Anyon condensation

Topological symmetry breaking phase transitions and commutative algebra objects in braided tensor categories

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Abstract

Put your abstract or summary here, if your university requires it.
Sometimes a scream is better than a thesis

Manfred Eigen
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In condensed matter physics, a fundamental problem is the determination of the low-temperature phases or orders of a system. Dating back to Lev Landau [21, 22], the theory of symmetry breaking phase transitions forms a cornerstone in this respect. Put very crudely, there are two dual aspects to the picture it provides: symmetry breaking and particle condensation. This general mechanism can actually be identified in a wide variety of situations and explains many phenomenon, ranging from exotic things like superconductivity – where the electric $U(1)$ symmetry is broken by a condensate of Cooper pairs – to everyday experiences as water becoming ice (Figure 1.1). Group theory provides the right language to discuss many aspects of symmetry breaking, and the classification of the different phases is essentially equivalent to the classification of subgroups of the symmetry group of the system.

![Figure 1.1: The structure of an ice crystal](image)

In the freezing of water, or any other liquid-to-solid transition, the molecules order in a regular lattice structure. This breaks the translational and rotational symmetry of the fluid state. Using group theory, one may see that there are 230 qualitatively different types of crystals corresponding to the 230 space groups.

In the past few decades, a growing interest has emerged to study topological phases. These phases of matter fall outside the conventional Landau symmetry breaking scheme,
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and are therefore of fundamental importance. Especially in two dimensions, topological phases offer access to fundamentally new physics. They can have anyonic quasiparticle excitations – particles that are neither bosons nor fermions – that might one day become an invaluable resource for quantum computers, as they offer a route to the fault-tolerant storage and manipulation of quantum information known as Topological Quantum Computation.

The best known physical realization of topological phases are the fractional quantum Hall fluids. These exotic states occur in two-dimensional electron gases in a strong perpendicular magnetic field. The quantum numbers that distinguish them are not related to symmetry in the usual way, but are conserved due to topological properties. The system is also said to exhibit topological order. Similar phases might occur in rotating Bose gases, high $T_c$ superconductors, and possibly many more systems. One can in fact show that there are infinitely many different topological phases possible, which suggests a world of possibilities if we ever gain enough control to engineer systems that realize any phase of choice.

From a mathematical point of view, interesting structures have entered the theory. The description of topological phases involves conformal field theory, topological quantum field theory, quantum groups and tensor categories, which are all heavily interlinked. They are studied by mathematicians and theoretical physicists alike and link topics like string theory, low-dimensional topology and knot theory.

In the forthcoming, we will be concerned with topological phases in 2+1 dimensions, with the goal to study phase transitions between these kind of phases. A general formalism, known as topological symmetry breaking or quantum group symmetry breaking, has been developed in the literature before. In this thesis, we put this in the context of unitary braided tensor categories, the mathematical structure underlying anyonic excitations in topological phases. We will discuss important classes of examples for which we do explicit calculations on the $F$-symbols and $R$-symbols, the data needed to define these structures. We connect the theory of topological symmetry breaking to a topic heavily studied by mathematicians, namely algebras in tensor categories, and argue that a commutative algebra is the same as a well-defined Bose condensate. Using this new viewpoint, we can calculate the topological $S$-matrix of the broken phases using only data from the original theory. This makes useful as an order parameter for topological order. Representative examples are worded out in detail.

1.1 Outline

This thesis is organized as follows.

In chapter two we discuss more of the detail of topological order, the fractional quantum Hall effect and some related topics. Important is the discussion about anyons that are special to planar systems. They can have fractional spin and obey braid statistics. The existence of these exotic particles is tightly connected to the topological properties of three-dimensional space which also allow us to tie our shoelaces. We give examples of wave functions for quantum Hall states that have been proposed
and some subsequent advances and the role of conformal field theory. The larger picture of how conformal field theory, topological field theory and topological order are related is touched upon. Some indicators of topological order have been proposed in the literature, namely ground state degeneracy and topological entanglement entropy. We will have a general discussion of those.

In chapter three we develop the formalism describing the structure underlying anyonic excitations in topological phases. Mathematicians know this as the study of unitary braided tensor categories. A graphical formalism is best suited for these kind of considerations. Important aspects are the fusion of quantum numbers and the braiding operation. Using the graphical formalism we will give a proof of the Verlinde formula. How to describe states and compute physical amplitudes and probabilities is subsequently considered.

Chapter four treats a few paradigmatic examples of anyon models. These serve to clarify how the preceding structures arise in a physical context and are used later to illustrate and develop new theory concerning phase transitions. We start with discussing the well known Fibonacci model. This is an important example since despite it’s simplicity, it is universal for quantum computation. Next we describe discrete gauge theories, the structure of which is given by the quantum double of a finite group. Discrete gauge theories were early examples of topological quantum field theories, and much of the theory concerning the breaking of quantum group symmetries was first developed in this context. Finally, we present some data that is important for a class of examples denoted $su(2)_k$. These are important for Chern-Simons theory and Wess-Zumino-Witten models of conformal field theory.

In the fifth chapter, we discuss quantum group symmetry breaking. This starts with a discussion about quantum groups. What kind of structures are these and why are they relevant to physics? The important property of quantum groups is that they provide a description of the particle spectrum of certain theories as the irreducible representations, as is a common scheme in quantum physics, but in addition describe the topological interactions between particles by some additional structures that they have. If read with a physicist’s pair of glasses, the axioms of quantum groups actually seem tailor made for this purpose. After this, we will review the quantum group symmetry breaking approach to phase transitions between topologically ordered phases. Not all excitations of the broken phase might be point-like. Some will pull strings in the condensate which gives a mechanism and get confined to the boundary. We will show the details in some paradigmatic examples. Relations concerning the topological entanglement entropy of multiphase systems are discussed and a universal quantity called the “quantum embedding index” is introduced.

In chapter six, we begin from scratch with the description phase transitions between topologically ordered phases using the full diagrammatic formalism from chapter three. This begins by the definition of a condensate obeying certain consistency conditions, which mathematicians would call the axioms for a commutative algebra. The particle spectrum of the broken theory and confinement are discussed. In principle, one would be able to calculate any diagram in the broken phase in terms of the data for the
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unbroken phase. As an example, we calculate the $S$-matrix.

Chapter seven describes a concept that plays an important role in the mathematical literature. This is known as the center of an algebra. In the examples we have used throughout, we will investigate what this is and if it has physical relevance.

Finally, chapter eight contains the conclusion and outlook.
1. INTRODUCTION
In this chapter, we give a brief discussion of some essential topics concerning topological phases and topological order in 2+1 dimensions. These are anyons, the FQH effect, some generalities of the mathematical structures that are involved, and indicators for topological order that have been proposed in the literature. The chapter serves as an more thorough introduction to motivate the main body of the text and to sketch a bit of the history the topic. Relevant references are included as much as possible.

2.1 Anyons

In the most mundane spacetime, that of 3+1 dimensions, particles fall in precisely two classes: bosons and fermions. These can either be defined by their exchange properties or spins. The importance of this difference between particles can of course hardly be overstated. It is at the root of the Pauli exclusion principle and Bose-Einstein condensation, and determines whether macroscopic numbers of particles obey Fermi-Dirac or Bose-Einstein statistics.

According to textbook quantum mechanics, the wave function of a multi-particle state of fermions is obtained by anti-symmetrising the tensor product of single-particle wave functions, while for bosons the wave function is symmetrised. This procedure long stood as a corner stone of quantum mechanics and is still a valid procedure in many applications. But one may object that in a quantum theory with indistinguishable particles, the labelling of particle coordinates has no physical significance and brings unobservable elements into the theory.

The consequences of above observation were first fully appreciate by Leinaas and Myrheim. In their seminal paper [24], they show that the exchange properties of particles are tightly connected to the topology of the configuration space of multi-particle systems. In fact, as was shown in [20], particle types correspond to unitary
representations of the fundamental group of this space.

Let us discuss this in a spacetime picture. The trajectories of particles can now conveniently be imagined by their world lines through spacetime. We will consider trajectories that leave the system in a configuration identical to the configuration it started in and that do not have intersecting world lines. These correspond to free propagation of the system without scattering events. If the particles are distinguishable, the world lines should start and end at the same spatial coordinates. If the particles are indistinguishable, the world lines are allowed to interchange spatial coordinates.

![Spacetime trajectories](image)

**Figure 2.1: Spacetime trajectories** - Two spacetime trajectories for a system of three particles with identical initial and final state. If the particles are of different type, world lines should start and end at the same coordinates in space (left), while for indistinguishable particles, permutations of spatial coordinates are allowed (right).

The trajectories fall in distinct topological classes, depending to whether or not they can be deformed into each other by smooth deformations of the world lines (without intermediate intersections). If space has three or more dimensions one can see that there is no topologically different notion of ‘under’ and ‘over’ crossing of world lines, but any exchange of coordinates is simply that: an exchange. So the exchange of particles is governed by the permutation group $S_N$.

However, when the particles live in a plane – so spacetime is three-dimensional – such a distinction of crossings becomes essential to incorporate the difference between interchanging the particles clockwise or counter-clockwise. The relevant group is not $S_N$ but $B_N$, Artins $N$-stranded braid group [2].

This group is generated by the two different interchanges or half-braidings

$$R_i = i_{i+1}^i, \quad R_i^{-1} = i_{i+1}^i$$

(2.1)

which are inverse to each other. Here, $1 \leq i \leq N - 1$ labels the strand of the braid.

These generators are subject to defining relations corresponding to topological manipulations of the braid. Pictorially they are given by
2.1 Anyons

We can also write them algebraically as

\[ R_i R_j = R_j R_i \quad \text{for} \quad |i - j| \geq 2 \]

\[ R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1} \quad \text{for} \quad 1 \leq i \leq N - 1 \]  (2.2)

The second relation is known as the famous Yang-Baxter equation.

As remarked before, the important difference between particles in the plane and particles in three-dimensional space is the difference between over and under crossings of world lines. The relation to statistics in higher dimensions represented on the level of groups by the fact that we can pass from the braid group \( B_N \) to the permutation group \( S_N \) by implying the relation \( R_i = R_{i-1}^{-1} \), or equivalently \((R_i)^2 = 1\). This one extra relation makes a huge difference for the properties of the group. While the permutation group is finite (\(|S_N| = N!\)), the braid group is infinite, even for two strands. The representation theory of the braid group is therefore much richer than that of the permutation group. This gives the possibility of highly non-trivial exchange statistics in 2+1 dimensions, also referred to as 'braid statistics'. Particles that obey these exotic braid statistics were dubbed \textit{anyons} by Wilczek [35]. Bosons and fermions, in this context, just correspond to two of an infinite number of possibilities. Since distinguishable anyons can also have nontrivial braiding, it is better to think of braiding statistics as a kind of topological interaction.

Bosons and fermions correspond to the two one-dimensional representations of \( S_N \), even and odd respectively. In principle, higher dimensional representations of \( S_N \), known as ‘parastatistics’ [?], could lead to more particle types, but it has been shown that these can be reduced to the one-dimensional representations at the cost of introducing additional quantum numbers [?].

One-dimensional representations of the braid group simply correspond to a choice of phase \( \exp(i\theta) \) assigned to an elementary interchange. Since, in this case the order of the exchange is unimportant because phases commute, anyons corresponding to one-dimensional representations are called Abelian. However, it is no longer true that higher dimensional representations of the braid group can be reduced. This gives rise to so called non-Abelian anyons, particles that implement higher dimensional braid
statistics. These non-Abelian anyons are the ones that can be used for Topological Quantum Computation as suggested by Kitaev in [18]. See [26,28] for a review.

2.2 The fractional quantum Hall effect

The most prominent system with anyonic excitations is the fractional Quantum Hall effect, so we include a brief discussion of some relevant aspects here. For more details, we refer to the literature. See e.g. [27,37] for a thorough introduction.

In 1879, Edwin Herbert Hall discovered a phenomenon in electrical conductors that we know today as the Hall effect. Under influence of an applied electric field, the electrons will start moving in the direction of the field, leading to current. But the application of a perpendicular magnetic field will, due to the Lorentz force, lead to a component to the current which is orthogonal to both the electric and magnetic field. This is the Hall effect.

As an idealization, we may consider a two-dimensional electron gas (2DEG) with coordinates \((x, y) = (x^1, x^2)\) that we submit to an electric field in the \(x\)-direction and a perpendicular magnetic field \(B\) in the \(z\) direction. The relation between the current and the electric field is given by the resistivity tensor \(\rho\) and the conductivity \(\sigma\), through the relations

\[
E^\mu = \rho_{\mu\nu} J^\nu, \quad J^\mu = \sigma_{\mu\nu} E^\nu
\] (2.3)

Assuming homogeneity of the system, one can use a relativistic argument to deduce that [16]

\[
\rho = \frac{B}{nec} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma = \frac{nec}{B} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\] (2.4)

with \(n\) the electron density and \(B\) the strength of the magnetic field. Remarkably, since \(\rho_{xx} = 0\) and \(\sigma_{xx} = 0\), the system behaves as a perfect insulator and a perfect conductor in the direction of the electric field simultaneously.

The Hall resistivity is defined as \(\rho_H = \rho_{xy}\). According to the equations above, we must have

\[
\rho_H = \frac{B}{nec}
\] (2.5)

The derivation of this equation leans critically on Lorentz invariance, but not on much more. Therefore, if there is no preferred reference frame, this result should be very robust. It should in particular hold whether we consider quantum or classical electrodynamics. It is therefore striking that it does not agree with experimental data. Experiments reveal that at low temperatures (\(\sim 10 \text{ mK}\)) and high magnetic field (\(\sim 10 \text{T}\)) the dependence of \(\rho_H\) on \(B\) is not linear, but in stead plateaus develop at precisely quantized values

\[
\rho_H = \frac{1}{\nu e^2}
\] (2.6)

The number \(\nu\) is called the filling fraction and is usually written as \(\nu = N_e/N_\Phi\). Here \(N_e\) is the number of electrons, while \(N_\Phi\) is the number of flux-quanta piercing the
2.2 The fractional quantum Hall effect

2DEG. To get a better understanding of the filling fraction, let us briefly point out where it comes from.

From analysing the single-particle Hamiltonian, one sees that the energy spectrum forms so called Landau levels

\[ E_n = (n + \frac{1}{2})\hbar\omega_c, \quad \omega_c = \frac{eB}{mc} \]  

(2.7)

In a finite sized system of area \( A \), the available states for each Landau level turns out to be [16]

\[ N_{\Phi} = A \frac{B}{\Phi_0} \]  

(2.8)

Here \( \Phi_0 = \frac{\hbar c}{e} \) is the the fundamental flux quantum, so apparently the number of available states for each energy level is the same as the number of flux-quanta that pierce the system. Hence, \( \nu = N_e/N_{\Phi} \) is the ratio between the number of filled energy states – the electrons – and the number of available energy states, which is why it is called the filling fraction.

Plateaus at integer values for the filling fraction were first found in 1980 by von Klitzing at al. [19], and are referred to as the integer quantum Hall (IQH). Plateaus at fractional filling fractions – found two years later by Tsui et al. [7] – are the hallmark for the fractional quantum Hall effect (FQHE). While the IQHE can essentially be understood from the single particle physics, neglecting the Coulomb interaction, the FQHE is an represents an intriguing interplay of many interacting electrons. The Coulomb interaction is crucial to explain it’s features.

![Figure 2.2: The quantum Hall effect](image)

**Figure 2.2: The quantum Hall effect** - The Hall resistance \( R_{xy} = R_H = V_y/I_x \) and the longitudinal resistance \( R_{xx} = V_x/I_x \) are plotted against a varying magnetic field. (In 2d, the resistance is the same as the resistivity: \( R = \sigma \).

An illustration of the measurement set-up is also given on the top left. Taken from ref. [32]
2. TOPOLOGICAL ORDER IN THE PLANE

Figure 2.2 shows a typical graph of the plateaus in the first Landau level, the \( \nu = 1/3 \) being the plateau that was first discovered. Note that the longitudinal resistivity \( \sigma_{xx} \) drops to zero at the plateaus, which means that the conductivity tensor is off-diagonal as in (2.4). Hence, a dissipationless transverse current flows in response to an applied electric field. In particular, when we thread an extra magnetic flux quantum through the medium, the system will expel a net charge of \( \nu e \) due to the induced magnetic field. Or, put differently, this will create a of charge \( -\nu e \) and one flux quantum, illustrating the intimate coupling between charge and flux in the quantum Hall effect. It is a first indication that these systems harvest elementary quasi-particle excitations that are charge-flux composites that, due to their internal Aharonov-Bohm interaction, can be anyons. The peculiar fact that these quasi-particles have fractional charge when \( \nu \) is non-integer has been experimentally verified in [17].

With the application of Topological Quantum Computing in mind, most interest lies in systems exhibiting non-Abelian anyons. In this respect, a series of plateaus that has been observed in the second Landau level [?] offers the most promising platform. Especially the \( \nu = \frac{5}{2} \) plateau has received much attention. See figure 2.3.

![Figure 2.3: Plateaus in the second Landau level - The resistivity is plotted against a varying magnetic field. The graph shows plateaus in the second Landau level. The filling fraction \( \nu = p/q \) with \( 2 < \nu < 4 \) is indicated. Taken from ref. [?].](image)

We will leave most of the theoretical subtleties concerning the quantum Hall effects untouched. But what we do want point out in the paragraph below is the relation of

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2.2 The fractional quantum Hall effect

wave function for the FQHE to conformal field theory (CFT) and topological quantum field theory, as it relates our discussion of topological symmetry breaking phase transitions.

