi) \subseteq \mathscr{O}_{rw}^{\equiv}$

□

Theorem 3.29. *The modal characterization of the rooted delay bisimulation satisfies (P).*

Proof. Given $\phi \in \mathscr{O}_{rd}$, we want to prove that $Sub(\phi) \subseteq \mathscr{O}_{rd}^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:

  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_{rd}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

  $\forall i \in I : \phi_i \in \mathscr{O}_{rd}$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $\forall i \in I : Sub(\phi_i) \subseteq \mathscr{O}_{rd}^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

  $\bigcup_{i\in I} Sub(\phi_i) \subseteq \mathscr{O}_{rd}^{\equiv} \wedge \{\bigwedge_{i\in I} \phi_i' \mid \phi_i' \in Sub(\phi_i)\} \subseteq \mathscr{O}_{rd}^{\equiv}$

  $\Longrightarrow$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_{rd}^{\equiv}$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_{rd}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\varphi \in \mathscr{O}_{rd}$
  $\implies$ [*Inductive Hypothesis*]
  $Sub(\varphi) \subseteq \mathscr{O}_{rd}^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\{\neg\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rd}^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_{rd}^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\langle\alpha\rangle\varphi$, *with* $\varphi \in \mathscr{O}_d$:
  $\langle\epsilon\rangle\langle\alpha\rangle\varphi \in \mathscr{O}_{rd}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Theorem 3.25*]
  $Sub(\varphi) \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$, *Modal Characterization of* $\mathscr{O}_d$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_d^{\equiv} \wedge \{\langle\epsilon\rangle\langle\alpha\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rd}^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rd}^{\equiv} \wedge \{\langle\epsilon\rangle\langle\alpha\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rd}^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_{rd}^{\equiv}$

- $\phi \equiv \varphi$, *with* $\varphi \in \mathscr{O}_d$:
  $\varphi \in \mathscr{O}_{rd}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Theorem 3.25*]
  $Sub(\varphi) \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $Sub(\phi) \subseteq \mathscr{O}_{rd}^{\equiv}$

□

Theorem 3.30. *The modal characterization of the rooted* η*-bisimulation satisfies (P).*

Proof. Given $\phi \in \mathscr{O}_{r\eta}$, we want to prove that $Sub(\phi) \subseteq \mathscr{O}_{r\eta}^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_{r\eta}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\forall i \in I : \phi_i \in \mathscr{O}_{r\eta}$
  $\implies$ [*Inductive Hypothesis*]
  $\forall i \in I : Sub(\phi_i) \subseteq \mathscr{O}_{r\eta}^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\bigcup_{i \in I} Sub(\phi_i) \subseteq \mathscr{O}_{r\eta}^{\equiv} \wedge \{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in Sub(\phi_i)\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_{r\eta}^{\equiv}$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_{r\eta}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\varphi \in \mathscr{O}_{r\eta}$
  $\implies$ [*Inductive Hypothesis*]
  $Sub(\varphi) \subseteq \mathscr{O}_{r\eta}^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\{\neg\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_{r\eta}^{\equiv}$

- $\phi \equiv \langle\alpha\rangle\langle\epsilon\rangle\varphi,$ *with* $\varphi \in \mathscr{O}_{\eta}$:
  $\langle\alpha\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_{r\eta}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\varphi \in \mathscr{O}_{\eta}$
  $\implies$ [*Theorem 3.26*]
  $Sub(\varphi) \subseteq \mathscr{O}_{\eta}^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$, *Modal Characterization of* $\mathscr{O}_{\eta}$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{\eta}^{\equiv} \wedge \{\langle\alpha\rangle\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{r\eta}^{\equiv} \wedge \{\langle\alpha\rangle\langle\epsilon\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_{r\eta}^{\equiv}$

- $\phi \equiv \varphi$, *with* $\varphi \in \mathscr{O}_\eta$:

  $\varphi \in \mathscr{O}_{r\eta}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $\varphi \in \mathscr{O}_\eta$

  $\Longrightarrow$ [*Theorem 3.26*]

  $Sub(\varphi) \subseteq \mathscr{O}_\eta^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $Sub(\phi) \subseteq \mathscr{O}_{r\eta}^{\equiv}$

□

Theorem 3.31. *The modal characterization of the rooted branching bisimulation satisfies (P).*

Proof. Given $\phi \in \mathscr{O}_{rb}$, we want to prove that $Sub(\phi) \subseteq \mathscr{O}_{rb}^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:

  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\forall i \in I : \phi_i \in \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $\forall i \in I : Sub(\phi_i) \subseteq \mathscr{O}_{rb}^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\bigcup_{i \in I} Sub(\phi_i) \subseteq \mathscr{O}_{rb}^{\equiv} \wedge \{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in Sub(\phi_i)\} \subseteq \mathscr{O}_{rb}^{\equiv}$

  $\Longrightarrow$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_{rb}^{\equiv}$

- $\phi \equiv \neg\varphi$:

  $\neg\varphi \in \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\varphi \in \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $Sub(\varphi) \subseteq \mathscr{O}_{rb}^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\{\neg\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rb}^{\equiv}$

  $\Longrightarrow$ [*Definition of* $Sub(\phi)$]

  $Sub(\phi) \subseteq \mathscr{O}_{rb}^{\equiv}$

- $\phi \equiv \langle\alpha\rangle\varphi$, *with* $\varphi \in \mathscr{O}_b$:
  $\langle\alpha\rangle\varphi \in \mathscr{O}_{rb}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $\varphi \in \mathscr{O}_b$
  $\implies$ [*Theorem 3.27*]
  $Sub(\varphi) \subseteq \mathscr{O}_b^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$, *Modal Characterization of* $\mathscr{O}_b$]
  $Sub(\varphi) \subseteq \mathscr{O}_b^{\equiv} \wedge \{\langle\alpha\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rb}^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $Sub(\varphi) \subseteq \mathscr{O}_{rb}^{\equiv} \wedge \{\langle\alpha\rangle\varphi' \mid \varphi' \in Sub(\varphi)\} \subseteq \mathscr{O}_{rb}^{\equiv}$
  $\implies$ [*Definition of* $Sub(\phi)$]
  $Sub(\phi) \subseteq \mathscr{O}_{rb}^{\equiv}$

- $\phi \equiv \varphi$, *with* $\varphi \in \mathscr{O}_b$:
  $\varphi \in \mathscr{O}_{rb}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $\varphi \in \mathscr{O}_b$
  $\implies$ [*Theorem 3.27*]
  $Sub(\varphi) \subseteq \mathscr{O}_b^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $Sub(\phi) \subseteq \mathscr{O}_{rb}^{\equiv}$

□

3.7.5 Relabelling Operator

The requirement found for the relabelling operator is the following:
(R) $\phi \in \mathscr{O} \implies \phi\{f^{-1}\} \subseteq \mathscr{O}$

Theorem 3.32. *The modal characterization of the weak bisimulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_w$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_w$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:
  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_w$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\forall i \in I : \phi_i \in \mathscr{O}_w$
  $\implies$ [*Inductive Hypothesis*]

$\forall i \in I : \phi\{f^{-1}\} \subseteq \mathscr{O}_w$
$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]
$\{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} \subseteq \mathscr{O}_w$
$\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]
$\phi_i\{f^{-1}\} \subseteq \mathscr{O}_w$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_w$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\varphi \in \mathscr{O}_w$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_w$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\{\bigwedge_{\varphi'\in\varphi\{f^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_w$
  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_w$
- $\phi \equiv \langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\varphi \in \mathscr{O}_w$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\varphi \in \mathscr{O}_w$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_w$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in \varphi\{f^{-1}\}\} \subseteq \mathscr{O}_w$
  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_w$
- $\phi \equiv \langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_w$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\varphi \in \mathscr{O}_w$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_w$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\{\langle\epsilon\rangle\langle a'\rangle\langle\epsilon\rangle\varphi' \mid f(a') = a \wedge \varphi' \in \varphi\{f^{-1}\}\} \subseteq \mathscr{O}_w$
  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_w$

□

Theorem 3.33. *The modal characterization of the delay bisimulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_d$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_d$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\forall i \in I : \phi_i \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{f^{-1}\} \subseteq \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} \subseteq \mathscr{O}_d$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi_i\{f^{-1}\} \subseteq \mathscr{O}_d$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\bigwedge_{\varphi' \in \varphi\{f^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_d$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_d$

- $\phi \equiv \langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\varphi \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in \varphi\{f^{-1}\}\} \subseteq \mathscr{O}_d$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_d$

- $\phi \equiv \langle\epsilon\rangle\langle a\rangle\varphi$:
  $\langle\epsilon\rangle\langle a\rangle\varphi \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\langle\epsilon\rangle\langle a'\rangle\varphi' \mid f(a') = a \wedge \varphi' \in \varphi\{f^{-1}\}\} \subseteq \mathscr{O}_d$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_d$

□

Theorem 3.34. *The modal characterization of the* η*-bisimulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_\eta$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_\eta$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:
  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_\eta$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\forall i \in I : \phi_i \in \mathscr{O}_\eta$
  $\implies$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{f^{-1}\} \subseteq \mathscr{O}_\eta$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} \subseteq \mathscr{O}_\eta$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi_i\{f^{-1}\} \subseteq \mathscr{O}_\eta$
- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_\eta$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\varphi \in \mathscr{O}_\eta$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_\eta$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\{\bigwedge_{\varphi'\in\varphi\{f^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_\eta$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_\eta$

- $\phi \equiv \langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\varphi \in \mathscr{O}_\eta$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\varphi \in \mathscr{O}_\eta$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_\eta$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\{\langle\epsilon\rangle\varphi' \mid \varphi' \in \varphi\{f^{-1}\}\} \subseteq \mathscr{O}_\eta$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_\eta$
- $\phi \equiv \langle\epsilon\rangle(\varphi\langle a\rangle\langle\epsilon\rangle\psi)$:
  $\langle\epsilon\rangle(\varphi\langle a\rangle\langle\epsilon\rangle\psi) \in \mathscr{O}_\eta$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\varphi, \psi \in \mathscr{O}_\eta$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_\eta \wedge \psi\{f^{-1}\} \subseteq \mathscr{O}_\eta$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\{\langle\epsilon\rangle(\varphi'\langle a'\rangle\langle\epsilon\rangle\psi') \mid f(a') = a \wedge \varphi' \in \varphi\{f^{-1}\} \wedge \psi' \in \psi\{f^{-1}\}\} \subseteq \mathscr{O}_\eta$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_\eta$

□

The same observation we did in the previous section about the $Sub()$ operator, applies to the operator $\{f^{-1}\}$ as well. So, in order to deal with $\langle\hat{\tau}\rangle$, we extend the definition of this function.

Definition 3.16. *We define:*

- $(\langle\hat{\tau}\rangle\phi)\{f^{-1}\} = \{\langle\hat{\tau}\rangle\phi' \mid \phi' \in \phi\{f^{-1}\}\}$

We now prove that, also with this extended definition, the requirement is still sufficient to guarantee that a given semantics is a congruence with respect to the relabelling operator. Namely, we prove that the property presented in Lemma 2.1 (extended in Lemma 3.11) is still valid.

Lemma 3.14. $p[f] \models \phi \iff \exists \phi' \in \phi\{f^{-1}\} : p \models \phi'$

Proof. We apply induction on the structure of ϕ. The cases $\phi \equiv \neg\varphi$, $\phi \equiv \bigwedge_{i\in I}\phi_i$, $\phi \equiv \langle x\rangle\varphi$, $\phi \equiv \langle\epsilon\rangle\varphi$ are already covered in the proofs of Lemmas 2.1 and 3.11.