2.2.1 Conformal and topological quantum field theory

In the course of time, a series of wave functions has been proposed to capture the physics of the fractional quantum Hall effect at different plateaus. The Laughlin wave function [23] was the first to be proposed. It can account for the plateaus at filling fraction \( \nu = 1/M \) for odd \( M \), and reads

\[
\Psi_L(z) = \prod_{i<j} (z_i - z_j)^M \exp \left[ \frac{1}{4} \sum_i |z_i|^2 \right] \tag{2.9}
\]

Here the \( z_i \) label the complex coordinates for \( N_e \) electrons.\(^1\)

The Laughlin state describes an electronic system where the electrons carry magnetic vortices. This attachment of vortices to electrons is paradigmatic for the various quantum Hall states that have been proposed in the literature. Note that to ensure a fermionic system, i.e. the wave function is anti-symmetric under interchange of the coordinates, we need \( M \) to be odd.

Skipping subsequent developments in the history of the FQHE (e.g. the Haldane-Halpern hierarchy [21,21,21] and the composite fermion approach by Jain [21]), we want to turn our attention to an observation first made by Moore and Read. In [21] they noted that the Laughlin wave function could be obtained as a correlator in a certain rational conformal field theory (RCFT). In fact, they argue that this relation should be general and conjecture that every FQHE state should be related to conformal field theory, which would give a means to classify FQHE states.

Without going into the details of conformal field theory (see e.g. [21,21] for introductory texts and [21,21] for further information), let us pause for a moment at the picture they sketch. They motivate their search for a formulation in terms of RCFT by pointing to the development of Ginzberg-Landau effective field theories for the FQHE effect where the action contains a Chern-Simons term. Witten showed [21] that there is a connection between Chern-Simon theory in a three dimensional bulk and a Wess-Zumino-Witten theory on the two-dimensional edge. Moore and Read argue that the long range effective field theory should essentially constitute a pure Chern-Simons theory and that a related RCFT should come in to play when we consider a two-dimensional edge of the spacetime. Interestingly, the RCFT can account for wave functions by means of correlators in a (2+0)d interpretation, but also governs the edge excitation in a (1+1)d interpretation.

Chern-Simons theory is an example of a topological quantum field theory (TQFT). This is apparent from the fact that the spacetime metric does not enter the Chern-

\(^1\)The magnetic length \( \ell_B = \sqrt{\frac{\hbar c}{eB}} \) is set to unity, or, equivalently, the complex coordinates are taken \( z = (x + iy)/\ell_B \).
Simons action

\[ S_{CS} = \frac{k}{4\pi} \int d^3x \, \text{Tr} \epsilon^\mu\nu\rho (a_\mu \partial_\nu a_\rho + \frac{2}{3} a_\mu a_\nu a_\rho) \].

(2.10)

A topological quantum field theory is generally a quantum field theory for which all correlation functions are invariant under arbitrary smooth deformations of the base manifold [?, ?].

The picture Moore and Read proposed for the FQHE is a general picture we have in mind when we discuss topological order. The long range physics is governed by an effective field theory that is a TQFT, hence we also speak of a topological phase. Often, the physics of a time slice and of gapless edge excitations is controlled by CFTs with the same topological order. Although this might not be accurate in all situations it provides a convenient view on things.

Figure 2.4: A global view on the TQFT/CFT connection - A schematic view of how TQFT and CFT at least in certain theories, are related. A TQFT governs the bulk of the cylinder. Time slices are described by a (2+0) dimensional CFT. An example of this are the FQHE wave functions constructed from CFT correlators. Gapless boundary excitations are described by a CFT with the same topological order when there is no edge reconstruction.

Moore and Read proposed the following wave function that could explain an even denominator FQH plateau

\[ \Psi_{\text{Pf}}(z) = \text{Pfaff} \left( \frac{1}{z_i - z_j} \right) \prod_{i<j} \exp \left[ \frac{1}{4} \sum_i |z_i|^2 \right] \]  

(2.11)

where Pfaff \( A_{ij} \) is the square root of the determinant of \( A \). The filling fraction \( \nu = 1/M \). It’s particular benefit lies in the fact that, for \( M = 2 \), it describes a fermionic system with \( \nu = 1/2 \). Fermionic systems with even denominators cannot be described by the Laughlin wave function and states derived from this. This Moore-Read state is a likely candidate for the plateau at \( \nu = 5/2 = 2 + \frac{1}{2} \), which has two completely filled Landau levels and one level at half filling. As it features quasi-holes that are
2.3 Other approaches to topological order

non-Abelian anyons and the \( \nu = 5/2 \) plateau is experimentally accessible, this is a promising candidate for the experimental verification of non-Abelian braid statistics. This experimental challenge does not seem to have been met decisively yet.

The Moore-Read state is constructed from the so called Ising CFT. Subsequent generalization of the ideas of Moore and Read was done by Read and Rezayi \[7\]. They constructed a series of states based on CFT related to \( su(2)_k \) WZW models (that include the Ising CFT), that could explain plateaus in the second Landau level. Interestingly, one of these states encompasses so called Fibonacci anyons which are universal for quantum computation.

2.3 Other approaches to topological order

The FQHE in 2DEGs is not the only place to look for (non-Abelian) anyons. Systems that might have states very similar to the FQH states discussed above are \( p_x + ip_y \) superconductors \[7, 8\] and rotating Bose gases \[7, 8, 9\], though the latter of course would display a bosonic version of the FQHE.

We should also note the existence of two lattice models that realize topological order, at this point. The first is known as the Kitaev model. In its most general form, this model features interacting spins placed on the edges of an arbitrary lattice \[18\]. The spins are labelled by the elements of a group. The Hamiltonian consists of mutually commuting projectors, and is therefore completely solvable. The projectors can be recognized as generalizations of either electric or magnetic interactions. The electric operators act on the edges joining at a vertex and ensure that the ground state transforms trivially under the action of the group, which can be seen as a sort of gauge transformation. The magnetic operators act on plaquettes and essentially project out states of trivial flux through the plaquette. In the simplest case, where the spins are labelled by \( \mathbb{Z}_2 \) and a square lattice is taken, the ground state can be readily identified as a loop gas.

This lattice model realizes a discrete gauge theory, that we discuss in the following chapter. For more information we refer to the literature. Especially \[7\] and \[8\] are interesting in the context of this thesis. They study the Kitaev model with boundary, which can be seen as an explicit symmetry breaking mechanism and relates to the phase transitions that we will discuss in subsequent chapters. These gauge theories might also be realizable in Josephson junction arrays \[7\].

The other paradigmatic lattice model for topological order was introduced by Levin and Wen under the name of string-net condensates \[7\]. This system lives on a trivalent lattice, where, again, spins are placed on edges and interact at vertices. The input is tensor category. We will discuss these structures in detail in \ref{chapter4}. The Hamiltonian has a similar structure as in the Kitaev model and is also completely solvable.

Remarkably, it has been shown that the Kitaev model can be mapped to a Levin-Wen type model, making the Kitaev model essentially a special case of the more general string-net models. What is interesting about this, is that the language of group theory is somehow insufficient to describe topological order in its fullest generality. They
argue that the language of tensor categories is more appropriate to discuss topological order, and indeed these structures appear in many facets of the theory.

In the next chapter we will discuss the role of quantum groups in (2+1)d. In many concrete situations these can be identified as the underlying symmetry structure. Regardless of the name, these are actually not groups but should be regarded as a generalization. Formally, they are certain type of algebras. An important point of this thesis is to reformulate results on phase transitions related to the breaking of quantum group symmetry in the language of tensor categories, again illustrating the importance of these structures in the discussion of topological order.
In the previous chapter we discussed aspects of topological order, such as anyonic quasi-particle excitations and the quantum Hall effect. In the present chapter we will have a discussion of the symmetries underlying topological order. Special to planar physics, especially when electric and magnetic degrees of freedom start to interact, is the occurrence of so called quantum group symmetry. Quantum groups generalize group symmetries in many ways. This full symmetry is often not apparent from a glance at the Hamiltonian or Lagrangian of the theory.

Symmetries of the Hamiltonian represented on the quantum level as operators that commute with it. Since commuting operators can be diagonalized simultaneously, the energy eigenstates can be labelled by quantum numbers coming from the symmetry operators. But it might happen that there is a bigger set of commuting operators. This gives an idea of how symmetries not directly apparent from the Hamiltonian or Lagrangian can materialize. In this chapter we will first illustrate the occurrence of quantum group symmetry in the context of discrete gauge theories. There, the electric and magnetic excitations can be treated on equal footing as irreducible representation of the quantum double of the residual gauge group. Then we give a more formal treatment of the constituents of quantum groups, or quasi-triangular Hopf algebras as they are also called. Finally, we discuss the quantum group $U_q[\mathfrak{su}(2)]$ that is related to Chern-Simons theory and the Wess-Zumino-Witten model for conformal field theory.

### 3.1 Discrete Gauge Theory

Suppose we start out with a (2+1)d Yang-Mills-Higgs theory with a gauge group $G$. We speak of a discrete gauge theory (DGT) when the continues gauge group $G$ is spontaneously broken down to a discrete, usually finite, subgroup $H$. Hence, this can be regarded as a gauge theory with residual gauge group $H$. 
The Higgs mechanism causes the gauge field to acquire a mass, making all local interactions freeze out in the low-energy regime, effectively leaving only topological interactions. In this limit, the theory becomes a TQFT \cite{8}. Below, we give a quick overview of the aspects that are important for our present purpose. See \cite{29} for an elaborate exposition on which this approach is based.

A DGT has electric particles labelled by the irreps \(\{\alpha\}\) of the residual gauge group, which, before we mod out by gauge transformations to get the physical Hilbert space, carry the corresponding representation module \(V_\alpha\) as an internal Hilbert space. Furthermore, a DGT also has magnetic particles, or fluxes, associated to topological defects.\footnote{We will assume representations to be unitary throughout the text.} We think of the fluxes as defined by their effect on charges in an Aharonov-Bohm type scattering experiment \cite{18}. An electric charge \(\alpha\) taken around a flux will feel the influence of the flux as a transformation in it’s internal Hilbert space \(V_\alpha\) by the action of some \(h \in H\). We denote the flux as a ket \(|h\rangle\) labelled by the transformation \(h \in H\) it induces. However, since a flux measurement followed by a gauge transformation should give the same result as applying a gauge transformation first and do the flux measurement in the transformed system we find \(gh = h'g\), where \(h, h'\) is the flux measured before or after the gauge \(g\) transformation respectively. Hence the gauge transformations has to act on the flux by conjugation, so the gauge invariant labelling is in fact by the conjugacy classes of \(H\) rather than just elements. Hence, while electric excitations carry representation modules as an internal Hilbert space, the internal space of fluxes is given by a conjugacy class.

Revealing the algebraic structure of the theory puts the above on a firmer footing. Define the flux measurement operators \(\{P_h\}_{h \in H}\) that satisfy the projector algebra

\[
P_h P_{h'} = \delta_{hh'} P_h
\]  

(3.1)

Since gauge transformations \(g \in H\) act on fluxes by conjugation, we must have

\[
g P_h P_{gh^{-1}} g^{-1} = g P_h g^{-1}
\]  

(3.2)

The full algebra of gauge transformations and flux projections is spanned by the combinations

\[
\{P_h g\}, \quad h, g \in H
\]  

(3.3)

which obey the multiplication rule

\[
P_h g P_{h'} g' = \delta_{h, gh^{-1}} P_h gg'
\]  

(3.4)

This is in fact the quantum double \(D(H)\) of \(H\) which can obtained from any finite group by a general construction due to Drinfel’d \cite{15}. Note that the multiplication rule says that this algebra has a unit \(1 = \sum_h P_h e\).

The full particle spectrum of a DGT with residual gauge group \(H\) can be recovered as the set of irreducible representations of \(D[H]\).\footnote{Note that we use electric and magnetic as general terminology for the two distinct types of particle-like excitations and it does not imply that the gauge group is \(U(1)\).} Indeed, one finds back the irreps
of $H$ corresponding to electric particles, as well as the conjugacy classes, belonging to fluxes. But there is a third class of irreps corresponding to flux-charge composites or dyons. Physically, this is very interesting, because due to the Aharonov-Bohm effect, these composites can have fractional spin and can be true anyons.

The representation theory of $D(H)$ was first worked out in [13], but we again follow [29]. It turns out that the irreducible representations of $D(H)$ can uniquely labelled by a conjugacy class $A$ of $H$, and an irreducible representation of the centralizer of some representing element of $A$. Thus, when the conjugacy class $\{e\}$ of the identity element is taken, or that of any other central element, we find the irreps of $H$. But apparently, for general non-trivial fluxes, not all transformations can be implemented straightforwardly on the charge part.

To make the action of $D(H)$ explicit, we will have to make a few choices. Pick some order for the elements of $A$ and write

$$A = \{ A_{h_1}, A_{h_2}, \ldots, A_{h_k} \}$$

(3.5)

Let $C(A)$ be the centralizer of $A_{h_1}$ and fix a set $X(A) = \{ A_{x_1}, A_{x_2}, \ldots, A_{x_k} \}$ of representatives for the equivalence classes $H/C(A)$, such that $A_{h_i} = A_{x_i} A_{h_1} A_{x_i}^{-1}$. Let us take $A_{x_1} = e$, the unit element of $H$, for convenience. These choices will effect some of the specifics in what follows but a different choice will lead to a unitary equivalent representation of $D(H)$.

The internal Hilbert space $V_A^\alpha$ of a particle with flux $A$ and centralizer charge $\alpha$ is now spanned by the quantum states

$$|A_{h_i}, \alpha v_j\rangle, \quad i = 1, \ldots, k, \quad j = 1, \ldots, \dim \alpha$$

(3.6)

where we have chosen a basis $\{|v_j\rangle\}$ for the centralizer representation $\alpha$. The action of an element $P_h g$ of $D(H)$ on these basis states, i.e. the effect of a gauge transformation $g$ followed by a flux measurement $P_h$, is given by

$$\Pi^A_{\alpha}(P_h g) |A_{h_i}, \alpha v_j\rangle = \delta_{h,g} A_{h_i} g^{-1} |g A_{h_i} g^{-1}, \Pi_\alpha(\tilde{g}) m_j \alpha v_m\rangle$$

(3.7)

(where summation over repeated indices is implied), with

$$\tilde{g} \equiv A_{x_i}^{-1} g A_{x_i}$$

(3.8)

and $A_{x_k}$ the element of $X(A)$ associated to $h_k = g A_{h_i} g^{-1}$. Note that the element $\tilde{g}$ commutes with $A_{h_1}$ as it should. Thus when acting on dyons, $\tilde{g}$ is the part of the transformation that slips through the conjugation of the flux, and which is subsequently implemented on the charge.

This gives an explicit description of the irreps of $D(H)$ including how the action of elements $P_h g$ can be worked out. This gives a description of the single-particle states of the theory. Before we discuss multi-particle states, let us make a remark concerning unitarity. In order to preserve the inner product of quantum states, we should have

$$\Pi^A_\alpha(P_h g)^\dagger = \Pi^A_\alpha(P_{g^{-1} h g} g^{-1})$$

(3.9)
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Hence it makes sense to define \((P_h g)^* = P_{g^{-1} h g}^{-1}\), and more generally

\[
\left( \sum_{h,g} c_{h,g} P_h g \right)^* = \sum_{h,g} c_{h,g}^* P_{g^{-1} h g}^{-1}
\]

where a \(*\) on the coefficients \(c_{h,g}\) just means complex conjugation. This actually gives a \(*\)-structure, making \(D[H]\) a \(*\)-algebra. Representations obeying (3.9) respect this \(*\)-structure and are the proper definition of a unitary representation.

3.1.1 Multi-particle states

To let \(D(H)\) act on a two-particle state, one needs a prescription of how an element \(P_h g\) acts on \(V^A \otimes V^B\). Corresponding to intuitive expectations, the element \(P_h g\) implements the gauge transformation \(g\) on both particles and then projects out the total flux. Thus we act on a state \(\left|A_{h_i}, \alpha_{v_j}\right> \right|B_{h_k}, \beta_{v_l}\rangle\) with

\[
\sum_{h', h''} \Pi^A_{h'}(P_{h'} g) \otimes \Pi^B_{h''}(P_{h''} g)
\]

Formally, this is nicely captured by the definition of a coproduct

\[
\Delta: D(H) \rightarrow D(H) \otimes D(H)
\]

by

\[
\Delta(P_h g) = \sum_{h' h'' = h} P_{h'} g \otimes P_{h''} g
\]

The action of \(P_h g\) on a two-particle state is then defined as the application of \(\Pi^A_{h'} \otimes \Pi^B_{h''}\) on \(\Delta(P_h g)\), which indeed gives (3.11).

The action on three-particle states is produced by first applying \(\Delta\) to produce an element of \(D[H] \otimes D[H]\) and then apply \(\Delta\) again on either the left or the right tensor leg. Starting out with the element \(P_h g\), either choice produces

\[
\sum_{h' h'' h''' = h} P_{h'} g \otimes P_{h''} g \otimes P_{h'''} g
\]

as an element of \(D[H]^{\otimes 3}\). Then we apply \(\Pi^A_{\alpha} \otimes \Pi^B_{\beta} \otimes \Pi^C_{\gamma}\) on this element to act on \(V^A_{h'} \otimes V^B_{h''} \otimes V^C_{h'''}\). Note that when we describe the transformation in words, it is again the application of the gauge transformation \(g\) on all three particles followed by a total flux measurement projecting out flux \(h\).

Extending the action of \(P_h g\) on states with more than three particles is now straightforward. The resulting transformation is again the global gauge transformation \(g\) followed by the flux measurement. On a formal level, this is achieved by consecutive application of the coproduct to get an element in \(D[h]^n\) and letting each tensor leg act on the corresponding particle.
3.1 Discrete Gauge Theory

The tensor product representations of $D(H)$ that we obtain using the coproduct are in general not irreducible but it can generally be decomposed as a direct sum of irreducible representations. This means that the space $V^A_\alpha \otimes V^B_\beta$ can be written as a direct sum of subspaces that do transform irreducibly, which leads to the decomposition rules

$$\Pi^A_\alpha \otimes \Pi^B_\beta = \bigoplus_{(C,\gamma)} N^{ABC}_\alpha\beta\gamma \Pi^C_\gamma$$

(3.15)

These so called fusion rules play an important role when we start discussing the general formalism for anyonic theories in the next chapter. Note that the representation $\Pi^e_{\{e\}0}$ has trivial fusion (where 0 denotes the trivial representation of $H$). This corresponds to the zero-particle state or vacuum of the theory. The map $\Pi^\epsilon_{\{e\}0} : D[H] \to \mathbb{C}$ can also be called the counit, and is then often denoted with $\epsilon$. This is a general constituent of a quantum group, and should satisfy certain compatibility conditions with regard to the comultiplication.

3.1.2 Topological interactions

As remarked before, in the low-energy limit, the excitations of a DGT interact solely by Aharonov-Bohm type, topological interactions. A very nice feature of DGT theories is that the precise form of these interactions can be deduced from intuitive reasoning.

![Figure 3.1: The transformation of fluxes](image)

- We start with two fluxes positioned next to each other in the plane, as depicted on the left. They carry flux $h_1$ and $h_2$, as can be measured by taking electric particles along the closed curve $C_1$ and $C_2$ respectively. The total flux is $h_1h_2$ as can be measured using curve $C_{12}$. The grey lines show Dirac strings attached to the vortices. The right picture shows the fluxes after a counter-clockwise interchange. Since the flux $h_2$ crosses the Dirac string attached to $h_1$, its value can change to $h'_2$. Because the total flux is conserved we must have $h'_2 h_1 = h_1 h_2$ hence we find that $h'_2 = h_1 h_2 h_1^{-1}$. So the half-braiding of fluxes leads to conjugation.

Suppose we have two fluxes with values $h_1$ and $h_2$. As is illustrated in figure 3.1, the requirement that the total flux has to be conserved leads to the conclusion that a counter-clockwise interchange results in commutation of $h_2$ by $h_1$. A charge crossing
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the $h_1$ Dirac line that is depicted in the figure picks up the action of $h_1$ in its internal Hilbert space. Hence we conclude that the general braiding operator acts as

$$R_{(A,\alpha),(B,\beta)} \left( |A_{h_1}, \alpha_{v_j} \rangle B_{h_m}, \beta_{v_n} \rangle \right) = |A_{h_4} B_{h_m} A_{h_1}^{-1}, \Pi_{\beta}(A_{h_1}) \beta_{v_n} \rangle |A_{h_1}, \alpha_{v_j} \rangle$$

(3.16)

This transformation be accomplished by composing the action of a special element in $D[H] \otimes D[H]$, called the universal $R$-matrix, on $V^A \otimes V^B$, with flipping the tensor legs. The universal $R$-matrix for $D[H]$ is

$$R \equiv \sum_{g,h} P_g e \otimes P_h g$$

(3.17)

Acting on a two-particle state, $R$ indeed measures the flux of the left particle and implements it on the right particle. We can now define

$$R_{(A,\alpha),(B,\beta)} = \tau(\Pi^A_\alpha \otimes \Pi^B_\beta)(R)$$

(3.18)

where $\tau$ is the flip of tensor legs. It is easy to check that this gives (3.16) when applied on two-particle states.

The universal $R$-matrix has the following properties.

$$\mathcal{R}\Delta(P_h g) = \Delta(P_h g)\mathcal{R}$$

(3.19)

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{23}\mathcal{R}_{12}$$

(3.20)

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{23}$$

(3.21)

(3.22)

Here $\mathcal{R}_{12} = \mathcal{R} \otimes 1$ and $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$. These consistency conditions ensure that the braiding is implemented consistently. The first relation tells that the braiding commutes with the action of the quantum double on two-particle states, hence it acts as multiplication by a complex number on the irreducible subspaces (Schur's lemma). The other two conditions are known as the quasi-triangularity conditions. They imply the Yang-Baxter equation for the $R$-matrix,

$$\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23}$$

(3.23)

On the level of representations, this leads to the Yang-Baxter equation for the braiding, thus implementing a representation of the braid group $B_n$ on the $n$-particle Hilbert space of $n$ identical particles.

To discuss the spin of general fluxes, charges and dyons in this context, imagine an excitation $(A, \alpha)$ as a composite where the flux $A$ and the centralizer charge $\alpha$ are kept apart by a tiny distance. The world line is now more like a ribbon with the flux and the centralizer charge attached to opposite edges. A $2\pi$ rotation corresponds to a full twist of the ribbon, such that the charge winds around the flux. This implements the flux on the charge. This is generally accomplished by the letting the element

$$\sum_h P_h h$$

(3.24)
act, which results in
\[ \sum_h \Pi_\alpha^A(P_h h | \hat{A}_{h_i} \hat{\alpha}_{v_j}) = | \hat{A}_{h_i}, \Pi_\alpha(h_1) \hat{\alpha}_{v_j} \rangle \]
(3.25)
as can be seen from working out the rule (3.7). Since the element \( h_1 \) commutes by definition with all elements of the centralizer \( C(A) \), we must have
\[ \Pi_\alpha(h_1) = e^{2\pi ih(A,\alpha)} 1_\alpha \]
(3.26)
by Schur’s lemma applied on the unitary irrep \( \alpha \). The phase \( \theta(A,\alpha) = \exp(2\pi ih(A,\alpha)) \)
is the spin of the \((A,\alpha)\)-particle. The element \( \sum h P_h \) is called the ribbon element of \( D[H] \). Note that only true dyons can have \( \theta(A,\alpha) \neq \pm 1 \).

### 3.1.3 Anti-particles

In DGTs every particle-type \((A,\alpha)\) has a unique conjugate particle-type \((\tilde{A},\tilde{\alpha})\) with the special property that they can fuse to the vacuum, given by the fusion rule
\[ \Pi_A \otimes \Pi_{\tilde{A}} = \Pi_0^{(e)} + \ldots \]
(3.27)
As a representation of \( D[H] \), we can give the structure of \((\tilde{A},\tilde{\alpha})\) explicitly. The representation module of \((\tilde{A},\tilde{\alpha})\) is just the dual of that of \((A,\alpha)\), i.e., \( V_{\tilde{A}}^{\tilde{\alpha}} = (V_A^\alpha)^* \). The action of \( P_h g \) on a state \( \langle \hat{A}_{h_i}, \hat{\alpha}_{v_j} | \) is given by
\[ \Pi_{\tilde{\alpha}}^A(P_h g): \langle \hat{A}_{h_i}, \hat{\alpha}_{v_j} | \rightarrow \langle \hat{A}_{h_i}, \hat{\alpha}_{v_j} | \Pi_{\tilde{\alpha}}^A(P_{g^{-1}h^{-1}g}^{-1}) \]
(3.28)
Indeed, we can find a one-dimensional subspace in \( V_A^\alpha \otimes V_{\tilde{A}}^{\tilde{\alpha}} \) that transforms like the vacuum under the action of \( D[H] \), namely the subspace spanned by \( \sum_{i,j} | \hat{A}_{h_i}, \hat{\alpha}_{v_j} \rangle \langle \hat{A}_{h_i}, \hat{\alpha}_{v_j} | \)
(3.29)
where \( 1_{(A,\alpha)} \) denotes the identity operator on \( V_A^\alpha \). In particular it commutes with the representation matrices. From this, it is easy to deduce that, as a state, it transforms like the vacuum. Recall that the vacuum representation is given by
\[ \Pi_0^{(e)}(P_h g) = \epsilon(P_h g) = \delta_{h,e} \]
(3.30)
Working out the action of \( P_h g \) on \( 1_{(A,\alpha)} \) gives
\[ \sum_{h'h''=h} \Pi_A^A(P_{h'} g) 1_{(A,\alpha)} \Pi_A^A(P_{g^{-1}h''^{-1}g}^{-1}) = \sum_{h'h''=h} \delta_{h',h''} \Pi_A^A(P_{h'} g^{-1}g 1_{(A,\alpha)}) \]
(3.31)
\[ = \sum_{h'} \delta_{h,e} \Pi_A^A(P_{h'}, e) 1_{(A,\alpha)} \]
(3.32)
\[ = \sum_{h'} \delta_{h,e} \Pi_A^A(P_{h'} e) 1_{(A,\alpha)} \]
(3.33)
which is indeed as claimed. Note the subtle difference between \((P_h g)^*\) and \(S(P_h g)\).

Again, it is a general feature that quantum group symmetry incorporates anti-particles in a natural way. For this we need a linear map \(S\) from the quantum group to itself, which is called the antipode, satisfying

\[
\sum_{(a)} S(a') a'' = 1 \epsilon(a) = \sum_{(a)} a' S(a'') \tag{3.34}
\]

for all elements \(a\) of the quantum group. Here we have made use of the Sweedler notation

\[
\Delta(a) = \sum a' \otimes a'' \tag{3.35}
\]

which generalizes

\[
\Delta(P_h g) = \sum_{h'h''} P_{h'} g \otimes P_{h''} g \tag{3.36}
\]

to arbitrary cases.

### 3.2 General remarks

The discussion above should give a general idea of quantum group symmetry. It is possible to give rigorous and general definitions of the essential structures that appeared above, and define in general what we mean by a quantum group. Let us give the usual mathematical nomenclature and some relevant references. Suppose we start with an associative unital algebra \(\mathcal{H}\). Definition of a comultiplication \(\Delta\) and counit \(\epsilon\) satisfying certain axioms gives a coalgebra structure, which, if it is compatible with the algebra structure, makes \(\mathcal{H}\) a bialgebra. If there is an antipode, this makes it a Hopf algebra. It can be shown that the antipode, if it exists, is unique, such that any bialgebra has at most one Hopf algebra structure. The involution or \(^*\)-structure necessary to define unitary representations makes it a Hopf-\(^*\)-algebra.

The most important feature in relation to planar physics is the possibility of non-trivial braiding. This is given by the existence of a universal \(R\)-matrix, which is an element of \(\mathcal{H} \otimes \mathcal{H}\). It has to satisfy certain consistency relations which ensure that the braiding of representations obey the Yang-Baxter equation. The name for a Hopf algebra with universal \(R\)-matrix is quasi-triangular.

It is nice to contemplate what is really new for quantum groups in comparison with the usual group symmetries abundant in physics. Let us consider two important cases, symmetries given by a finite group \(H\) and symmetries given by a continuous Lie group \(G\).

In stead of the group \(H\), we can alternatively work with the group algebra \(\mathbb{C}[H]\) without losing any information. This is just the complex algebra generated by the group elements with the multiplication given by the group operation.

\[
\mathbb{C}[H] = \bigoplus_{g \in H} \mathbb{C} g, \quad \left( \sum_g c_g g \right) \cdot \left( \sum_{g'} c'_{g'} g' \right) = \sum_{g, g'} c_g c'_{g'} gg' \tag{3.37}
\]

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This is in fact a Hopf algebra with comultiplication, counit and antipode
\[ \Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} \]  
(3.38)
It is quasi-triangular, with universal \( R \)-matrix
\[ R = e \otimes e \]  
(3.39)
where \( e \) is the unit element of \( H \). Since this universal \( R \)-matrix is trivial so to say, braiding just amount to flipping the tensor legs. This gives to the usual statistics in higher dimensions. We might call \( \mathbb{C}[H] \) triangular, in stead of quasi-triangular.

For Lie groups \( G \) the situation is similar. In stead of the Lie group, one usually works with the Lie algebra \( \mathfrak{g} \). We usually allow for normal products \( xy \) apart from the bracket \( [x, y] \) and incorporate a unit 1 element, which is very natural if we think of this Lie algebra as an algebra of symmetry operators. Formally, this leads to the universal enveloping algebra \( \mathbb{C}[\mathfrak{g}] \), which generated by the unit 1 and the elements of \( \mathfrak{g} \) subject to the relations
\[ xy - yx = [x, y] \]  
(3.40)
It becomes a (quasi-)triangular Hopf algebra by defining
\[ \Delta(1) = 1 \otimes 1 \quad \Delta(x) = x \otimes 1 + 1 \otimes x \]  
(3.41)
\[ \epsilon(1) = 1 \quad \epsilon(x) = 0 \]  
(3.42)
\[ S(1) = 1 \quad S(x) = -x \]  
(3.43)
\[ R = 1 \otimes 1 \]  
(3.44)
which can be seen as the infinitesimal version of the definitions given for \( \mathbb{C}[H] \).

When we use the term quantum group, we have quasi-triangular Hopf algebras in mind, with a non-trivial \( R \)-matrix leading to interesting braiding. Some authors seem prefer to use the term quantum group only for quasi-triangular Hopf algebras that occur as \( q \)-deformations of semi-simple Lie algebras discussed below, or to include quasi-Hopf algebras, for which the coproduct does not precisely satisfy the axioms for a coalgebra, but do almost. For more general information on quantum groups and precise definitions, we refer to the literature, for example [?, ?].

### 3.3 Chern-Simons theory and \( U_q[\mathfrak{su}(2)] \)

The quantum doubles that can be obtained via the Drinfel’d construction form an important class of examples of quantum groups. The other important class comes from a semi-simple Lie algebra by “\( q \)-deformation” of the universal enveloping algebra. We will discuss an important example, \( U_q[\mathfrak{su}(2)] \) that comes from the Lie algebra \( \mathfrak{su}(2) \), briefly. It plays an important role in the quantization of Chern-Simons theory with gauge group \( SU(2) \) [?] and in the Wess-Zumino-Witten models for CFT [?, ?] based on the affine algebra of \( \mathfrak{su}(2) \) at level \( k \). We follow [31].
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One can view $U_q[\mathfrak{su}(2)]$ as the algebra generated by the unit 1 and the three elements $H, L^+$ and $L^-$, which satisfy the relations

$$[H, L^\pm] = \pm 2L^\pm$$
$$[L^+, L^-] = \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}}$$

where $q$ is a formal parameter that may be set to any non-zero complex number. The coproduct $\Delta$ is given on the generators by

$$\Delta(1) = 1 \otimes 1$$
$$\Delta(H) = 1 \otimes H + H \otimes 1$$
$$\Delta(L^\pm) = L^\pm \otimes q^{H/4} + q^{-H/4} \otimes L^\pm$$

The counit and antipode are

$$\epsilon(1) = 1, \quad \epsilon(H) = 0, \quad \epsilon(L^\pm) = 0$$
$$S(1) = 1, \quad S(H) = -H, \quad S(L^\pm) = -q^{\mp 1/4}L^\pm$$

There is a $*$-structure given by

$$(L^\pm)^* = L^\mp, \quad H^* = H$$

giving rise to the notion of unitary representations. Without the $*$-structure, it is perhaps better to call this algebra $U_q[\mathfrak{sl}(2)]$ in stead of $U_q[\mathfrak{su}(2)]$, under which name this quantum group usually appears in the literature.

In the limit $q \to 1$, we recover the definitions for the universal enveloping $U[\mathfrak{su}(2)]$ algebra of $\mathfrak{su}(2)$, for instance

$$[L^+, L^-] = H$$

Therefore, it is sensible to talk about a $q$-deformation of $U[\mathfrak{su}(2)]$.

If $q$ is not a root of unity, the representation theory is very similar to that of $U[\mathfrak{su}(2)]$. For every half-integer $j \in \frac{1}{2}\mathbb{Z}$ there is an irreducible highest weight representation of dimension $d = 2j + 1$ with highest weight $\lambda = 2j$. The representation modules $V^\lambda$ have an orthonormal basis

$$|j, m\rangle, \quad j \in \{-j, -j + 1, \ldots, j - 1, j\}$$

The action of the generators is

$$\Pi^\lambda(H) |j, m\rangle = 2m |j, m\rangle$$
$$L^\pm |j, m\rangle = \sqrt{|j \mp m\rangle_q [j \pm m \pm 1]} |j, m \pm 1\rangle$$

Here we have used the notation

$$[m]_q = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}$$
for the so called $q$-numbers that enter the formula to the commutation relation $[L^+, L^-] = [H]_q$.

A formal expression for the universal $R$-matrix of $U_q[su(2)]$ is

$$R = q^{H \otimes H} \sum_{n=0}^{\infty} \frac{1 - q^{-1}}{[n]_q!} q^{n(1-n)/4}(q^{nH/4}(L^+)^n) \otimes (q^{-nH/4}(L^-)^n)$$

From this expression, one can work out the action on representations. This produces a non-trivial braiding in a similar fashion as we saw for DGT and the quantum double.

In relation to Chern-Simons theory and the WZW models, the most interesting case is, however, when $q$ is not a root of unity. In that case, the representation theory of $U_q[su(2)]$ changes drastically. We will not go in to the details of the mathematical tricks needed to find the right notion representations that describe charges in these physical theories. These can be read in for instance [3, 31] or [31], but the right definition was first discovered in [1]. In this case, we denote the resulting theory as $su(2)_k$, where $k$ is related to $q$ as $q = \exp\left(\frac{2\pi i}{k+2}\right)$. We will come back to these theories in chapter 5.
3. QUANTUM GROUPS IN PLANAR PHYSICS
As discussed in the previous chapter, quantum group symmetry can account for fusion and non-trivial braiding of excitations in (2+1)-dimensional systems. But the physical states usually correspond to the unitary irreps rather than the internal states of the representation modules. Also, in the case of $U_q[\mathfrak{su}(2)]$ modules for $q$ a root of unity, the identification of representations with particle types is not straightforward. One may ask if a rigorous formalism exists that treats these theories on the level of the excitations rather than the underlying symmetries. The answer is yes.