- $\phi \equiv \langle\hat{\tau}\rangle\varphi$:
  $p[f] \models \phi$
  $\iff$
  $p[f] \models \varphi \vee (p[f] \xrightarrow{\tau} p'[f] \wedge p'[f] \models \varphi)$
  $\iff$ [*Inductive Hypothesis*]
  $\exists\varphi' \in \varphi\{f^{-1}\} : p \models \varphi' \vee (p[f] \xrightarrow{\tau} p'[f] \wedge \exists\varphi' \in \varphi\{f^{-1}\} : p' \models \varphi')$
  $\iff$ [(REL) *and Definiton of* f]
  $\exists\varphi' \in \varphi\{f^{-1}\} : p \models \varphi' \vee (p \xrightarrow{\tau} p' \wedge \exists\varphi' \in \varphi\{f^{-1}\} : p' \models \varphi')$
  $\iff$
  $\exists\varphi' \in \varphi\{f^{-1}\} : p \models \varphi' \vee \exists\varphi' \in \varphi\{f^{-1}\} : p \models \langle\tau\rangle\varphi'$
  $\iff$
  $\exists\varphi' \in \varphi\{f^{-1}\} : p \models \langle\hat{\tau}\rangle\varphi'$
  $\iff$ [*Definition of* $\phi\{f^{-1}\}$]
  $\exists\phi' \in \phi\{f^{-1}\} : p \models \phi'$

□

Theorem 3.35. *The modal characterization of the branching bisimulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_b$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_b$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\forall i \in I : \phi_i \in \mathscr{O}_b$
  $\implies$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{f^{-1}\} \subseteq \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} \subseteq \mathscr{O}_b$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi_i\{f^{-1}\} \subseteq \mathscr{O}_b$
- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\varphi \in \mathscr{O}_b$
  $\implies$ [*Inductive Hypothesis*]

$\varphi\{f^{-1}\} \subseteq \mathscr{O}_b$
$\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
$\{\bigwedge_{\varphi' \in \varphi\{f^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_b$
$\implies$ [*Definition of* $\phi\{f^{-1}\}$]
$\phi\{f^{-1}\} \subseteq \mathscr{O}_b$

- $\phi \equiv \langle\epsilon\rangle(\varphi\langle\hat{\tau}\rangle\psi)$:
  $\langle\epsilon\rangle(\varphi\langle\hat{\tau}\rangle\psi) \in \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\varphi, \psi \in \mathscr{O}_b$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_b \wedge \psi\{f^{-1}\} \subseteq \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\{\langle\epsilon\rangle(\varphi'\langle\hat{\tau}\rangle\psi') \mid f(a') = a \wedge \varphi' \in \varphi\{f^{-1}\} \wedge \psi' \in \psi\{f^{-1}\}\} \subseteq \mathscr{O}_b$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_b$

- $\phi \equiv \langle\epsilon\rangle(\varphi\langle a\rangle\psi)$:
  $\langle\epsilon\rangle(\varphi\langle a\rangle\psi) \in \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\varphi, \psi \in \mathscr{O}_b$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_b \wedge \psi\{f^{-1}\} \subseteq \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\{\langle\epsilon\rangle(\varphi'\langle a'\rangle\psi') \mid f(a') = a \wedge \varphi' \in \varphi\{f^{-1}\} \wedge \psi' \in \psi\{f^{-1}\}\} \subseteq \mathscr{O}_b$
  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_b$

$\square$

Theorem 3.36. *The modal characterization of the rooted weak bisimulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{rw}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{rw}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_{rw}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
  $\forall i \in I : \phi_i \in \mathscr{O}_{rw}$

$\implies$ [*Inductive Hypothesis*]
$\forall i \in I : \phi\{f^{-1}\} \subseteq \mathscr{O}_{rw}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$\{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} \subseteq \mathscr{O}_{rw}$
$\implies$ [*Definition of* $\phi\{f^{-1}\}$]
$\phi_i\{f^{-1}\} \subseteq \mathscr{O}_{rw}$

- $\phi \equiv \neg\varphi$:
$\neg\varphi \in \mathscr{O}_{rw}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$\varphi \in \mathscr{O}_{rw}$
$\implies$ [*Inductive Hypothesis*]
$\varphi\{f^{-1}\} \subseteq \mathscr{O}_{rw}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$\{\bigwedge_{\varphi' \in \varphi\{f^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_{rw}$
$\implies$ [*Definition of* $\phi\{f^{-1}\}$]
$\phi\{f^{-1}\} \subseteq \mathscr{O}_{rw}$

- $\phi \equiv \langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\varphi,\ with\ \varphi \in \mathscr{O}_w$:
$\langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_{rw}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$\varphi \in \mathscr{O}_w$
$\implies$ [*Theorem 3.32*]
$\varphi\{f^{-1}\} \subseteq \mathscr{O}_w$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$\{\langle\epsilon\rangle\langle\alpha'\rangle\langle\epsilon\rangle\varphi' \mid f(\alpha') = \alpha \wedge \varphi' \in \varphi\{f^{-1}\}\} \subseteq \mathscr{O}_{rw}$
$\implies$ [*Definition of* $\phi\{f^{-1}\}$]
$\phi\{f^{-1}\} \subseteq \mathscr{O}_{rw}$

- $\phi \equiv \varphi,\ with\ \varphi \in \mathscr{O}_w$:
$\varphi \in \mathscr{O}_{rw}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$\varphi \in \mathscr{O}_w$
$\implies$ [*Theorem 3.32*]
$\varphi\{f^{-1}\} \subseteq \mathscr{O}_w$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
$\phi\{f^{-1}\} \subseteq \mathscr{O}_{rw}$

□

Theorem 3.37. *The modal characterization of the rooted delay bisimulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{rd}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{rd}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\forall i \in I : \phi_i \in \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{f^{-1}\} \subseteq \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} \subseteq \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi_i\{f^{-1}\} \subseteq \mathscr{O}_{rd}$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\varphi \in \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\{\bigwedge_{\varphi' \in \varphi\{f^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{rd}$

- $\phi \equiv \langle\epsilon\rangle\langle\alpha\rangle\varphi,$ *with* $\varphi \in \mathscr{O}_d$:
  $\langle\epsilon\rangle\langle\alpha\rangle\varphi \in \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\varphi \in \mathscr{O}_d$
  $\Longrightarrow$ [*Theorem 3.33*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_d$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
  $\{\langle\epsilon\rangle\langle\alpha'\rangle\varphi' \mid f(\alpha') = \alpha \wedge \varphi' \in \varphi\{f^{-1}\}\} \subseteq \mathscr{O}_{rd}$
  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{rd}$