Mathematically, the representations of a quantum group (often) form a modular tensor category, a structure that may be defined independently. In this chapter we introduce many generalities of these tensor categories as they arise in physical applications. These capture the topological properties, including the addition of quantum numbers (fusion), of many theories in a very general way and allow for a graphical formalism which can be regarded as a kind of topological Feynman diagrams. They can be constructed from quantum groups, but also arise directly from a (rational) CFT and can be used describe the excitations in Kitaev and Levin-Wen type lattice models directly. They are generally related to topological quantum field theories.

In stead of relying heavily on the language of category theory, we have chosen a route along the lines of [11, 28]. This means that we focus on concreteness and calculability, in favour of formal language. For the mathematically inclined, this basically means that we assume the category to be strict and construct everything in terms of simple objects. A paper in the mathematical physics literature that treats tensor categories in a similar fashion is [?] where symmetries of the $F$-symbols (to be defined below) are derived. In the appendix we outline how the discussion below is related to a more formal treatment. Proper mathematical textbooks are [?, 9].

We have particularly relied on [11] for the discussion below.
4. ANYONS AND TENSOR CATEGORIES

4.1 Fusing and splitting

For a general theory, the particle-like excitations fall in different topological classes or sectors. For simplicity, we treat these sectors as elementary excitations and assume there is finite number of them. These are the first ingredients of a modular tensor category, which in a physical context has to be unitary, or, more generally, a braided tensor category.

One can vary the assumptions on the kind of tensor category in consideration according to context and each set of assumptions gives a slightly different mathematical structure which has it’s own name. To avoid getting stuck in nomenclature, we will generally refer to a ‘the theory’ in this chapter. The term anyon models has also appeared in the physical literature for unitary braided tensor categories.

4.1.1 Fusion multiplicities

We start with a theory $\mathcal{A}$ with a finite collection of topological sectors $\mathcal{C}_A = \{a, b, c, \ldots \}$. We also refer to these sectors as particle-types, (anyonic) charges, (particle) labels or sometimes simply particles or anyons. They obey a set of fusion rules that we write as

$$a \times b = \sum_{c \in \mathcal{C}_A} N_{a b}^c c$$  \hspace{1cm} (4.1)

(The domain of the sum will be left implicit from now on.) Here the $N_{a b}^c$ are non-negative integers called fusion multiplicities. These determine the possibilities for the total charge when two anyons are combined (fused). The total charge of two anyons with respective labels $a$ and $b$ can be any of the $c$ with $N_{a b}^c \neq 0$. If $N_{a b}^c \neq 0$, we also speak of the fusion channel $c$ of $a$ and $b$ and may denote this as $c \in a \times b$.

If the state-space of a pair of particles is multidimensional this gives rise to non-Abelian anyons.\(^1\) We say, therefore, that the theory is non-Abelian if there are charges $a$ and $b$ that have

$$\sum_c N_{a b}^c > 1$$  \hspace{1cm} (4.2)

The particle labels together with the fusion rules (4.1) generate the fusion algebra of the theory. In this context we sometimes denote the basis states by kets $|a\rangle, |b\rangle, |c\rangle$.

The fusion algebra has to obey certain conditions which have a clear physical interpretation. We require that there is a unit, i.e. a unique trivial particle, the vacuum, $0 \in \mathcal{A}$ that fuses trivially with all labels: $0 \times a = a = a \times 0$ for all $a$. Furthermore, fusion should be an associative operation, $(a \times b) \times c = a \times (b \times c)$, such that the total anyonic charge is a well-defined notion. In terms of the fusion multiplicities, these conditions

\(^1\)There is a subtlety to the term non-Abelian anyons. Identical particles with a multidimensional state space for a pair might still have Abelian braiding. Hence, on could say that this is a necessary but not sufficient condition, but we will not bothered with the distinction.
can translate to
\[ N_{ba}^b = N_{a0}^b = \delta_{ab} \] (4.3)
\[ \sum_e N_{eb} N_{ed}^d = \sum_f N_{af}^d N_{bc}^f \] (4.4)

We also require that the fusion rules are commutative, such that \( a \times b = b \times a \). This should hold for (2+1)-dimensional systems, since there is no way to define left and right unambiguously. We can lift this requirement if we consider charges of a (1+1)d system, for example when we look at boundary excitations. Finally, each charge \( a \in C_A \) should have a unique conjugate charge \( \bar{a} \in C_A \) or anti-particle such that \( a \) and \( \bar{a} \) can annihilate
\[ a \times \bar{a} = 0 + \sum_{c \neq 0} N_{ab}^c \] (4.5)

This does not mean that \( a \) and \( \bar{a} \) can only fuse to the vacuum, there might be \( c \neq 0 \) with \( N_{\bar{a}a}^c \neq 0 \). Note that \( \bar{\bar{a}} = a \).

The following relations for the fusion multiplicities also follow from these conditions on fusion
\[ N_{ab}^0 = \delta_{\bar{a}a} \] (4.6)
\[ N_{ab}^c = N_{ba}^c = N_{bc}^c = N_{\bar{a}c}^c \] (4.7)

The first line is just the anti-particle property. The second line can be derived using commutativity and associativity.

It is useful to define the fusion matrices \( N_a \) with
\[ (N_a)_{bc} = N_{ab}^c \] (4.8)
The fusion rules give
\[ N_a N_b = \sum_c N_{ab}^c N_c \] (4.9)
which shows that the fusion matrices provide a representation of the fusion algebra. This is just the representation of the fusion algebra that is obtained by letting it act on itself via the fusion product. Associativity of the fusion rules translates to commutativity of the fusion matrices, such that
\[ N_a N_b = N_b N_a \] (4.10)

The explicit diagonalization of the fusion matrices is a very interesting result, that we come back to in section 4.1.2

### 4.1.2 Diagrams and F-symbols

Special about the formalism based on category theory in comparison to the quantum group approach is that internal states are totally left out of the picture. This makes
4. ANYONS AND TENSOR CATEGORIES

sense because, in physical settings, to the gauge invariant states we still have to mod out by the quantum group. Operators on anyons are best described in a diagrammatic formalism.\textsuperscript{1} They form states in certain vector spaces. The formalism gives rules to manipulate the diagrams, and thus do diagrammatic calculations. Manipulating diagrams according to the rules can be thought of as rewriting the same state, for example, by changing to another basis. One alters the representation, but not the state (operator) itself.

Each anyonic charge label is associated with a directed line. This is the identity operator for the anyon, or in the language of category theory, the identity morphism. It is often useful to think of it as the world line of the anyon propagating in time, which we will take as flowing upward. Reversing the orientation of a line segment is equivalent to conjugating the charge labelling it, so that

\[ \begin{array}{c}
\begin{array}{c}
\uparrow \\
a
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\uparrow \\
\bar{a}
\end{array}
\end{array} \quad \text{(4.11)}
\]

The second most elementary operators are directly related to fusion, and, the dual process, splitting of anyons. To every triple of charge labels \((a,b,c)\) assign a complex vector space \(V_{ab}^c\) of dimension \(N_{ab}^c\). This is called a fusion space. States of the fusion space \(V_{ab}^c\) are denoted by fusion vertices with labels corresponding to the charges on the outer legs

\[ \begin{array}{c}
\begin{array}{c}
\mu
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\downarrow \\
a
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\downarrow \\
b
\end{array}
\end{array} \end{array} \begin{array}{c}
\begin{array}{c}
\uparrow \\
c
\end{array}
\end{array} \quad \mu = 1, \ldots, N_{ab}^c \quad \text{(4.12)}
\]

The dual of the fusion space is denoted \(V_{ab}^{bc}\) and is called a splitting space. The states of the splitting space \(V_{ab}^{bc}\) are denoted by splitting vertices like

\[ \begin{array}{c}
\begin{array}{c}
\mu
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\downarrow \\
a
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\downarrow \\
b
\end{array}
\end{array} \end{array} \begin{array}{c}
\begin{array}{c}
\uparrow \\
c
\end{array}
\end{array} \quad \mu = 1, \ldots, N_{ab}^{bc} \quad \text{(4.13)}
\]

When \(N_{ab}^{bc} = 1\) we can leave the index \(\mu\) implicit.\textsuperscript{2}

The 0-line can be inserted and removed from diagrams at will, reflecting the properties of the vacuum, and is therefore often left out or 'invisible'. When made explicit, we draw vacuum-lines dotted. Hence, we have

\[ \begin{array}{c}
\begin{array}{c}
0 \times \\
a
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\uparrow \\
a
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\downarrow \\
a
\end{array}
\end{array} \quad \text{(4.14)}
\]

and similar for the corresponding fusion vertices.

\textsuperscript{1}In the mathematical literature this is known as graphical calculus, introduced by Reshetikhin and Turaev [?].

\textsuperscript{2}In subsequent chapters, we will assume that all \(N_{ab}^{bc} = 0, 1\) hence diagrams will have no labels at the vertices, but only at the charge lines. For completeness, however, we leave the \(\mu\)'s in tact in this chapter.
In terms of quantum group representations, splitting vertices correspond to the embedding of an irrep into a tensor product representation, while fusion vertices should be thought of as the projection onto an irreducible subspace of a tensor product representation.

General anyon operators can be made by stacking fusion and splitting vertices and taking linear combinations. The intermediate charges of connected charge lines have to agree and the vertices should be allowed by the fusion rules, otherwise the whole diagram evaluates to zero. This gives a diagrammatic encoding of charge conservation.

As a general notation we write \( V_{a_1,\ldots,a_n}^{a_1',\ldots,a_n'} \) for the vector space of operators that take \( n \) anyons with charges \( a_1',\ldots,a_n' \) as input and which produce \( m \) anyons of charges \( a_1,\ldots,a_m \).

An important example is \( V_{abc}^{d} \). Operators of this space can be made by stacking two splitting vertices on top of each other. The choice we have of connecting the top vertex either on the right or on the left of the bottom vertex leads to two different bases of \( V_{abc}^{d} \). The change of basis in these spaces is given by so called \( F \)-symbols\(^1\), which are an important piece of data for these models. They are defined by the diagrammatic equation

\[
\begin{align*}
\begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array}
\begin{array}{c}
b \\
a \\
c
\end{array}
\begin{array}{c}
\epsilon \\
\delta \\
\gamma
\end{array}
\begin{array}{c}
d
\end{array}
= \sum_{f,\mu,\nu} \left[ F_{d}^{abc} \right]_{(e,\alpha,\beta)(f,\mu,\nu)}
\begin{array}{c}
a \alpha \\
b \\
c \\
\epsilon
\end{array}
\begin{array}{c}
b' \\
\nu \\
\mu
\end{array}
\begin{array}{c}
\delta \\
\gamma \\
\delta
\end{array}
\begin{array}{c}
d
\end{array}
\end{align*}
\] (4.15)

Like all other diagrammatic equations in the formalism, this ‘\( F \)-move’ can be used locally, within bigger diagrams to perform calculations. If a diagram on the right is not permitted by the fusion rules, we put the corresponding \( F \)-symbol to zero.

The \( F \)-symbols have to satisfy certain consistency conditions, called the pentagon relations, that we will come back to later. There is a certain gauge freedom in the \( F \)-symbols, reflecting the fact that we can perform a unitary change of basis in the elementary splitting space \( V_{ab}^{c} \) without changing the theory, which will also be discussed later.

We could just as well have introduced the \( F \)-symbols in terms of fusion states in \( V_{d}^{abc} \). The \( F \)-move for fusion states is governed by the adjoint of the \( F \)-move as introduced above. Unitarity of the model amounts to

\[
\left[ (F_{d}^{abc})^{-1} \right]_{(f,\mu,\nu)(e,\alpha,\beta)} = \left[ (F_{d}^{abc})^{*} \right]_{(e,\alpha,\beta)(f,\mu,\nu)} = \left[ (F_{d}^{abc})^{-1} \right]_{(f,\mu,\nu)(e,\alpha,\beta)}
\] (4.16)

This can be seen by taking the adjoint of the diagrammatic equation for the \( F \)-symbols. In general, the adjoint of a diagram in a unitary theory is taken by reflecting it in the horizontal plane, such that top and bottom interchange, and then reversing the orientation of all arrows.

If one of the upper outer legs of the tree occurring in the \( F \)-move equation is labelled by the vacuum, it is essentially just a splitting vertex and the \( F \)-move should leave it unchanged.

\(^1\)The \( F \)-symbols are sometimes called recoupling coefficients or quantum 6j-symbols. The \( F \)-symbols are the analogue of the 6j-symbols from the theory of angular momentum, hence the latter name.
unchanged. Thus, for example \( F_{0bc}^{ef} = \delta_{eb} \delta_{fc} \), and similarly if the middle or right upper index equals 0. Note that when \( d = 0 \) there is only one non zero \( F \)-symbol for fixed \( a, b, c \), namely \( F_{0}^{abc} \), but this can be non-trivial.

The pairing of \( V_{\alpha}^{bc} \) and \( V_{\alpha}^{ce} \) or the inner product of fusion / splitting states, is again denoted by stacking the appropriate diagrams. We think of the elements of \( V_{\alpha}^{bc} \) as kets and of the elements of \( V_{\alpha}^{ce} \) as bras. Composition of operators is written from bottom to top, in accordance with the flow of time. The conservation of anyonic charge is also taken into account in the diagrams, which gives

\[
\begin{align*}
\begin{array}{c}
\alpha \\
\beta
\end{array}
\begin{array}{c}
\beta' \\
\alpha'
\end{array}
\begin{array}{c}
\mu \\
\mu'
\end{array}
\begin{array}{c}
\mu'' \\
\mu'''
\end{array} = \delta_{\alpha\beta'} \delta_{\beta\gamma'} \delta_{\gamma\mu'} \delta_{\mu\mu''} \delta_{\mu''\mu'''} \sqrt{\frac{d_{\alpha} d_{\beta} d_{\mu}}{d_{\gamma}}}
\end{align*}
\]

(4.17)

Here, the quantum dimension \( d_{a} \) is defined for each particle label \( a \) as

\[
d_{a} = \left| [F_{a}^{\alpha\alpha} \mid 00] \right|^{-1}
\]

(4.18)

This is a very important quantity in all that follows. From the properties of the \( F \)-symbol immediately find that \( d_{0} = 1 \). The normalization factor \( \sqrt{\frac{d_{\alpha} d_{\beta}}{d_{\mu}}} \) is inserted in this ‘diagrammatic bracket’ to make the diagrams invariant under isotopy, i.e. under bending of the charge lines (end points should be left fixed).

For horizontal bending, keeping in tact the flow of charge in time, isotopy invariance is trivial. Vertical bending however, introduces events of particle creation and annihilation. Invariance under this kind of bending is almost realized by introducing the normalization of the vertex bracket, but a subtle issue remains.

Consider the calculation

\[
\begin{align*}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
\beta
\end{array}
\begin{array}{c}
\mu
\end{array}
\begin{array}{c}
\mu'
\end{array}
\begin{array}{c}
\mu''
\end{array} = [F_{a}^{\alpha\alpha} \mid 00] \delta_{\alpha \beta'} \delta_{\beta \gamma'} \delta_{\gamma \mu'} \delta_{\mu \mu''} \sqrt{\frac{d_{\alpha} d_{\beta}}{d_{\mu}}} = d_{a} [F_{a}^{\alpha\alpha} \mid 00] \delta_{\alpha \beta'} \delta_{\beta \gamma'} \delta_{\gamma \mu'} \delta_{\mu \mu''} \sqrt{\frac{d_{\alpha} d_{\beta}}{d_{\mu}}}
\end{align*}
\]

(4.19)

which uses the \( F \)-move and (4.17). By the definition of \( d_{a} \), the coefficient on the right has unit norm. However, \( [F_{a}^{\alpha\alpha} \mid 00] \) might have a non-trivial phase

\[
\text{fs}_{a} = d_{a} [F_{a}^{\alpha\alpha} \mid 00]
\]

(4.20)

As we will see later, for \( a \neq \bar{a} \) the phase \( \text{fs}_{a} = \text{fs}_{\bar{a}}^{*} \) can be set to unity by a gauge transformation. But when \( a \) is self dual, \( \text{fs}_{a} = \pm 1 \) is a gauge invariant quantity known as the Frobenius-Schur indicator. Hence, in the appropriate gauge, isotopy invariance is realized up to a sign.

To account for this sign, we introduce cup and cap diagrams with a direction given by a flag. These are defined as

\[
\begin{align*}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
\beta
\end{array} = \text{fs}_{a} [a \mid \bar{a} \mid 0] = [a \mid \bar{a} \mid 0 \mid 0]
\end{align*}
\]

(4.21)
4.1 Fusing and splitting

and

\[ a \left( \begin{array}{c} \bar{a} \\ \bar{a} \end{array} \right) = fs_a \left( \begin{array}{c} \bar{a} \\ \bar{a} \end{array} \right) = \left( \begin{array}{c} \bar{a} \\ \bar{a} \end{array} \right) \quad (4.22) \]

Bending a line vertically is now taken to include the introduction of a cap/cup pair with oppositely directed flags. Then we have the following equality

\[ a \left( \begin{array}{c} \bar{a} \\ \bar{a} \end{array} \right) = \left( \begin{array}{c} \bar{a} \\ \bar{a} \end{array} \right) = \left( \begin{array}{c} \bar{a} \\ \bar{a} \end{array} \right) \quad (4.23) \]

With these conventions in place, isotopy invariance of the diagrams is realized. If the cups and caps are paired up with opposite flags in diagrams, we leave them implicit. This, shows that loops evaluate as the corresponding quantum dimension

\[ a \left( \begin{array}{c} \bar{a} \\ \bar{a} \end{array} \right) = d \quad (4.24) \]

when we combine this with relation (4.17).

In accordance with the normalization for the inner product, the identity operator on a pair of anyons can now be written as

\[ \left( \begin{array}{c} a \\ a \end{array} \right) = \sum_{\mu, \nu} \sqrt{d_{\mu} d_{\nu}} \left( \begin{array}{c} a \\ a \end{array} \right) \quad (4.25) \]

It is very convenient to define \( F \)-symbols for diagrams with two-anyon input and two-anyon output. This is done by the diagrammatic equations

\[ \left( \begin{array}{c} a \\ a \end{array} \right) = \sum_{\mu, \nu} \sqrt{d_{\mu} d_{\nu}} \left( \begin{array}{c} a \\ a \end{array} \right) \quad (4.26) \]

Equation (4.25) gives

\[ [F_{ab}]_0 (\alpha, \beta)(f, \mu, \nu) = \sqrt{d_{\alpha} d_{\beta} d_{f} d_{d}} \delta_{\mu, \nu} \quad (4.27) \]

And more generally we have

\[ [F_{cd}]_a (\epsilon, \alpha, \beta)(f, \mu, \nu) = \sqrt{d_{\epsilon} d_{f} d_{d}} [F_{ab}]_0 (\alpha, \beta)(a, \epsilon, \alpha, \beta)(f, \mu, \nu) \quad (4.28) \]

These alternative \( F \)-moves can be used to change a splitting vertex with one leg bend down into a fusion vertex, etcetera. This gives equalities like

\[ \left( \begin{array}{c} c \\ \bar{a} \end{array} \right) = \sum_{\mu, \nu} \left( \begin{array}{c} c \\ \bar{a} \end{array} \right) \quad (4.29) \]
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By looking at the adjoint versions, more equalities may be derived. For the general spaces of operators $V^{a_1\ldots a_m}_{a'_1\ldots a'_n}$ one usually agrees on a standard basis. The choice generally made is the basis made by stacking vertices on the left, i.e. consisting of operators

\[
\begin{array}{c}
\vdots
\end{array}
\]

Any operator in the theory can in principle be expressed in terms of the standard basis elements. This is a very important remark. It implies, for example, that any operator we write down that acts on an anyon of charge $a$ is, effectively, nothing but multiplication by a complex number since the space $V^a_a$ is one-dimensional. Furthermore, many equalities can be deduced by comparing coefficients of the corresponding equations expressed in terms of the standard basis. But for other applications, it is sometimes tedious to write everything in terms of the standard basis, and it is much more convenient to leave the diagrams intact.

4.1.3 Gauge freedom

As remarked before, there is a certain freedom present in any anyon model. This corresponds to the choice of bases in the $V^{ab}_{cd}$. We can apply a unitary transformation in the spaces $V^{ab}_{cd}$ without changing the theory. Denote such a unitary transformation by $u^{ab}_{cd}$. This gives new basis states

\[
\begin{array}{c}
\vdots
\end{array}
\]

Any operator in the theory can in principle be expressed in terms of the standard basis elements. This is a very important remark. It implies, for example, that any operator we write down that acts on an anyon of charge $a$ is, effectively, nothing but multiplication by a complex number since the space $V^a_a$ is one-dimensional. Furthermore, many equalities can be deduced by comparing coefficients of the corresponding equations expressed in terms of the standard basis. But for other applications, it is sometimes tedious to write everything in terms of the standard basis, and it is much more convenient to leave the diagrams intact.

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\[
\begin{array}{c}
\vdots
\end{array}
\]

The effect of this transformation on the $F$-symbols is

\[
[F^abc_d]_{(e,\alpha,\beta')(f,\mu',\nu')} = \sum_{\alpha,\beta,\mu,\nu} [(u^{ab}_e)^{-1}]_{\alpha'\alpha}[(u^{bc}_d)^{-1}]_{\beta'\beta}[F^abc_d]_{(e,\alpha,\beta)(f,\mu,\nu)}[u^{bc}_{e}]_{\mu'\mu}[u^{af}_{d}]_{\nu'\nu}
\]

When there are no multiplicities, the transformations $u^{ab}_{cd}$ are just complex phases. In this case, which is of most practical interest, we have the freedom to redefine the $F$-symbols following

\[
[F^abc_d]_{ef} = \frac{u^{bc}_{ef}u^{af}_{df}}{u^{bc}_{de}u^{af}_{df}}[F^abc_d]_{ef}
\]
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We can use this freedom to switch to a more convenient set of $F$-symbols. Note that we do not allow $u_a^0$ and $u^0$ to differ from unity due to (4.14).

From (4.34) it is straightforward that we can remove the phase $f_{a\bar{a}}$ of $[F_{a\bar{a}}]_{00}$ for all $a \neq \bar{a}$, by taking $u_{a\bar{a}} = f_a$ and $u_{\bar{a}a} = 1$. So there is a choice of gauge for which all Frobenius-Schur indicators are equal to unity, except for self-conjugate particles as claimed before.

The gauge freedom discussed here should not be mistaken for gauge freedom in the sense of gauge theories. We merely wish to express that there is a redundancy in the theory.

4.1.4 Tensor product and quantum trance

The tensor product of two operators acting on well separated groups of anyons is made by juxtaposition of the corresponding diagrams. For a general operator we introduce the notation

\[
X \in V_{A_1 \ldots A_m}^{A'_1 \ldots A'_n} = \bigoplus_{a_1^a \ldots a_m^a} V_{a_1 \ldots a_m}^{a'_1 \ldots a'_m}
\]

where we used capital labels to denote sums over all anyonic charges such that $X$ is defined to act on any input of $n$ anyons, giving some $m$ anyon output. The tensor product of two operators $X$ and $Y$ is then given by the diagram

\[
X \otimes Y = X \quad Y
\]

While it is of course possible to rewrite this in terms of the standard basis, it is usually much more convenient to keep the separation explicit.

In the mathematical literature, one often sees equation (4.24) as the definition of the quantum dimension. It is a special case of the quantum trace applied to the identity operator. The quantum trace of a general operator $X$, denoted as $\hat{\text{Tr}} X$, is formed diagrammatically by closing the diagram with loops that match the outgoing lines with the incoming lines at the same position

\[
\hat{\text{Tr}} X = \hat{\text{Tr}} \left[ \begin{array}{c} A_1 \ldots A_m \\ A'_1 \ldots A'_n \end{array} \right] = \hat{\text{Tr}} \left[ \begin{array}{c} A_1 \ldots A_m \\ A'_1 \ldots A'_n \end{array} \right]
\]

The fact that the quantum dimension is the quantum trace of the identity operator is analogous to the fact that the trace of the identity matrix gives the dimension of the
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vector space it acts on. In cases where we can speak of an internal Hilbert space of the
anyon, as in DGT, the quantum trace is indeed the same as the trace over this internal
space.

From the diagrams it is clear that this definition of trace and tensor product behave
in the appropriate way. for two operators \( X \) and \( Y \), we have \( \hat{\text{Tr}}(X \otimes Y) = \hat{\text{Tr}}(X) \hat{\text{Tr}}(Y) \).

We can also define the partial trace, and ‘trace out’ a subsystem. In this case only
a part of the outer lines is closed with loops, either on the outer left or on the outer
right side.

4.1.5 Topological Hilbert space

The topological Hilbert space of a collection of anyons with charges \( a_1, \ldots, a_n \)
with total charge \( c \) is the space \( V^{a_1, \ldots, a_n}_c \). It is interesting to study the asymptotic behaviour
of the topological Hilbert space of \( n \) anyons of charge \( a \). This is effectively given by
the quantum dimension \( d_a \), as discussed below. This discussion is based on [28].

An important relation for the quantum dimension is

\[
d_a d_b = \sum_{c, \mu} N_{ab}^c d_c d_a d_b = \sum_{c} N^c_{ab} d_c \tag{4.38}
\]

The \( d_a \) form a one-dimensional representation of the fusion algebra, so to say. Now
consider the vector \( |\omega\rangle = \sum_d d_a |a\rangle \) in the fusion algebra. Then this relation shows that
\( |\omega\rangle \) is a common eigenvector of the fusion matrices, and we have

\[
N_a |\omega\rangle = d_a |\omega\rangle \tag{4.39}
\]

Since the matrices \( N_a \) have non-negative integer entries and \( |\omega\rangle \) has only positive coeffi-
cients, \( d_a \) must be the largest eigenvalue. Note that this in particular means that we do
not need the \( F \)-symbols to deduce \( d_a \), but that the fusion rules are sufficient. We can
rephrase the above observation as the statement that \( |\omega\rangle \) is a common Perron-Frobenius
eigenvector of the \( N_a \).

The significance for the topological Hilbert space is the following. Suppose we have
a collection of \( n \) anyons of charge \( a \) which have total charge \( b \). Denote the dimension
of the topological Hilbert space \( V_{ab}^b \) by \( N_{ab}^b \). We have

\[
N_{ab}^b = \sum_{\{ b_i \}} N_{aa}^{b_1} N_{ab_1}^{b_2} \ldots N_{ab(n-2)}^{b(n-2)} \tag{4.40}
\]

The matrix \( N_a \) can be diagonalized as

\[
N_a = \frac{d_a}{\text{Tr}} |\omega\rangle \langle \omega| + \ldots \tag{4.41}
\]

38
where
\[ \mathcal{D} = \sqrt{\langle \omega | \omega \rangle} = \sqrt{\sum_a d_a^2} \] (4.42)
and we have left out sub-leading terms. So for large \( n \), we obtain
\[ N_{ab...a} \approx d^a_b d^b_a / \mathcal{D}^2 \] (4.43)
Therefore we see that \( d_a \) controls the rate of growth for the \( n \) particle topological Hilbert space.

Corresponding to \( |\omega\rangle \), we have the charge line labelled by \( \omega \). Since \( d_a = d_{\bar{a}} \), the labels \( a \) and \( \bar{a} \) appear with identical weights. Therefore we can leave out the orientation of the charge line, and write
\[ \omega = \sum_a d_a \] (4.44)
This gives
\[ \omega = \mathcal{D}^2 \] (4.45)

We would like to point the significance of the label \( O\text{label} \) when we consider a DGT. In the representation theory of groups, one often considers the group algebra \( \mathbb{C}[H] \) of, say, a finite group \( H \) as a representation. We can just let \( H \) act on it via the group operation. It is a well known fact that \( \mathbb{C}[H] \) can be written as a sum of irreducible representation which each occur with multiplicity given by the dimension of the irrep, \( \mathbb{C}[h] \simeq d_a V^a \). In fact, the same holds for quantum double representations, \( D[H] \simeq \sum_{(A,a)} d_{(A,a)} V^A_a \). Hence, for these theories \( \omega = D[H] \).

### 4.2 Braiding

The defining properties of anyonic excitations are their non-trivial spin and braiding properties. These are also incorporated in the formalism by means of diagrammatic relations.

The effect of two anyons switching places in the system is governed by the braiding operators, which are written as
\[ R_{ab} = \begin{array}{c} a \\ b \end{array}, \quad R_{ab}^\dagger = R_{ab}^{-1} = \begin{array}{c} a \\ b \end{array} \] (4.46)
They are defined by a set of \( R \)-symbols which give the effect on basis states of \( V^{ab}_c \). The diagrammatic relation, or \( R \)-moves, are
\[ \begin{array}{c} b \\ c \end{array}^{\mu} = \sum_{\nu} [R_c^{ab}]_{\mu\nu} \begin{array}{c} b \\ c \end{array}^{\nu} \] (4.47)
\[ \begin{array}{c} a \\ c \end{array}^{\nu} = \sum_{\mu} [R_c^{ab}]_{\nu\mu} \begin{array}{c} a \\ c \end{array}^{\mu} \] (4.48)
4. ANYONS AND TENSOR CATEGORIES

The full braiding operator can thus be represented as

\[ a \bigotimes b = \sum_{c, \mu, \nu} \sqrt{d_c} [R^{ab}_{c}]_{\mu \nu} \bigotimes c \bigotimes \mu \bigotimes b \]

(4.49)

The similar equations hold for the inverse \( R \)-move. Unitarity of the \( R \)-moves amounts to

\[ [(R^{ab}_{c})^{-1}]_{\mu \nu} = [(R^{ab\dagger}_{c})]_{\mu \nu} = [R^{ab}_{c}]_{\nu \mu} \]

(4.50)

For braiding and fusion to be consistent, the \( R \)-moves have to obey the hexagon equations, discussed below. These relations imply that braiding commutes with fusion, which means that in diagrams lines may be passed over and under vertices. They imply the usual Yang-Baxter relation for braids. Together with triviality of fusion with the vacuum they also imply

\[ R^{a0}_{a} = R^{0b}_{b} = 1 \]

(4.51)

so that the vacuum braids trivial, as it should.

The topological spin \( \theta_a \), also called twist factor, is associated with a \( 2\pi \) rotation of an anyon of charge \( a \) and defined by

\[ \theta_a = \theta_{\bar{a}} = d_{\hat{a}}^{-1} \text{Tr} R_{aa} = \text{fs}_{a}[R_{1\bar{2}a}]^* \]

(4.52)

Diagrammatically, it can be used to remove twists from a diagram:

\[ a \bigotimes a = \theta_a \bigotimes a = a \bigotimes a \]

(4.53)

and

\[ a \bigotimes a = \theta_a^* \bigotimes a = \theta_a \bigotimes a \]

(4.54)

Because the \( \theta_a \) are not necessarily trivial, it is better to think of the lines in diagrams as ribbons for which a twist is really a non-trivial operation, as illustrated in figure ??.

\[ \bigotimes \bigotimes = \bigotimes \bigotimes \]

When applicable, the twist factor is related to the (ordinary angular momentum) spin or CFT conformal scaling dimension \( h_a \) of \( a \), as

\[ \theta_a = e^{i2\pi h_a} \]

(4.55)
4.3 Pentagon and Hexagon relations

The effect of a double braiding – or monodromy – of two anyons is governed by the monodromy equation

\[ \sum_{\lambda} [R_{\lambda}^{ab}]_{\mu \lambda} [R_{\lambda}^{ba}]_{\lambda \nu} = \frac{\theta_c}{\theta_a \theta_b} \delta_{\mu, \nu} \] (4.56)

which diagrammatically gives

\[ \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{\mu}
\end{array} \stackrel{\theta_c}{\longrightarrow} \begin{array}{c}
\text{c} \\
\text{a} \\
\text{b} \\
\text{\mu}
\end{array} = \begin{array}{c}
\text{c} \\
\text{\mu}
\end{array} \quad (4.57)

It is a matter of topological manipulation to see that this equation holds. This is easier to see when we draw the ribbon version.

We can also write down the full monodromy operator

\[ \begin{array}{c}
\text{a} \\
\text{b} \\
\text{\mu}
\end{array} = \sum_{c,\mu} \theta_c \begin{array}{c}
\text{c} \\
\text{\mu}
\end{array} \sqrt{d_c} \begin{array}{c}
\text{a} \\
\text{b}
\end{array} \quad (4.58)\]

which corresponds to one particle encircling the other, or equivalently a $2\pi$ rotation of a pair of particles.

4.3 Pentagon and Hexagon relations

With the definition of fusion rules, $F$-symbols and $R$-symbols the topological data is complete and all diagrams can be calculated. In order to ensure that different ways of manipulating a diagram always lead to the same answer, some consistency conditions on the $F$-symbols and $R$-symbols have to be satisfied. These are the pentagon and hexagon relations or equations. The MacLane’s coherence theorem from category theory guarantees that no other consistency checks are necessary [?].

The pentagon equations are a condition on the $F$-symbols. They can be understood as the equivalence of two sequences of moves, as shown in figure 4.1, which can be neatly organized in a pentagon. The resulting equation is

\[ \sum_{\delta} [F_{\delta}^{fcdg}]_{(g,\beta,\gamma)(l,\lambda,\delta)} [F_{\delta}^{ab}]_{(f,\alpha,\delta)(k,\mu,\nu)} = \sum_{h,\rho,\sigma,\psi} [F_{\delta}^{abcd}]_{(f,\alpha,\beta)(h,\rho,\sigma)} [F_{\delta}^{abhd}]_{(g,\beta,\gamma)(l,\lambda,\delta)} [F_{\delta}^{bhdg}]_{(h,\rho,\psi)(l,\lambda,\mu)} \] (4.59)
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Figure 4.1: The pentagon relations - The pentagon relation say that the two possible sequences of manipulating the basis states of $V_{abc}^{e}$ using $F$-moves give the same result.

The hexagon relations are similarly understood as the equality of two sequences of manipulation. They guarantee consistence of fusion and braiding. The diagrammatic relations are illustrated in figure 4.2. The resulting equations are

$$
\sum_{\lambda,\gamma}[R_{e}^{\alpha\lambda}]\alpha\lambda[F_{d}^{abc}](e,\lambda,\beta)(g,\gamma,\nu)[R_{g}^{\beta\mu}]\gamma\mu
$$

(4.60)

$$
= \sum_{f,\sigma,\delta,\psi}[F_{d}^{abc}](e,\alpha,\beta)[R_{f}^{\delta\mu}]\delta\mu[F_{d}^{abc}](f,\sigma,\psi)(g,\mu,\nu)
$$

(4.61)

and

$$
\sum_{\lambda,\gamma}[(R_{e}^{\alpha\lambda})^{-1}]\alpha\lambda[F_{d}^{abc}](e,\lambda,\beta)(g,\gamma,\nu)[(R_{g}^{\beta\mu})^{-1}]\gamma\mu
$$

(4.62)

$$
= \sum_{f,\sigma,\delta,\psi}[F_{d}^{abc}](e,\alpha,\beta)[(R_{f}^{\delta\mu})^{-1}]\delta\mu[F_{d}^{abc}](f,\sigma,\psi)(g,\mu,\nu)
$$

(4.63)

The pentagon and hexagon equations first appeared in this form in the physical literature in [?].