- $\phi \equiv \varphi$, *with* $\varphi \in \mathscr{O}_d$:

  $\varphi \in \mathscr{O}_{rd}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

  $\varphi \in \mathscr{O}_d$

  $\implies$ [*Theorem 3.33*]

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_d$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{rd}$

□

Theorem 3.38. *The modal characterization of the rooted* η*-bisimulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{r\eta}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{r\eta}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:

  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_{r\eta}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $\forall i \in I : \phi_i \in \mathscr{O}_{r\eta}$

  $\implies$ [*Inductive Hypothesis*]

  $\forall i \in I : \phi\{f^{-1}\} \subseteq \mathscr{O}_{r\eta}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $\{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} \subseteq \mathscr{O}_{r\eta}$

  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]

  $\phi_i\{f^{-1}\} \subseteq \mathscr{O}_{r\eta}$

- $\phi \equiv \neg\varphi$:

  $\neg\varphi \in \mathscr{O}_{r\eta}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $\varphi \in \mathscr{O}_{r\eta}$

  $\implies$ [*Inductive Hypothesis*]

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{r\eta}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $\{\bigwedge_{\varphi' \in \varphi\{f^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_{r\eta}$

  $\implies$ [*Definition of* $\phi\{f^{-1}\}$]

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{r\eta}$

- $\phi \equiv \langle\alpha\rangle\langle\epsilon\rangle\varphi,$ *with* $\varphi \in \mathscr{O}_\eta$:
  $\langle\alpha\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_{r\eta}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\varphi \in \mathscr{O}_\eta$
  $\Longrightarrow$ [*Theorem 3.34*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_\eta$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\{\langle\alpha'\rangle\langle\epsilon\rangle\varphi' \mid f(\alpha') = \alpha \wedge \varphi' \in \varphi\{f^{-1}\}\} \subseteq \mathscr{O}_{r\eta}$
  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{r\eta}$

- $\phi \equiv \varphi,$ *with* $\varphi \in \mathscr{O}_\eta$:
  $\varphi \in \mathscr{O}_{r\eta}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\varphi \in \mathscr{O}_\eta$
  $\Longrightarrow$ [*Theorem 3.34*]
  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_\eta$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{r\eta}$

□

Theorem 3.39. *The modal characterization of the rooted branching bisimulation satisfies (R).*

Proof. Given $\phi \in \mathscr{O}_{rb}$, we want to prove that $\phi\{f^{-1}\} \subseteq \mathscr{O}_{rb}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I}\phi_i$:
  $\bigwedge_{i\in I}\phi_i \in \mathscr{O}_{rb}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $\forall i \in I : \phi_i \in \mathscr{O}_{rb}$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{f^{-1}\} \subseteq \mathscr{O}_{rb}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $\{\bigwedge_{i\in I}\phi'_i \mid \phi'_i \in \phi_i\{f^{-1}\}\} \subseteq \mathscr{O}_{rb}$
  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]
  $\phi_i\{f^{-1}\} \subseteq \mathscr{O}_{rb}$

- $\phi \equiv \neg\varphi$:

  $\neg\varphi \in \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\varphi \in \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\{\bigwedge_{\varphi' \in \varphi\{f^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{rb}$

- $\phi \equiv \langle\alpha\rangle\varphi,\ with\ \varphi \in \mathscr{O}_b$:

  $\langle\alpha\rangle\varphi \in \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\varphi \in \mathscr{O}_b$

  $\Longrightarrow$ [*Theorem 3.35*]

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_b$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\{\langle\alpha'\rangle\varphi' \mid f(\alpha') = \alpha \wedge \varphi' \in \varphi\{f^{-1}\}\} \subseteq \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Definition of* $\phi\{f^{-1}\}$]

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{rb}$

- $\phi \equiv \varphi,\ with\ \varphi \in \mathscr{O}_b$:

  $\varphi \in \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\varphi \in \mathscr{O}_b$

  $\Longrightarrow$ [*Theorem 3.35*]

  $\varphi\{f^{-1}\} \subseteq \mathscr{O}_b$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\phi\{f^{-1}\} \subseteq \mathscr{O}_{rb}$

$\square$

3.7.6 Abstraction Operator

The requirement found for the abstraction operator is the following:
(AB) $\phi \in \mathscr{O} \implies \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}^{\equiv}$

Theorem 3.40. *The modal characterization of the weak bisimulation satisfies (AB).*

Proof. Given $\phi \in \mathscr{O}_w$, we want to prove that $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_w$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\forall i \in I : \phi_i \in \mathscr{O}_w$
  $\implies$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\{\bigwedge_{i \in I} \phi'_i \mid \phi'_i \in \phi_i\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_w^{\equiv}$
  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi_i\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$
- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_w$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\varphi \in \mathscr{O}_w$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\{\bigwedge_{\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_w^{\equiv}$
  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$
- $\phi \equiv \langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\varphi \in \mathscr{O}_w$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
  $\varphi \in \mathscr{O}_w$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]

$\{\langle\epsilon\rangle\langle h_0\rangle \dots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i \in H, \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_w^{\equiv}$
$\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
$\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\varphi$:
  - $a \notin H$:
    $\langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_w$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
    $\varphi \in \mathscr{O}_w$
    $\implies$ [*Inductive Hypothesis*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_w$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \dots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\langle a\rangle\langle\epsilon\rangle\langle h'_0\rangle \dots \langle\epsilon\rangle\langle h'_k\rangle\langle\epsilon\rangle\varphi' \mid h_i, h'_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_w^{\equiv}$
    $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
    $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$
  - $a \in H$:
    $\phi\{\mathcal{T}_H^{-1}\} = \varnothing \implies \phi\{\mathcal{T}_H^{-1}\} \in \mathscr{O}_w^{\equiv}$

□

Theorem 3.41. *The modal characterization of the delay bisimulation satisfies (AB).*