4.4 Modularity

The consistent definition fusion rules and unitary $F$-symbols and $R$-symbols gives a unitary braided tensor category. But in relation to topological quantum field theories, models obeying an additional condition are of particular interest. These are known as modular tensor categories. The characteristic property is that the topological $S$-matrix is non-degenerate.

The topological $S$-matrix, defined by

$$
S_{ab} = \frac{1}{D} \left( \begin{array}{c} a \\ b \end{array} \right)
$$

(4.64)
4.4 Modularity

Figure 4.2: The hexagon relations - The hexagon relations relate fusion to braiding. They imply the Yang-Baxter equation.

encodes a wealth of information about the theory. By applying the ribbon property we find

\[ S_{ab} = \frac{1}{D} \sum_c N_{ab}^c \frac{\theta_c}{\theta_a \theta_b} d_c \]  

(4.65)

From this expression, we derive that \( S_{ab} = S_{ba} = S_{ab}^{\ast} \) and \( S_{0a} = \frac{1}{D} \) \( d_a \). A very useful equality is

\[ a \begin{array}{c} \uparrow \\ \rightarrow \\ \rightarrow \\ \downarrow b \end{array} = \frac{S_{ab}}{S_{0b}^{\ast}} b \]  

(4.66)

This holds, because both left and right side are elements of the one-dimensional vector space \( V_b \), hence must be proportional. The constant of proportionality is checked by taking the quantum trace. This is a convenient trick to prove equalities of single-anyon operators. Often, these immediately imply a corresponding result for multi-anyon operators by decomposing these using (4.25).

Modular tensor categories are defined by the non-degeneracy of the \( S \)-matrix. When the theory is unitary, as is the case for physical theories, the \( S \)-matrix is in fact a unitary matrix. Together with the matrix \( T \) with coefficients

\[ T_{ab} = e^{2\pi c/24} d_a \delta_{ab} \]  

(4.67)

it forms a representation of the modular group \( SL(2, \mathbb{Z}) \) with defining relations \((ST)^3 = S^2 \) and \( S^4 = 1 \). Here \( c \) is the topological central charge, which is equal to the CFT central charge mod 24 when applicable. It can be determined mod 8 from the twist factors and quantum dimensions by the relation

\[ \exp \left( \frac{2\pi i c}{8} \right) = \frac{1}{D} \sum_a d_a^2 \theta_a \]  

(4.68)
4. ANYONS AND TENSOR CATEGORIES

This is all well known, but since it is an interesting application of the diagrammatic formalism we include a derivation based on [9] of the relations between $S$ and $T$ below.

Let

$$p^\pm = \sum_a d_a^2 \theta_a^\pm$$

(4.69)

The following equations hold

$$\omega \begin{tikzpicture} [baseline=(current bounding box.center)]
\draw [thick, -stealth] (0,0) -- (2,0);
\draw [thick, -stealth] (2,0) -- (4,0);
\end{tikzpicture} = p^+ \theta_b^{-1}$$

(4.70)

To show these equalities, note that the left and right hand side of these equations must be proportional, since they are both single-anyon operators. Hence we take the quantum trace on either side to check the proportionality constant. For the equation involving $p^+$, the quantum trace of the right hand side is evidently $p^+ d_b / \theta_b$, while the left hand side gives

$$\sum_a d_a \theta_a D S_{ab} = \sum_{a,c} N_{ab}^c d_a d_c \theta_c / \theta_b$$

(4.71)

$$= \sum_c d_c^2 \theta_c d_b / \theta_b$$

(4.72)

$$= p^+ d_b / \theta_b$$

(4.73)

Here we used that $\sum_a N_{ab}^c d_a = \sum_a N_{ab}^3 d_a = d_b d_c$ by (4.38), the symmetries of the fusion coefficients, and the fact that $d_a = d_a$. As a consequence we get

$$\omega \begin{tikzpicture} [baseline=(current bounding box.center)]
\draw [thick, -stealth] (0,0) -- (2,0);
\draw [thick, -stealth] (2,0) -- (4,0);
\end{tikzpicture} = p^+ \theta_a^{-1} \theta_b^{-1}$$

(4.74)

To see this, apply the ribbon property on the double clockwise braid on the right hand side, i.e. on the operator $(R_{ab}^\dagger)^2$, and compare with (4.70).

Define

$$T_{ab}^\prime = \theta_a \delta_{ab}, \quad C_{ab} = \delta_{ab}$$

(4.75)

Thus $T'$ is just $T$ with the phase in front left out, and $C$ is the charge conjugation matrix. Clearly, $C$ commutes with $S$ and with $T'$, and $C^2 = 1$. But more interestingly

1. $(ST')^3 = p^+ S^2 C / \mathcal{D}$
2. $(ST'^{-1})^3 = p^- S^2 / \mathcal{D}$
3. $S^2 = p^+ p^- C / \mathcal{D}^2$
4.4 Modularity

To prove the first relation, start with the identity

\[ \omega_a \uparrow \uparrow \rightarrow \uparrow \uparrow \rightarrow \rightarrow \rightarrow a = p^+ \theta_a^{-1} \theta_c^{-1} \]  \hspace{1cm} (4.76)

which follows from (4.74). The right hand side is equal to

\[ p^+ \theta_a^{-1} \theta_c^{-1} \frac{S_{ca}}{S_{0a}} \]  \hspace{1cm} (4.77)

while the left hand side can be computed as

\[ \sum_b d_b \theta_b \sum_{\alpha \beta} b \uparrow \uparrow \rightarrow a = \sum_b d_b \theta_b \frac{S_{bc}}{S_{0b}} \sum_{\alpha \beta} b \uparrow \uparrow \rightarrow a = D \sum_b \theta_b S_{bc} \frac{S_{ba}}{S_{0a}} \]  \hspace{1cm} (4.78)

Equating the coefficients gives \( D \sum_b \theta_b S_{bc} S_{ba} = p^+ \theta_a^{-1} \theta_c^{-1} S_{ac} \), which can be seen to be equivalent to the matrix equation \( DST'S = p^+ T'^{-1} SCT'^{-1} \) written in components. Multiplying from left by \( ST' \) and from the right by \( T' \) and dividing out \( D \) produces the first statement. The second statement is derived similarly.

The third statement follows from the first two. We first show that \( S^2 T' = T'S^2 \). Since \( S, T' \) are symmetric, by taking the transpose of the equality \( (ST'^{-1})^3 = p^+ S^2 / D \), we find that \( (ST'^{-1})^3 = T'S^3 \), or \( ST'^{-1}ST'^{-1}S = T'^{-1}ST'^{-1}ST' \). Plugging this into the following version of the second equality, \( \frac{p^+}{p^+} ST'^{-1}ST'^{-1}S = T'S^2 \), we find

\[ T' S^2 = \frac{D}{p} T'^{-1} ST'^{-1} ST'^{-1} ST' \]  \hspace{1cm} (4.79)

\[ = S^{-1} \left( \frac{D}{p} ST'^{-1} ST'^{-1} ST'^{-1} \right) ST' \]  \hspace{1cm} (4.80)

\[ = S^{-1} (S^2) ST' \]  \hspace{1cm} (4.81)

\[ = S^2 T' \]  \hspace{1cm} (4.82)

so \( S^2 \) and \( T' \) indeed commute. Now multiply the identities

\[ T'^{-1} ST'^{-1} ST'^{-1} = p^- S / D \]  \hspace{1cm} and  \hspace{1cm} \[ T' ST' T' = p^+ S C / D \]  \hspace{1cm} (4.83)

Pulling factors of \( S^2 \) through \( T' \)'s and cancelling \( T'T'^{-1} \) pairs, this leaves

\[ S^4 = p^+ p^- S^2 C / D^2 \]  \hspace{1cm} (4.84)

which proves the third statement.
This third statement has an important consequence. It implies that loops labelled by $\omega$ have the killing property. This somewhat hostile name comes from the fact that all non-trivial charge lines going through a loop labelled by $\omega = \sum_a d_a a$ get killed in the following way.

$$\omega \uparrow a = p^+ p^- \delta_{0a} \quad (4.85)$$

This is again proved by comparing the quantum trace of the left and right hand side. The trace of the left hand side gives

$$\mathcal{D} \sum_b d_b S_{ba} = \mathcal{D}^2 \sum_b S_{0b} S_{ba} = p^+ p^- C_{0a} = p^+ p^- \delta_{0a} \quad (4.86)$$

which is clearly the same as the trace of the right hand side. Thus loops labelled by $\omega$ give a natural projector in the theory. Taking $a = 0$ we get the following result

$$p^+ p^- = \mathcal{D}^2 \quad (4.87)$$

If we now define

$$T = \zeta^{-1} T', \quad \zeta = (p^+ / p^-)^{1/2} \quad (4.88)$$

we find that $S$ and $T$ obey the relations of the modular group, as was alluded to before. This also gives the implicit definition of the topological central charge $c$ in terms of the quantum dimensions and twist factors of the theory.

### 4.4.1 Verlinde formula

The famous Verlinde formula, which first appeared in the context of CFT in [34], expresses the fusion coefficients in terms of the $S$-matrix as

$$N^{c}_{ab} = \sum_x S_{ax} S_{bx} S_{0x} / S_{0x} \quad (4.89)$$

Because the fusion matrices by associativity of fusion, a basis of common eigenvectors is expected. The Verlinde formula says that this basis is essentially given by the columns of the $S$-matrix. We will prove the Verlinde formula below.

Define

$$\Lambda_{bc}^{(a)} = \lambda_b^{(a)} \delta_{bc}, \quad \lambda_b^{(a)} = S_{ab} / S_{0b} \quad (4.90)$$

Recall that the fusion matrices are given by $(N_a)_{bc} = N_{ab}^{c}$. We will show that

$$N_a S = S \Lambda^{(a)} \quad (4.91)$$

---

1. This projector has recently been used to provide a spacetime picture for the Levin-Wen models in an interesting work [12]
by manipulating the following diagram:

\[ \begin{array}{c}
  a \\
  \uparrow \\
  b \\
  \downarrow \\
  x \\
\end{array} \]

(4.92)

On the one hand, we can apply equality (4.66) twice to get

\[
\frac{S_{ax} S_{bx}}{S_{0x} S_{0x}} \bigg|_x
\]

(4.93)

On the other hand, we can first fuse the encircling anyons with charge \( a \) and \( b \) and then apply (4.66), to get

\[
\sum_d N_{ab}^d S_{dx} \bigg|_x
\]

(4.94)

This gives the equality

\[
\sum_d N_{ab}^d S_{dx} = S_{ax} S_{bx} \frac{S_{bx}}{S_{0x}}
\]

(4.95)

which is precisely (4.91) in components. When we multiply both sides by \( S_{cx} \), sum over \( x \), and use that \( S^2 = C \) or \( \sum_x S_{dx} S_{cx} = \delta_{dc} \), we find (4.89).

### 4.5 States and amplitudes

To incorporate quantum states of anyons in the formalism, we have to keep track of the creation history. General states for a system with anyons of charge \( a_1, \ldots, a_n \) can be written down as

\[
|\Psi\rangle = \sum_{a_1, a_2, a_3, a_4, c} \frac{\psi_{a_1, a_2, a_3, a_4, c}}{(d_{a_1} d_{a_2} d_{a_3} d_{a_4} d_c)^{1/4}} \bigg| a_1 \, \overrightarrow{a_2} \, \overrightarrow{a_3} \, \overleftarrow{a_4} \bigg|
\]

(4.96)

where we have suppressed the Greek indices labelling the vertices. The normalization factor is included such that \( \langle \Psi | \Psi \rangle = 1 \) iff

\[
\sum_{a_1, a_2, a_3, a_4, c} |\psi_{a_1, a_2, a_3, a_4, c}|^2 = 1
\]

(4.97)

where \( \langle \Psi | \) is of course defined by conjugating the diagram

\[
\langle \Psi | = \sum_{a_1, a_2, a_3, a_4, c} \frac{\psi_{a_1, a_2, a_3, a_4, c}^*}{(d_{a_1} d_{a_2} d_{a_3} d_{a_4} d_c)^{1/4}} \bigg| a_1 \, \overleftarrow{a_2} \, \overrightarrow{a_3} \, \overrightarrow{a_4} \bigg|
\]

(4.98)

and the bracket is formed by stacking the diagrams on top of each other. Amplitudes for anyon operators can now be calculated in the usual way. We sandwich the operator
4. ANYONS AND TENSOR CATEGORIES

between the ket and bra of the state and calculate the value of the diagram. As an example, let us consider the monodromy operator. Suppose we create two anyon pairs \( \bar{a}a \) and \( \bar{b}b \). This gives the tensor product state

\[
|\Psi\rangle = \frac{1}{\sqrt{d_a d_b}} \sqrt{\bar{a} a} \sqrt{\bar{b} b}
\]

(4.99)

The amplitudes for the monodromy of the inner \( ab \)-pair gives rise to the monodromy matrix

\[
M_{ab} = \frac{1}{d_a d_b} \sum_c N_c^a \theta_{ab} \theta_{cb} d_c
\]

(4.100)

which, as the diagram shows, is closely related to the \( S \)-matrix. Interferometry experiments are possibly able to measure matrix elements of the monodromy matrix in the near future for interesting systems, such as in the quantum Hall effect. Since the matrix is highly constrained by the various \( S \)-matrix relations, measuring only a few matrix elements might make it possible to construct the full matrix and discover the topological order. See [7, 8, 9, 10] for a theoretical treatment of interferometry experiments in a language similar to the presentation here.

One can also adopt a density matrix formalism which can handle systems of non-trivial total charge. By writing \( \rho = |\Psi\rangle \langle \Psi| \) diagrammatically, we find that the density matrix corresponding to a pure state is of the form

\[
\rho = \sum_{a_1, a_2, a_3, a_4, a'_1, a'_2, a'_3, a'_4} \frac{\rho(a_1 a_2 a_3 a_4, a'_1 a'_2 a'_3 a'_4)}{(d_a d_b d_c d_{a'_1} d_{a'_2} d_{a'_3} d_{a'_4})^{1/2}}
\]

(4.101)

General density matrices, corresponding to mixed states or states with non-trivial anyonic charge, can be constructed from these by tracing out anyons of the system using the quantum trace. For example, a general two-anyon density matrix is of the form

\[
\sum_{a, b, a', b', c} \rho_{a, b, a', b', c} \frac{(d_a d_b d_{a'} d_{b'})^{1/2}}{(d_a d_b d_{a'} d_{b'})^{1/2}}
\]

(4.102)

which can be obtained from the pure state density matrix

\[
\sum_{a, b, c} \frac{\rho(a, b, c)}{(d_a d_b d_c d_{a'} d_{b'})^{1/2}}
\]

(4.103)

by tracing out the right anyon.
Let us end this chapter by discussing the full braid operator $B$ or $B$-move. It is defined by

$$
\begin{array}{c}
\begin{array}{c}
\stackrel{a}{\downarrow} \quad \stackrel{b}{\downarrow} \quad \stackrel{c}{\downarrow} \\
\bullet \\
\stackrel{d}{\downarrow} \\
\end{array}
\end{array}
\quad = \sum_f [B^{abc}_d]_{ef}
\begin{array}{c}
\begin{array}{c}
\stackrel{a}{\downarrow} \quad \stackrel{b}{\downarrow} \quad \stackrel{c}{\downarrow} \\
\bullet \\
\stackrel{d}{\downarrow} \\
\end{array}
\end{array}

(4.104)

Reinstalling the Greek indices, we find that

$$
[B^{abc}_d]_{(e,\alpha,\beta)(f,\mu,\nu)} = \sum_{g,\gamma,\delta,\eta} [F^{acb}_d]_{(e,\alpha,\beta)(g,\gamma,\delta)} [R^{cb}_g]_{\gamma \eta} [(F^{abc}_d)^{-1}]_{(g,\eta,\delta)(f,\mu,\nu)}

(4.105)

This operator captures the effect of braiding on general anyonic states. The monodromy matrix is for example recovered as the $(0,0)$ of the appropriate square of this operator,

$$
M_{ab} = \sum_{f,\mu,\nu} [B^{\bar{a}ba}_b]_{0,(f,\mu,\nu)} [B^{\bar{a}ba}_b]_{(f,\mu,\nu),0}

(4.106)
4. ANYONS AND TENSOR CATEGORIES
CHAPTER 5

Examples of anyon models

In principle, the unitary braided tensor categories discussed in the last chapter can be constructed in a three step procedure: First, fix a set of particle labels and fusion rules. Next, solve the pentagon relation to find a consistent set of $F$-symbols. Finally, solve the hexagon relations to find $R$-symbols. If no solution to the pentagon relations exists, the declared fusion rules are inconsistent with local quantum mechanics and no model with these fusion rules exists. When we do find a solution to the pentagon relations, but not to the hexagon relations, there is no way to define a consistent braiding for the anyons which excludes the model with these fusion rules as an effective theory for bulk anyons but, as we will see, might describe boundary effects of a medium carrying anyons. If there are multiple solutions for the pentagon/hexagon relations that are not related by a gauge transformation, there are apparently non-equivalent anyon models that obey the same fusion rules. A theorem known as Ocneanu rigidity ensures that, for a given set of fusion rules, there are only finitely many non-equivalent anyon models.