Proof. Given $\phi \in \mathscr{O}_d$, we want to prove that $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:
  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\forall i \in I : \phi_i \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in \phi_i\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi_i\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\bigwedge_{\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\varphi \in \mathscr{O}_d$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\varphi \in \mathscr{O}_d$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
  $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i \in H, \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_d^{\equiv}$
  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\langle a\rangle\varphi$:

  - $a \notin H$:
    $\langle\epsilon\rangle\langle a\rangle\varphi \in \mathscr{O}_d$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
    $\varphi \in \mathscr{O}_d$
    $\implies$ [*Inductive Hypothesis*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\langle a\rangle\varphi' \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_d^{\equiv}$
    $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
    $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$
  - $a \in H$:
    $\phi\{\mathcal{T}_H^{-1}\} = \varnothing \implies \phi\{\mathcal{T}_H^{-1}\} \in \mathscr{O}_d^{\equiv}$

□

Theorem 3.42. *The modal characterization of the η-bisimulation satisfies (AB).*

Proof. Given $\phi \in \mathscr{O}_\eta$, we want to prove that $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:
  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_\eta$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\forall i \in I : \phi_i \in \mathscr{O}_\eta$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\{\bigwedge_{i\in I} \phi_i' \mid \phi_i' \in \phi_i\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_\eta^{\equiv}$
  $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi_i\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^{\equiv}$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_\eta$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\varphi \in \mathscr{O}_\eta$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\{\bigwedge_{\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_\eta^{\equiv}$
  $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\varphi$:
  $\langle\epsilon\rangle\varphi \in \mathscr{O}_\eta$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\varphi \in \mathscr{O}_\eta$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_\eta$]
  $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i \in H, \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_\eta^{\equiv}$
  $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle(\varphi\langle a\rangle\langle\epsilon\rangle\psi)$:
  - $a \notin H$:
    $\langle\epsilon\rangle(\varphi\langle a\rangle\langle\epsilon\rangle\psi)$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
    $\varphi, \psi \in \mathscr{O}_\eta$
    $\implies$ [*Inductive Hypothesis*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^\equiv \wedge \psi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^\equiv$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_\eta$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle(\varphi'\langle a\rangle\langle\epsilon\rangle\langle h'_0\rangle \ldots \langle\epsilon\rangle\langle h'_k\rangle\langle\epsilon\rangle\psi') \mid h_i, h'_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} \wedge \psi' \in \psi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_\eta^\equiv$
    $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
    $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_\eta^\equiv$
  - $a \in H$:
    $\phi\{\mathcal{T}_H^{-1}\} = \varnothing \implies \phi\{\mathcal{T}_H^{-1}\} \in \mathscr{O}_\eta^\equiv$

□

The same observation we did in the previous sections about the $Sub()$ and $\{f^{-1}\}$ operators, applies to the operator $\{\mathcal{T}_H^{-1}\}$ as well. So, in order to deal with $\langle\hat{\tau}\rangle$, we extend the definition of this function.

Definition 3.17. *We define:*

- $(\langle\hat{\tau}\rangle\phi)\{\mathcal{T}_H^{-1}\} = \{\langle\hat{\tau}\rangle\phi' \mid \phi' \in \phi\{\mathcal{T}_H^{-1}\}\} \cup \{\langle h\rangle\phi' \mid h \in H \wedge \phi' \in \phi\{\mathcal{T}_H^{-1}\}\}$

Again, we need to prove that extending the operator's semantic, the requirement (AB) is still effective. In order to do it, we prove that, also for this case, the Lemma 3.12 is still valid.

Lemma 3.15. $\mathcal{T}_H(p) \models \phi \iff \exists\phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$

Proof. We apply induction on the structure of ϕ. The cases $\phi \equiv \neg\varphi$, $\phi \equiv \bigwedge_{i\in I} \phi_i$, $\phi \equiv \langle x\rangle\varphi$, $\phi \equiv \langle\epsilon\rangle\varphi$ are already covered in the proofs of Lemma 3.12.

- $\phi \equiv \langle\hat{\tau}\rangle\varphi$:
  $\mathcal{T}_H(p) \models \phi$
  $\iff$
  $\mathcal{T}_H(p) \models \varphi \vee (\mathcal{T}_H(p) \xrightarrow{\tau} \mathcal{T}_H(p') \wedge \mathcal{T}_H(p') \models \varphi)$
  $\iff$ [(ABS2), (ABS3)]
  $\mathcal{T}_H(p) \models \varphi \vee ((p \xrightarrow{\tau} p' \vee \exists h \in H : p \xrightarrow{h} p') \wedge \mathcal{T}_H(p') \models \varphi)$
  $\iff$ [*Inductive Hypothesis*]
  $(\exists\varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \models \varphi') \vee (\exists\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}, h \in H : p \models \langle h\rangle\varphi') \vee$
  $(\exists\varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \models \langle\tau\rangle\varphi')$

$\iff$
$(\exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} : p \models \langle \hat{\tau} \rangle \varphi') \vee (\exists \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}, h \in H : p \models \langle h \rangle \varphi')$
$\iff$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
$\exists \phi' \in \phi\{\mathcal{T}_H^{-1}\} : p \models \phi'$

□

Theorem 3.43. *The modal characterization of the branching bisimulation satisfies (AB).*

Proof. Given $\phi \in \mathscr{O}_b$, we want to prove that $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:
  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\forall i \in I : \phi_i \in \mathscr{O}_b$
  $\implies$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\{\bigwedge_{i \in I} \phi_i' \mid \phi_i' \in \phi_i\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_b^{\equiv}$
  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi_i\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$
- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\varphi \in \mathscr{O}_b$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\{\bigwedge_{\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_b^{\equiv}$
  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$
- $\phi \equiv \langle \epsilon \rangle (\varphi \langle \hat{\tau} \rangle \psi)$:
  $\langle \epsilon \rangle (\varphi \langle \hat{\tau} \rangle \psi) \in \mathscr{O}_b$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_b$]
  $\varphi, \psi \in \mathscr{O}_b$

$\Longrightarrow$ [*Inductive Hypothesis*]
$\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv} \wedge \psi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$
$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_b$]
$\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle(\varphi'\langle\hat{\tau}\rangle\psi') \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} \wedge \psi' \in \psi\{\mathcal{T}_H^{-1}\}\} \cup$
$\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle(\varphi'\langle h\rangle\psi') \mid h_i, h \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} \wedge \psi' \in \psi\{\mathcal{T}_H^{-1}\}\}$
$\subseteq \mathscr{O}_b^{\equiv}$
$\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
$\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle(\varphi\langle a\rangle\psi)$:
  - $a \notin H$:
    $\langle\epsilon\rangle(\varphi\langle a\rangle\psi)$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_b$]
    $\varphi, \psi \in \mathscr{O}_b$
    $\Longrightarrow$ [*Inductive Hypothesis*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv} \wedge \psi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_b$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle(\varphi'\langle a\rangle\psi') \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\} \wedge \psi' \in \psi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_b^{\equiv}$
    $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
    $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$
  - $a \in H$:
    $\phi\{\mathcal{T}_H^{-1}\} = \varnothing \implies \phi\{\mathcal{T}_H^{-1}\} \in \mathscr{O}_b^{\equiv}$

□

Theorem 3.44. *The modal characterization of the rooted weak bisimulation satisfies (AB).*

Proof. Given $\phi \in \mathscr{O}_{rw}$, we want to prove that $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rw}^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:
  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_{rw}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
  $\forall i \in I : \phi_i \in \mathscr{O}_{rw}$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rw}^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
  $\{\bigwedge_{i\in I} \phi'_i \mid \phi'_i \in \phi_i\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rw}^{\equiv}$
  $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi_i\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rw}^{\equiv}$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_{rw}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
  $\varphi \in \mathscr{O}_{rw}$
  $\Longrightarrow$ [*Inductive Hypothesis*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rw}^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
  $\{\bigwedge_{\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_{rw}^{\equiv}$
  $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rw}^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\varphi$, *with* $\varphi \in \mathscr{O}_w$:
  - $\alpha = \tau$:
    $\langle\epsilon\rangle\langle\tau\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_{rw}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
    $\varphi \in \mathscr{O}_w$
    $\Longrightarrow$ [*Theorem 3.40*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]
    $\{\langle\epsilon\rangle\langle h'_0\rangle \ldots \langle\epsilon\rangle\langle h'_k\rangle\langle\epsilon\rangle\varphi' \mid h'_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_w^{\equiv}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$, *Modal Characterization of* $\mathscr{O}_{rw}$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\langle\epsilon\rangle\langle h'_0\rangle \ldots \langle\epsilon\rangle\langle h'_k\rangle\langle\epsilon\rangle\varphi' \mid h_i, h'_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_w^{\equiv} \wedge \{\langle\epsilon\rangle\langle\tau\rangle\langle\epsilon\rangle\langle h'_0\rangle \ldots \langle\epsilon\rangle\langle h'_k\rangle\langle\epsilon\rangle\varphi' \mid h'_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rw}^{\equiv}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\langle h'_0\rangle \ldots \langle\epsilon\rangle\langle h'_k\rangle\langle\epsilon\rangle\varphi' \mid h_i, h'_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \cup \{\langle\epsilon\rangle\langle\tau\rangle\langle\epsilon\rangle\langle h'_0\rangle \ldots \langle\epsilon\rangle\langle h'_k\rangle\langle\epsilon\rangle\varphi' \mid h'_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rw}^{\equiv}$
    $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
    $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rw}^{\equiv}$
  - $\alpha \notin H \cup \{\tau\}$:
    $\langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_{rw}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]
    $\varphi \in \mathscr{O}_w$
    $\Longrightarrow$ [*Theorem 3.40*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_w$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\langle h'_0\rangle \ldots \langle\epsilon\rangle\langle h'_k\rangle\langle\epsilon\rangle\varphi' \mid h_i, h'_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_w^{\equiv}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

$\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\langle\alpha\rangle\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i, h_i' \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rw}^{\equiv}$
$\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
$\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rw}^{\equiv}$

– $\alpha \in H$:
$\phi\{\mathcal{T}_H^{-1}\} = \varnothing \implies \phi\{\mathcal{T}_H^{-1}\} \in \mathscr{O}_{rw}^{\equiv}$

- $\phi \equiv \varphi$, *with* $\varphi \in \mathscr{O}_w$:

  $\varphi \in \mathscr{O}_{rw}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

  $\varphi \in \mathscr{O}_w$

  $\implies$ [*Theorem 3.40*]

  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_w^{\equiv}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rw}$]

  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rw}^{\equiv}$

□

Theorem 3.45. *The modal characterization of the rooted delay bisimulation satisfies (AB).*

Proof. Given $\phi \in \mathscr{O}_{rd}$, we want to prove that $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rd}^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:

  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_{rd}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

  $\forall i \in I : \phi_i \in \mathscr{O}_{rd}$

  $\implies$ [*Inductive Hypothesis*]

  $\forall i \in I : \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rd}^{\equiv}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

  $\{\bigwedge_{i\in I} \phi_i' \mid \phi_i' \in \phi_i\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rd}^{\equiv}$

  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]

  $\phi_i\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rd}^{\equiv}$

- $\phi \equiv \neg\varphi$:

  $\neg\varphi \in \mathscr{O}_{rd}$

  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

  $\varphi \in \mathscr{O}_{rd}$

  $\implies$ [*Inductive Hypothesis*]

$\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rd}^{\equiv}$
$\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
$\{\bigwedge_{\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_{rd}^{\equiv}$
$\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
$\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rd}^{\equiv}$