Although the procedure outlined above can in principle be used to find all anyon models, in practice it is not feasible to find many interesting models this way. The reason is that the number of equations in the pentagon relations, as well as the number of variables, grows very rapidly with the number of charges in the theory. For small numbers of charges, some classification results have been obtained by direct attack of the pentagon/hexagon relations. For example, all models with up to four charges were classified in [30]. Also, in [11], solutions to the pentagon and hexagon relations were found explicitly for a number of interesting fusion rules by solving them directly.

Most known models, however, were constructed via a completely different route, namely as representation categories of quantum group that we discussed before. In this chapter we will show how to obtain the fusion rules, $F$-symbols and $R$-symbols for the quantum double. We will also give this data for the $\mathfrak{su}(2)_k$ theories related to $U_q[\mathfrak{su}(2)]$, Chern-Simons theory and the WZW-models and we flesh out the details for
some representative examples that will be used later. But we will start by discussing
the well know Fibonacci anyons, for which the pentagon and hexagon equations can be
solved by hand.

5.1 Fibonacci

Suppose we have only one non-trivial charge besides the vacuum, which we label by 1.
The two possibilities for the one non-trivial fusion rule are $1 \times 1 = 0$ and $1 \times 1 = 0 + 1$.
The first gives an Abelian theory that is not so interesting, so we take the second rule.
This fixes the theory completely, as we will see, and gives rise to the so called Fibonacci
anyons. The discussion below is based on [28].

The $F$-symbols we need to find are $[F_{11}^{11}]_{ef}$ and $[F_{11}^{11}]_{ef}$. The matrix $[F_{11}^{11}]_{ef}$, has
only one non-zero component, which we can set to unity. This gives

\[ [F_{11}^{11}]_{ef} = \delta_{e1} \delta_{f1} \] (5.1)

This leaves us with the task to find the $2 \times 2$ matrix $F$ with $F_{ef} = [F_{01}^{11}]_{ef}$. Since we
are looking for a unitary theory, we search an $F$ of the form

\[ F = \begin{pmatrix} z & w \\ w^* & z \end{pmatrix} \] (5.2)

with $|z|^2 + |w|^2 = 1$. Without multiplicities in the theory, the pentagon relation simplify to

\[ [F^{cd}_{e}]_{gl} [F^{bd}_{e}]_{fk} = \sum_h [F^{abc}_{d}]_{gf} [F^{ahd}_{e}]_{gh} [F^{bcd}_{k}]_{hl} \] (5.3)

Now take $a = b = c = d = e = f = k = 1$ and $g = l = 0$. The pentagon equation for this
combination of labels is

\[ [F_{11}^{11}]_{00} = [F_{11}^{11}]_{01} [F_{11}^{11}]_{10} \] (5.4)

which gives $z = |w|^2$. This leads us to the general solution

\[ F = \begin{pmatrix} \phi^{-1} & e^{i\theta} \sqrt{\phi^{-1}} \\ e^{-i\theta} \sqrt{\phi^{-1}} & -\phi^{-1} \end{pmatrix} \] (5.5)

where $\phi = \frac{1+\sqrt{5}}{2}$, i.e the golden ratio. The phase $e^{i\theta}$ can be set to unity with a gauge
transformation. One can check that indeed all pentagon relations are now satisfied.

Next we will solve the hexagon relations. Without multiplicities, these take the following form.

\[ R^{ec}_{f} [F^{abc}_{d}]_{eg} R^{bc}_{g} = \sum_f [F^{cde}_{f}]_{ef} R^{fc}_{d} [F^{abc}_{d}]_{fg} \] (5.6)

\[ (R^{ec}_{f})^{-1} [F^{abc}_{d}]_{eg} (R^{bc}_{g})^{-1} = \sum_f [F^{cab}_{c}]_{ef} (R^{fc}_{d})^{-1} [F^{abc}_{d}]_{fg} \] (5.7)
Since the vacuum braids trivially, we only need to find the $R$-symbols with the two upper indices equal to 1, so only $R_0 = R_1^{01}$ and $R_1 = R_1^{11}$ corresponding to a pair of Fibonacci anyons having trivial or non-trivial total charge respectively. We find

$$R_0 = e^{-4\pi i/5}, \quad R_1 = e^{3\pi i/5}$$  \hspace{1cm} (5.8)$$

We can read off the quantum dimension $d_1 = 1 + \sqrt{5}$ from $F$, and find that then non-trivial particle has spin $\theta_1 = f s_1 R_1^{10} = e^{2\pi i/5}$.

Now let $N_0^n = N_1^{1n}$ be the dimension of the topological Hilbert space of $n$ Fibonacci anyons with trivial total charge. It is easy to see that

$$N_0^1 = 0, \quad N_0^2 = 1, \quad N_0^3 = 1, \quad N_0^4 = 2, \quad N_0^5 = 3$$  \hspace{1cm} (5.9)$$

when looking at the number of fusion states explicitly. In general, the $N_0^n$ obey a simple recursion relation. If the first two particles have trivial total charge, the $n - 2$ other particles can fuse in $N_0^{n-2}$ different ways. When the charge of the first two particles is 1, the other $n - 2$ charges can combine with this charge in $N_0^{n-1}$ ways. So the recursion relation is

$$N_0^n = N_0^{n-1} + N_0^{n-2}$$  \hspace{1cm} (5.10)$$

which is well known to produce the Fibonacci numbers, the first of which we already gave above. This is why these anyons are given their specific name.

### 5.2 Quantum double

Using the facts about the quantum double $D[H]$ and it’s representations from chapter 3, it is possible to deduce fusion rules, $F$-symbols and $R$-symbols, and thus to obtain a description of the fusion braiding properties of the excitations in discrete gauge theories in the language of the previous chapter.

The connection with the graphical formalism and the algebraic discussion is as follows. Diagrams correspond to operators that commute with the action of $D[H]$ in the internal space, i.e. gauge invariant operators. The identity operator of the carrier space $V_\alpha^A$ is represented by a charge line labelled by $a = (A, \alpha)$ directed upwards. The splitting vertex labelled by $a = (A, \alpha)$, $b = (B, \beta)$ on top and $c = (C, \gamma)$ on the bottom is identified with the inclusion of $V_\gamma^C$ in $V_\alpha^A \otimes V_\beta^B$, up to a scale factor. The dual fusion vertex, on the other hand, is associated with the projection onto $V_\gamma^C$ as irreducible subspace of $V_\alpha^A \otimes V_\beta^B$. If there are multiple copies of $V_\gamma^C$ present in $V_\alpha^A \otimes V_\beta^B$, we can label them with indices $\mu$. Note that a unitary transformation on the $\mu$’s in $\oplus_{\mu}(V_\gamma^C)_\mu$ leaves the representations in tact. The quantum dimension and quantum trace correspond to the usual dimension and trace of the modules $V_\alpha^A$.

$$\xymatrix{ a \text{ identity,} & b \text{ inclusion,} & c \text{ projection} \ar@{-}[d]^{\mu} }$$  \hspace{1cm} (5.11)$$

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5. EXAMPLES OF ANYON MODELS

Throughout this section, it is very convenient to condense the notation, as discussion above already show. We agree to write \( a, b, c, \ldots \) for the particle-labels \((A, \alpha), (B, \beta), (C, \gamma), \ldots\). To not confuse the internal space \( V^A_\alpha \) with the fusion / splitting spaces, we will also refer to it as \( a \), and similar for the other particle labels. The representation maps are denoted \( \Pi^a = \Pi^A_\alpha \), etcetera. We can label the basis states of \( a \) by indices \( i = 1, \ldots, d_a \). We denote the basis states of \( a \) therefore as

\[
\{ |a, i \rangle \}, \quad i = 1, \ldots, d_a
\]  

leaving the particle label explicit, in stead of \( | h_j, \alpha v_k \rangle \) as we did before. The coordinate functions of the representation matrices are denoted \( \Pi^a_{ij} \). The fusion product \( a \times b \) means the tensor product representation.

5.2.1 Fusion rules

The fusion coefficients \( N^c_{ab} \), corresponding to the decomposition

\[
a \times b = \Pi^a \otimes \Pi^b = \bigoplus_c \Pi^c \sum_c N^c_{ab} c
\]  

can be obtained using certain orthogonality relations for the characters of representation. These are well known in the case of groups, but generalize to representations of the quantum double. It means that for unitary representations we have

\[
\frac{1}{|H|} \sum_{h,g} \left( \Pi^a(P_h g) \right) \left( \Pi^b(P_h g)^* \right) = \delta_{ab}
\]  

In fact, more general orthogonality relations for the coordinate functions \( \Phi^a_{ij} \). They obey

\[
\sum_{h,g \in H} \Pi^a_{ij}(P_h g)^* \Pi^b_{kl}(P_h g) = \delta_{a,b} \delta_{i,k} \delta_{j,l}
\]  

This follows from Woronowicz’s theory of pseudogroups [36] and generalize the well known Schur orthogonality relations for groups. We will use them heavily in the calculation of \( F \)-symbols.

From the orthogonality of characters, we see that the fusion multiplicities may be obtained from the representations matrices as

\[
N^c_{ab} = \frac{1}{|H|} \sum_{h,g} \left( \Pi^a \otimes \Pi^b(\Delta(P_h g)) \right) \left( \Pi^c(P_h g)^* \right)
\]  

\[
= \sum_{h,g} \sum_{h' h'' = h} \left( \sum_{i,j} \Pi^a_{ij}(P_{h'} g) \Pi^b_{ij}(P_{h''} g) \right) \left( \sum_k \Pi^c_{kk}(P_h g) \right)
\]

Next, we show how to calculate \( F \)-symbols.
5.2 Quantum double

5.2.2 Computing the $F$-symbols

Consider the diagrammatic equation for the $F$-moves:

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\hline
\text{c}
\end{array}
\begin{array}{c}
\text{b} \\
\hline
\text{d}
\end{array}
\begin{array}{c}
\text{e} \\
\hline
\text{f}
\end{array}
\end{array}
\begin{array}{c}
\text{a} \\
\hline
\text{c}
\end{array}
\begin{array}{c}
\text{b} \\
\hline
\text{d}
\end{array}
\begin{array}{c}
\text{e} \\
\hline
\text{f}
\end{array}
\end{array}
= \sum_{f} [F_{d}^{a b c}]_{e f}
\]

(5.18)

We see now that the $F$-symbols, relate two different ways to embed the irrep $d$ in the triple tensor product $a \times b \times c$. We can either first embed $d$ in $e \times c$ and then embed $e$ in $a \times b$, or we start with embedding $d$ in $a \times f$ and then embed $f$ in $b \times c$. The two different paths are related by a transformation inside the subspace of charge $d$ in $a \times b \times c$. This is transformation is what the $F$-moves describe.

The $F$-symbols are conveniently calculated using Clebsch-Gordan coefficients, which describe how the irreps are precisely embedded in tensor product representations. When there are no fusion multiplicities, that is all $N_{a b}^{c} = 0, 1$, we can calculate the Clebsch-Gordan symbols using by generalization the projection operator technique (a technique well known from the theory of group representations). We will therefore assume that the theory is multiplicity free. This is the case, for example, when $H = \bar{D}_{2}$, the group of unit quaternions described in [?] which we took as an example to test our calculations.

When there are no fusion multiplicities, there is a unique basis

\[
\{ |c, k\rangle \}, \quad N_{a b}^{c} = 1, \quad k = 1, \ldots, d_{c}
\]

(5.19)
in $a \times b$ corresponding to the decomposition in irreps. Of course there is also the standard inner product basis

\[
|a, i\rangle |b, j\rangle, \quad i = 1, \ldots, d_{a}; j = 1, \ldots, d_{b}
\]

(5.20)
The Clebsch-Gordan coefficients, sometimes also called $3j$-symbols, precisely give the relation between these two. They are defined by

\[
|c, k\rangle = \sum_{i, j} \binom{a}{i} \binom{b}{j} \binom{c}{k} |a, i\rangle |b, j\rangle,
\]

(5.21)
The inverse of this relation is written as

\[
|a, i\rangle |b, j\rangle = \sum_{c, k} \binom{a}{i} \binom{b}{j} \binom{c}{k} |c, k\rangle.
\]

(5.22)

To calculate the actual coefficients, we will use projectors $\mathcal{P}_{ij}^{a}$ defined by

\[
\mathcal{P}_{ij}^{a} = \frac{d_{a}}{|H|} \sum_{h, g \in H} \Pi_{ij}^{a}(P_{h} g)^{*} \Pi(P_{h} g),
\]

(5.23)

where the last $\Pi(P_{h} g)$ stands form the appropriate representation of the $P_{h} g$ element of the quantum double, depending on what it is acting on. These projectors, which can be applied in any representation, act as

\[
\mathcal{P}_{ij}^{a} |b, k\rangle = \delta_{a, b} \delta_{j, k} |a, i\rangle
\]

(5.24)
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as a consequence of the orthogonality relation (5.15). Applying the projector to a direct product of two states and using equation (5.22), gives

\[ P^c_{lk} |a, i\rangle \langle b, j| = \sum_{i', j'} \left( \begin{array}{ccc} c & a & b \\ k & i & j' \end{array} \right) \left( \begin{array}{ccc} a & b & c \\ i' & j & l \end{array} \right) |a, i'\rangle \langle b, j'|. \]  

(5.25)

By using the definition (5.23), this is seen to be equal to

\[ P^c_{lk} |a, i\rangle \langle b, j| = d_c |h\rangle \sum_{h, g \in H} \Pi^a_{i'}(P_h g) \Pi^b_{j'}(P_{h'} g) |a, i'\rangle \langle b, j'|, \]  

(5.26)

where we have inserted the definition of the comultiplication to act on the product state.

Equating expressions (5.25) and (5.26), we obtain

\[ \left( \begin{array}{ccc} c & a & b \\ k & i & j \end{array} \right) \left( \begin{array}{ccc} a & b & c \\ i' & j' & l \end{array} \right) = d_c |h\rangle \sum_{h, g \in H} \Pi^a_{i'}(P_h g) \Pi^b_{j'}(P_{h'} g) |a, i'\rangle \langle b, j'|. \]  

(5.27)

Unitarity amounts to

\[ \left( \begin{array}{ccc} c & a & b \\ k & i & j \end{array} \right) = \left( \begin{array}{ccc} a & b & c \\ i & j & k \end{array} \right)^*. \]  

(5.28)

Now pick some triple \((i, j, k)\) such that

\[ \frac{d_c}{|H|} \sum_{h, g \in H} \Pi^c_{kk'}(P_h g) \sum_{h' h'' = h} \Pi^a_{i'}(P_{h'} g) \Pi^b_{j'}(P_{h''} g) \]  

(5.29)

is non-zero. From equation (5.27) with \(i = i', j = j'\) and \(k = k'\) and from the unitarity condition (5.28) it follows that it is real and positive. This fixes one of the Clebsch-Gordan

\[ \left( \begin{array}{ccc} a & b & c \\ i & j & k \end{array} \right) = \left( \begin{array}{ccc} a & b & c \\ i & j & k \end{array} \right)^* \]  

(5.30)

We can use equation (5.27) to calculate all others, resulting in

\[ \left( \begin{array}{ccc} a & b & c \\ i' & j' & k' \end{array} \right) = \sqrt{\frac{d_c}{|H|} \sum_{h, g \in H} \sum_{h' h'' = h} \Pi^c_{kk'}(P_h g) \Pi^a_{i'}(P_{h'} g) \Pi^b_{j'}(P_{h''} g)} \]  

(5.31)
5.2 Quantum double

From the Clebsch-Gordan symbols it is straightforward to calculate the $F$-symbols. The definitions imply the relation

$$
\sum_{f,n} F_{f}^{abc} \left( \begin{array}{ccc} b & c & f \\ j & k & n \end{array} \right) \left( \begin{array}{ccc} a & f & d \\ m & k & \ell \end{array} \right) = \left( \begin{array}{ccc} a & b & e \\ i & j & m \end{array} \right) \left( \begin{array}{ccc} e & c & d \\ m & k & \ell \end{array} \right)
$$

(5.32)

It follows from (??) that

$$
\sum_{i,j} \left( \begin{array}{ccc} a & b & c \\ i & j & k \end{array} \right) \left( \begin{array}{ccc} a & b & c' \\ i & j & k' \end{array} \right)^* = \delta_{cc'} \delta_{kk'}
$$

(5.33)

Together, these give the following expression for the $F$-symbols in terms of the Clebsch-Gordan coefficients

$$
\left[ F_d^{abc} \right]_{e,f} = \sum_{i,j,k,m,n} \left( \begin{array}{ccc} b & c & f \\ j & k & n \end{array} \right) \left( \begin{array}{ccc} a & f & d \\ m & k & \ell \end{array} \right) \left( \begin{array}{ccc} a & b & e \\ i & j & m \end{array} \right)^* \left( \begin{array}{ccc} e & c & d \\ m & k & \ell \end{array} \right)^*
$$

(5.34)

(Note that $\ell$ can be chosen freely.)

In principle, one can now run a computer program to calculate the Clebsch-Gordan coefficients and subsequently the $F$-symbols from the above description.

5.2.3 Braiding

Recall that the braiding for the quantum double is defined through the action of the universal $R$-matrix

$$
\mathcal{R} = \sum_{h,g \in H} P_g e \otimes P_h g
$$

(5.35)

and subsequent flipping of the tensor legs. Because $\Delta(P_h g)\mathcal{R} = \mathcal{R}\Delta(P_h g)$, the effect of braiding on irreducible subspaces $c$ of $a \times b \simeq b \times a$ is in principle just multiplication by a complex number $R_{ab}^{c} \mu\nu$ by Schur’s lemma. But if some $c$ occurs in $a \times b$ with multiplicity greater than one, it can in fact act as a unitary transformation $[R_{ab}^{c}]_{\mu\nu}$ in the subspace of total charge $c$.