- $\phi \equiv \langle\epsilon\rangle\langle\alpha\rangle\varphi$, *with* $\varphi \in \mathscr{O}_d$:
  - $\alpha = \tau$:
    $\langle\epsilon\rangle\langle\tau\rangle\varphi \in \mathscr{O}_{rd}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
    $\varphi \in \mathscr{O}_d$
    $\implies$ [*Theorem 3.41*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_d$, *Modal Characterization of* $\mathscr{O}_{rd}$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \dots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_d^{\equiv} \wedge \{\langle\epsilon\rangle\langle\tau\rangle\varphi' \mid \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rd}^{\equiv}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \dots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \cup \{\langle\epsilon\rangle\langle\tau\rangle\varphi' \mid \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rd}^{\equiv}$
    $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
    $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rd}^{\equiv}$
  - $\alpha \notin H \cup \{\tau\}$:
    $\langle\epsilon\rangle\langle\alpha\rangle\varphi \in \mathscr{O}_{rd}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
    $\varphi \in \mathscr{O}_d$
    $\implies$ [*Theorem 3.41*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_d$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \dots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\langle\alpha\rangle\varphi' \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_d^{\equiv}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]
    $\{\langle\epsilon\rangle\langle h_0\rangle \dots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\langle\alpha\rangle\varphi' \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rd}^{\equiv}$
    $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
    $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rd}^{\equiv}$
  - $\alpha \in H$:
    $\phi\{\mathcal{T}_H^{-1}\} = \varnothing \implies \phi\{\mathcal{T}_H^{-1}\} \in \mathscr{O}_{rd}^{\equiv}$
- $\phi \equiv \varphi$, *with* $\varphi \in \mathscr{O}_d$:
  $\varphi \in \mathscr{O}_{rd}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

$\varphi \in \mathscr{O}_d$

$\Longrightarrow$ [*Theorem 3.41*]

$\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_d^{\equiv}$

$\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rd}$]

$\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rd}^{\equiv}$

□

Theorem 3.46. *The modal characterization of the rooted* η*-bisimulation satisfies (AB).*

Proof. Given $\phi \in \mathscr{O}_{r\eta}$, we want to prove that $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i \in I} \phi_i$:

  $\bigwedge_{i \in I} \phi_i \in \mathscr{O}_{r\eta}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $\forall i \in I : \phi_i \in \mathscr{O}_{r\eta}$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $\forall i \in I : \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $\{\bigwedge_{i \in I} \phi_i' \mid \phi_i' \in \phi_i\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$

  $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]

  $\phi_i\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$

- $\phi \equiv \neg\varphi$:

  $\neg\varphi \in \mathscr{O}_{r\eta}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $\varphi \in \mathscr{O}_{r\eta}$

  $\Longrightarrow$ [*Inductive Hypothesis*]

  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]

  $\{\bigwedge_{\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$

  $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]

  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$

- $\phi \equiv \langle\alpha\rangle\langle\epsilon\rangle\varphi$, *with* $\varphi \in \mathscr{O}_\eta$:

- $\alpha = \tau$:
  $\langle\tau\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_{r\eta}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\varphi \in \mathscr{O}_{\eta}$
  $\Longrightarrow$ [*Theorem 3.42*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{\eta}^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{\eta}$]
  $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{\eta}^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\{\langle\tau\rangle\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \cup \{\langle h\rangle\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h, h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$
  $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$
- $\alpha \notin H \cup \{\tau\}$:
  $\langle\alpha\rangle\langle\epsilon\rangle\varphi \in \mathscr{O}_{r\eta}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\varphi \in \mathscr{O}_{\eta}$
  $\Longrightarrow$ [*Theorem 3.42*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{\eta}^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{\eta}$]
  $\{\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{\eta}^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\{\langle\alpha\rangle\langle\epsilon\rangle\langle h_0\rangle \ldots \langle\epsilon\rangle\langle h_m\rangle\langle\epsilon\rangle\varphi' \mid h_i \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$
  $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$
- $\alpha \in H$:
  $\phi\{\mathcal{T}_H^{-1}\} = \varnothing \implies \phi\{\mathcal{T}_H^{-1}\} \in \mathscr{O}_{r\eta}^{\equiv}$

- $\phi \equiv \varphi$, *with* $\varphi \in \mathscr{O}_{\eta}$:
  $\varphi \in \mathscr{O}_{r\eta}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\varphi \in \mathscr{O}_{\eta}$
  $\Longrightarrow$ [*Theorem 3.42*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{\eta}^{\equiv}$
  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{r\eta}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{r\eta}^{\equiv}$

□

Theorem 3.47. *The modal characterization of the rooted branching bisimulation satisfies (AB).*

Proof. Given $\phi \in \mathscr{O}_{rb}$, we want to prove that $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rb}^{\equiv}$. We apply induction on the grammar production rules.

- $\phi \equiv \bigwedge_{i\in I} \phi_i$:
  $\bigwedge_{i\in I} \phi_i \in \mathscr{O}_{rb}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $\forall i \in I : \phi_i \in \mathscr{O}_{rb}$
  $\implies$ [*Inductive Hypothesis*]
  $\forall i \in I : \phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rb}^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $\{\bigwedge_{i\in I} \phi_i' \mid \phi_i' \in \phi_i\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rb}^{\equiv}$
  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi_i\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rb}^{\equiv}$

- $\phi \equiv \neg\varphi$:
  $\neg\varphi \in \mathscr{O}_{rb}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $\varphi \in \mathscr{O}_{rb}$
  $\implies$ [*Inductive Hypothesis*]
  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rb}^{\equiv}$
  $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
  $\{\bigwedge_{\varphi' \in \varphi\{\mathcal{T}_H^{-1}\}} \neg\varphi'\} \subseteq \mathscr{O}_{rb}^{\equiv}$
  $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rb}^{\equiv}$

- $\phi \equiv \langle\alpha\rangle\varphi$, *with* $\varphi \in \mathscr{O}_b$:

  - $\alpha = \tau$:
    $\langle\tau\rangle\varphi \in \mathscr{O}_{rb}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
    $\varphi \in \mathscr{O}_b$
    $\implies$ [*Theorem 3.43*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$
    $\implies$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
    $\{\langle\tau\rangle\varphi' \mid \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \cup \{\langle h\rangle\varphi' \mid h \in H \wedge \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rb}^{\equiv}$
    $\implies$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
    $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rb}^{\equiv}$

  - $\alpha \notin H \cup \{\tau\}$:
    $\langle\alpha\rangle\varphi \in \mathscr{O}_{rb}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
    $\varphi \in \mathscr{O}_b$
    $\Longrightarrow$ [*Theorem 3.43*]
    $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$
    $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]
    $\{\langle\alpha\rangle\varphi' \mid \varphi' \in \varphi\{\mathcal{T}_H^{-1}\}\} \subseteq \mathscr{O}_{rb}^{\equiv}$
    $\Longrightarrow$ [*Definition of* $\phi\{\mathcal{T}_H^{-1}\}$]
    $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rb}^{\equiv}$
  - $\alpha \in H$:
    $\phi\{\mathcal{T}_H^{-1}\} = \varnothing \implies \phi\{\mathcal{T}_H^{-1}\} \in \mathscr{O}_{rb}^{\equiv}$

- $\phi \equiv \varphi$, *with* $\varphi \in \mathscr{O}_b$:

  $\varphi \in \mathscr{O}_{rb}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\varphi \in \mathscr{O}_b$

  $\Longrightarrow$ [*Theorem 3.43*]

  $\varphi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_b^{\equiv}$

  $\Longrightarrow$ [*Modal Characterization of* $\mathscr{O}_{rb}$]

  $\phi\{\mathcal{T}_H^{-1}\} \subseteq \mathscr{O}_{rb}^{\equiv}$

$\square$

Conclusions

In this work, we investigated the congruence properties of several process algebraic operators with regard to a spectrum of behavioural semantics. We built upon previous results provided in [10], extending the range of analyzed operators and extending those results to abstract semantics. To the best of our knowledge it is the first attempt to face this quest in the context of abstract semantics.

For all these operators, general conditions have been provided in order to guarantee congruence of process equivalences defined by means of a modal characterization. We studied these constraints in relation with a wide range of semantics presented in the literature, both concrete and abstract ones. Although our conditions are sufficient and by no means necessary, they proved to yield the expected results with regard to all the considered semantics.

The results presented in this work can be further extended by including other operators (e.g. sequential composition, merge with communication) or considering different modal languages.

Bibliography

[1] L. Aceto, W.J. Fokkink and C. Verhoef.
Structural operational semantics.
Handbook of Process Algebra, North-Holland, Amsterdam, 2001. pp. 197–292.

[2] J.C.M. Baeten and C. Verhoef.
A congruence theorem for structured operational semantics with predicates.
E. Best, editor, CONCUR' 93, LNCS 715, Springer-Verlag, 1993. pp. 477-492.

[3] J.C.M. Baeten.
A brief history of process algebra.
Theoretical Computer Science 335.2, 2005. pp. 131-146.

[4] B. Bloom, S. Istrail and A.R. Meyer.
Bisimulation Can't be Traced: Preliminary Report.
Proceedings of the ACM Symposium on Principles of Programming Languages, 1988. pp. 229-239.

[5] B. Bloom, W.J. Fokkink and R.J. van Glabbeek.
Precongruence formats for decorated trace semantics.
ACM Transactions on Computational Logic 5.1, 2004. pp. 26-78.

[6] T. Chen, W.J. Fokkink and R.J. van Glabbeek.
On the axiomatizability of impossible futures.
Logical Methods in Computer Science 11 (3:17), 2015.

[7] R. De Simone.
Higher-level synchronising devices in Meije-SCCS.
Theoretical computer science 37, 1985. pp. 245-267.

[8] W.J. Fokkink, R.J. van Glabbeek and P. de Wind.
Divide and congruence: From decomposition of modal formulas to preservation of branching and η-bisimilarity.
Information and Computation 214, 2012. pp. 59-85.

[9] W.J. Fokkink and R.J. van Glabbeek.
Divide and Congruence II: Delay and Weak Bisimilarity.
Symposium on Logic in Computer Science, IEEE, 2016.

[10] M. Gazda and W.J. Fokkink.
Congruence from the operator's point of view: Compositionality requirements on process semantics.
Proceedings of the 7th Workshop on Structural Operational Semantics - SOS'10, Paris, Electronic Proceedings in Theoretical Computer Science 32, 2010. pp. 15-25.

[11] R.J. van Glabbeek.
Bounded nondeterminism and the approximation induction principle in process algebra.
Proceedings of the 4th Symposium on Theoretical Aspects of Computer Science, Springer Berlin Heidelberg, 1987. pp. 336-347.

[12] R.J. van Glabbeek.
The linear time-branching time spectrum.
J.C.M. Baeten and J.W. Klop, editors, CONCUR' 90, LNCS 458, Springer, 1990. pp. 278–297.

[13] R.J. van Glabbeek.
The linear time-branching time spectrum II.
E. Best, editor, CONCUR' 93, LNCS 715, Springer-Verlag, 1993, pp. 66-81.

[14] J.F. Groote.
Transition system specifications with negative premises.
Theoretical Computer Science 118.2, 1993. pp. 263-299.

[15] J.F. Groote and F. Vaandrager.
Structured operational semantics and bisimulation as a congruence.
Information and computation 100.2, 1992. pp. 202-260.

[16] M. Hennessy and R. Milner.
On observing nondeterminism and concurrency.
Lecture Notes in Computer Science 85, Springer Berlin Heidelberg, 1980.

[17] M. Hennessy and R. Milner.
Algebraic laws for nondeterminism and concurrency.
Journal of the ACM (JACM) 32, 1985. pp. 137-161.

[18] R.M. Keller.
Formal verification of parallel programs.
Communications of the ACM 19.7, 1976. pp. 371-384.

[19] R. Milner.
A calculus of communicating systems.
Lecture Notes in Computer Science 92, Springer Verlag, 1980.

[20] G.D. Plotkin.
A structural approach to operational semantics.
In Journal of Logic and Algebraic Programming 60.61, 2004. pp. 17-139.

[21] D. Veglioni and R. de Nicola.
Possible worlds process algebras.
D. Sangiorgi and R. de Simone, editors, CONCUR' 98, LNCS 1466, Springer, 1998. pp. 179-193.

[22] C. Verhoef.
A congruence theorem for structured operational semantics with predicates and negative premises.
Nordic Journal of Computing 2.2, 1995. pp. 274-302.