We can derive the $R$-symbols by composing the braiding braiding with the inclusion of $c$. Denote the representation matrix of the action of $\mathcal{R}$ on a state $|a, i\rangle |b, j\rangle$ as $\mathcal{R}^{a,b}_{(ij),(j')}$, i.e.

$$
\mathcal{R}^{a,b}_{(ij),(j')} = \Pi_{i'i}^a \otimes \Pi_{j'j}^b (\mathcal{R})
$$

(5.36)

For the simple case where there are no multiplicities of $c$, we get

$$
R_{ab}^{c} = \sum_{i,i'=1}^{d_a} \sum_{j,j'=1}^{d_b} \left( \begin{array}{ccc} a & b & c \\ i & j & \ell \end{array} \right) \left( \begin{array}{ccc} b & a & c' \\ j & j' & \ell' \end{array} \right)^* \mathcal{R}^{a,b}_{(ij),(j')} \mathcal{R}^{a,b}_{(ij),(j')}
$$

(5.37)

where again $\ell$ can be chosen freely.

If multiplicities of $c$ do occur, we can still obtain the $[R_{ab}^{c}]_{\mu\nu}$ by first including the $\mu^{th}$ copy of $c$ and next project onto the $\nu^{th}$ copy. This can be done by introducing more
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general Clebsch-Gordan coefficients that also an index that keeps track of the copy of c. The formula then becomes

\[
[R^a_{c}]_{j\mu, j' \nu} = \sum_{i,n} d_n \sum_{i',n'} (a, b | c, \mu)_{i, \ell} (c, \nu | b, a)_{j', \ell'} R_{ij, (i', j', n, n')}
\]

(5.38)

5.3 The \( su(2)_k \) theories

The data for \( su(2)_k \) theories, given this name because they appear in the WZW-models based on the affine algebra of \( su(2) \) at level \( k \), can be derived from the representation theory of \( U_q[su(2)] \) where \( q \) is a root of unity and related to \( k \) by as \( q = e^{i \frac{2\pi}{k+1}} \). As for general values of \( q \) a similar structure for the representations appears as in the representation theory of \( su(2) \), with integers replaced by the \( q \)-numbers \( [n]_q = \frac{q^n - q^{-n}}{q^{1/2} - q^{-1/2}} \). But, when considering the appropriate category of representations, the highest values for the highest weights get truncated. There are only a finite number of irreducible representations corresponding to the finite set of charge labels. We will just state the general formulas as they appeared in [31], where the authors refer to [?] for the calculation.

The charges of \( su(2)_k \) are labelled by integers \( a = 0, 1, \ldots, k \) that are related to the highest weights \( j \) as \( a = 2j \) (a notation also frequently used). The fusion rules are given by a truncated version of the usual addition rules for \( SU(2) \)-spin,

\[
a \times b = \sum_c N^c_{ab} c = |a-b| + |a-b| + 2 + \cdots + \min\{a+b, 2k-a-b\}
\]

(5.40)
i.e. \( N^c_{ab} = 1 \) when \( |a-b| \leq c \leq \min\{a+b, 2k-a-b\} \) and \( a + b + c = 0 \) (mod 2, and zero otherwise.

For the \( F \)-symbols, one has the general formula

\[
[F^a_{d}bc]_{ef} = (-1)^{(a+b+c+d)/2} \sqrt{[e+1]_q[f+2]_q} \{a, b \mid c, d \mid e, f \}
\]

(5.41)

where

\[
\begin{align*}
\{a, b \mid c, d \mid e, f \} &= \Delta(a, b, c) \Delta(e, c, d) \Delta(b, c, f) \Delta(a, f, d) \\
&\times \sum_z (-1)^{[z]} [1 + 1]_q [\frac{a+b+c}{2}]_q [\frac{a+b+c+d}{2}]_q [\frac{b+c+f}{2}]_q [\frac{b+c+d+f}{2}]_q [\frac{b+c+d+f+1}{2}]_q [\frac{a+b+c}{2}]_q [\frac{a+b+c+d}{2}]_q [\frac{b+c+d+f}{2}]_q [\frac{b+c+d+f+1}{2}]_q
\end{align*}
\]

(5.42)

with

\[
\Delta(a, b, c) = \sqrt{[\frac{a+b+c}{2}]_q [\frac{a+b+c}{2}]_q [\frac{a+b+c}{2}]_q [\frac{a+b+c+d}{2}]_q [\frac{a+b+c+d+d}{2}]_q [\frac{a+b+c+d+d+1}{2}]_q}
\]

(5.43)

and \( [n]_q! = \prod_{m=1}^{n} [m]_q \).
The sum over $z$ should run over all integers for which the $q$-factorials are well-defined. This depends on the level $k$. The expression for $\Delta$ is only well defined for admissible triples $(a,b,c)$, by which we mean that $a + b + c = 0 \pmod{2}$ and $|a-b| \leq c \leq a + b$ (we will take it to be zero for other triples, implementing consistency with the fusion rules). Note that $\Delta$ is invariant under permutations of the input.

The $R$-symbols are given by the general equation

$$R_{c}^{ab} = (-1)^{c-a-b}q^{\frac{1}{k}(c(c+2)-a(a+2)-b(b+2))}$$

This gives topological spins

$$\theta_a = e^{2\pi i \frac{a(a+2)}{k+2}}$$

The quantum dimensions are

$$d_a = \frac{\sin \left( \frac{(a+1)\pi}{k+2} \right)}{\sin \left( \frac{\pi}{k+2} \right)}$$

For later reference, we will work out the values of these functions for some representative values of $k$.

$k = 2$  The simplest non-trivial case occurs for $k = 2$ which is closely related to the Ising model. The charges are denoted $0, 1, 2$, with non-trivial fusion rules

$$1 \times 1 = 0 + 2, \quad 1 \times 2 = 1, \quad 2 \times 2 = 0$$

The relevant $q$-numbers and factorials are

<table>
<thead>
<tr>
<th>$n$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[n]_q$</td>
<td>$0$</td>
<td>$1$</td>
<td>$\sqrt{2}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[n]_q!$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{2}$</td>
</tr>
</tbody>
</table>

This gives quantum dimensions and spins

$$\begin{array}{c|cc}
\text{su}(2)_2 \\
\hline
0 & d_0 = 1 & h_0 = 0 \\
1 & d_1 = 2 & h_1 = \frac{3}{2} \\
2 & d_2 = 1 & h_2 = \frac{1}{2} \\
\end{array}$$

Using the general formula, we can calculate the $F$-symbols. Let us calculate $[F_{111}^{111}]_{00}$. The relevant $\Delta$’s are

$$\Delta(1,1,0) = \Delta(1,0,1) = \Delta(0,1,1) = \sqrt{\frac{[0]_q![0]_q![1]_q!}{[2]_q!}} = \frac{1}{2^{1/4}}$$
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This gives
\[
\left\{ \begin{array}{c|c}
1 & 1 \\
1 & 1 \\
0 & 0 \\
\end{array} \right\} = \frac{1}{2} \sum_{z} \frac{(-1)^{z} [z+1]_{q}!}{([z+1]_{q}!)^{4} [2-z]_{q}! [1-z]_{q}!} \\
= \frac{1}{2} \frac{[-2]_{q}!}{([0]_{q}!)^{4} [1]_{q}! [0]_{q}!} \\
= -1/\sqrt{2}
\]
(5.51)

which produces
\[
[F_{1}^{111}]_{00} = -1/\sqrt{2}
\]
(5.52)

By similar calculations we find
\[
[F_{1}^{111}]_{ef} = \begin{pmatrix}
-1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}, \quad e, f = 0, 2
\]
(5.53)

\( k = 4 \) For \( \mathfrak{su}(2)_{4} \) has been a very important example while working on this thesis. Since the prime goal was to study Bose condensation, it is of particular interest because it is one of the simplest models with a non-trivial boson.

The charges of \( \mathfrak{su}(2)_{4} \) are labeled by 0, 1, 2, 4 with the fusion rules given by the general formula, for example
\[
2 \times 2 = 0 + 2 + 4, \quad 1 \times 3 = 2 + 4
\]
(5.54)

The relevant \( q \)-numbers and \( q \)-factorials are
\[
\begin{array}{c|ccccc}
\text{n} & 0 & 1 & 2 & 3 & 4 \\
\hline
[n]_{q} & 0 & 1 & \sqrt{3} & 2 & \sqrt{3} \\
[n]_{q}! & 1 & 1 & \sqrt{3} & 2\sqrt{3} & 6
\end{array}
\]
(5.55)

This gives soins and quantum dimensions
\[
\mathfrak{su}(2)_{4} \\
\begin{array}{c|cc}
\text{n} & d_{n} & h_{n} \\
\hline
0 & d_{0} = 1 & h_{0} = 0 \\
1 & d_{1} = \sqrt{3} & h_{1} = \frac{1}{\sqrt{3}} \\
2 & d_{2} = 2 & h_{2} = \frac{1}{2} \\
3 & d_{3} = \sqrt{3} & h_{3} = \frac{1}{\sqrt{3}} \\
4 & d_{4} = 1 & h_{4} = 1
\end{array}
\]
(5.56)

In chapter 6 we need some specific values for the \( F \)-symbols, hence we calculate these here. These are \([F_{2}^{242}]_{22}, [F_{0}^{242}]_{22}, [F_{2}^{242}]_{22}, \) or actually \([F_{22}^{22}]_{40}, [F_{22}^{22}]_{42} \) and \([F_{22}^{22}]_{44} \). The relevant \( \Delta \)'s are
\[
\Delta (2, 2, 2) = 1/\sqrt{6}, \quad \Delta (2, 2, 0) = 1/\sqrt{2}, \quad \Delta (2, 2, 4) = 1/\sqrt{2}
\]
(5.57)
5.3 The $su(2)_k$ theories

We get

$$[F_{242}^{242}]_{22} = 2 \Delta(2, 4, 2) \Delta(2, 2, 4) \Delta(4, 2, 2) \Delta(2, 2, 4)$$  \hspace{1cm} (5.58)

$$\times \sum_z (z+1)_q! (-1)^z(4-z)_q! (6-z)_q! (4-3z)_q! (6-3z)_q!$$  \hspace{1cm} (5.59)

$$= 2 \times \frac{1}{4} \times \frac{[5]_q!}{([2]_q!)^2}$$  \hspace{1cm} (5.60)

$$= 1$$  \hspace{1cm} (5.61)

Similarly,

$$[F_{022}^{242}]_{22} = 2 \Delta(2, 4, 2) \Delta(2, 2, 0) \Delta(4, 2, 2) \Delta(2, 2, 0)$$  \hspace{1cm} (5.62)

$$\times \sum_z (z+1)_q! (-1)^z(z-3)_q! (z-2)_q! (z-1)_q!$$  \hspace{1cm} (5.63)

$$= 2 \times \frac{1}{4} \times \frac{[5]_q!}{([2]_q!)^2}$$  \hspace{1cm} (5.64)

$$= 1$$  \hspace{1cm} (5.65)

and finally,

$$[F_{222}^{242}]_{22} = -2 \Delta(2, 4, 2) \Delta(2, 2, 2) \Delta(4, 2, 2) \Delta(2, 2, 2)$$  \hspace{1cm} (5.66)

$$\times \sum_z (z+1)_q! (-1)^z(z-3)_q! (z-2)_q! (z-1)_q!$$  \hspace{1cm} (5.67)

$$= -2 \times \frac{1}{12} \times \frac{[5]_q!}{([0]_q!)^3([1]_q!)^4}$$  \hspace{1cm} (5.68)

$$= -1$$  \hspace{1cm} (5.69)

By

$$[F_{cd}^{ab}]_{(e,\alpha,\beta)(f,\mu,\nu)} = \sqrt{d_e d_f} [F_{fe}^{cd}]_{(a,\alpha,\mu)(b,\beta,\nu)}$$  \hspace{1cm} (5.70)

this gives

$$[F_{222}^{222}]_{40} = \frac{1}{2}, \quad [F_{222}^{222}]_{42} = -\frac{1}{\sqrt{2}}, \quad [F_{222}^{222}]_{44} = \frac{1}{2}$$  \hspace{1cm} (5.71)

$k=10$ Also $su(2)_k$ was of particular interest to us. While having a reasonably small number of charges – eleven – it is a much more intricate example than $su(2)_4$. There is a non-trivial bosonic charge, labelled 6 and the result of condensation was known. Hence it presented a fine test case for the theory of subsequent chapters.
5. EXAMPLES OF ANYON MODELS

We list the quantum dimensions and spins here.

<table>
<thead>
<tr>
<th>$\mathfrak{su}(2)_{10}$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$d_0 = 1$</td>
<td>$h_0 = 0$</td>
</tr>
<tr>
<td>1</td>
<td>$d_1 = \sqrt{s + \sqrt{3}}$</td>
<td>$h_1 = \frac{1}{16}$</td>
</tr>
<tr>
<td>2</td>
<td>$d_2 = 1 + \sqrt{3}$</td>
<td>$h_2 = \frac{1}{6}$</td>
</tr>
<tr>
<td>3</td>
<td>$d_3 = \sqrt{2} + \sqrt{2 + \sqrt{3}}$</td>
<td>$h_3 = \frac{3}{16}$</td>
</tr>
<tr>
<td>4</td>
<td>$d_4 = 2 + \sqrt{3}$</td>
<td>$h_4 = \frac{3}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>$d_5 = 2\sqrt{2 + \sqrt{3}}$</td>
<td>$h_5 = \frac{35}{38}$</td>
</tr>
<tr>
<td>6</td>
<td>$d_6 = 2 + \sqrt{3}$</td>
<td>$h_6 = 1$</td>
</tr>
<tr>
<td>7</td>
<td>$d_7 = \sqrt{2} + \sqrt{2 + \sqrt{3}}$</td>
<td>$h_7 = \frac{21}{16}$</td>
</tr>
<tr>
<td>8</td>
<td>$d_8 = 1 + \sqrt{3}$</td>
<td>$h_8 = \frac{3}{3}$</td>
</tr>
<tr>
<td>9</td>
<td>$d_9 = \sqrt{2 + \sqrt{3}}$</td>
<td>$h_9 = \frac{33}{2}$</td>
</tr>
<tr>
<td>10</td>
<td>$d_{10} = 1$</td>
<td>$h_{10} = \frac{1}{2}$</td>
</tr>
</tbody>
</table>

The fusion rules and $F$-symbols can be calculated using the general formulae. Calculating the $F$-symbols by hand becomes very cumbersome, hence we implemented the formulae in Mathematica to do calculations on $\mathfrak{su}(2)_{10}$ and general $\mathfrak{su}(2)_k$ models.
In this chapter we start the study of phase transitions between topologically ordered phases, the main goal of this thesis. The theory developed in the previous chapters is used to extend the formalism for topological symmetry breaking. This should enable one to construct a full diagrammatic formalism for topological phases that are obtained by means of topological symmetry breaking.

We start by a quick review of the topological symmetry breaking approach, which is based on quantum group symmetry breaking by a bosonic condensate. The main idea is that a mechanism of symmetry breaking, very similar to spontaneous symmetry breaking known from the ordinary theory of phase transitions, can account for many transitions between topological phases, only the symmetry is given by a quantum group. This was first explored in the context discrete gauge theories in [3, 6]. Later, the formalism was refined and applied to nematic phases in liquid crystals in [4, 5, 25]. These earlier approaches had the drawback that they were not applicable to treat cases most interesting for the FQHE, where non-integer quantum dimensions occur. In [7], a general scheme was laid out that can also handle these cases.

An important feature of this general scheme is that the explicit description of the quantum group disappears and everything is done on the level of the representations, or particle labels. In mathematical terminology, one might say that the representation category of the quantum group is put center stage while the quantum group itself is put to the background. It could therefore be expected that there is a connection with the mathematical literature on (modular) tensor categories.

The generalization of the notion of a subgroup to the appropriate category theoretical context is a commutative algebra, as argued in [?]. The results of [7] and [?] are in perfect agreement, although the language is very different in many respects. This motivated the study of the mathematical literature to see what physically meaningful results could be extracted.
6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

6.1 Topological symmetry breaking

We will outline the formalism of topological symmetry breaking, as described in detail in [7], below.

The topological symmetry breaking scheme has two steps, or three stages. We start with a (2+1) dimensional system exhibiting topological order with underlying quantum group symmetry labelling the topological sectors. This is the unbroken phase described by the quantum group or theory $\mathcal{A}$ (we use the notation $\mathcal{A}$ for both the quantum group and it’s category of representations, if no confusion is likely to arise).

![Diagram of the topological symmetry breaking scheme.

The first step of the topological symmetry breaking scheme is the formation of a bosonic condensate. For this to happen, the particle spectrum of the $\mathcal{A}$-theory should of course include a bosonic excitation. The formation of a condensate reduces the symmetry $\mathcal{A}$ to some subalgebra $\mathcal{T}$ of $\mathcal{A}$ which can be thought of as the stabilizer of the condensate. This $\mathcal{T}$-algebra represents the second stage of the symmetry breaking scheme. The precise definition of $\mathcal{T}$ has been discussed for quantum double theories and generalizations thereof [3–6,25]. Since quantum groups are not groups, some subtle issues occur. Which of the structures of $\mathcal{A}$ must be inherited by $\mathcal{T}$? It is a priori unclear if $\mathcal{T}$ has the full structure a quantum group. In fact, allowing the freedom that $\mathcal{T}$ does not give consistent braiding gives rise to one of the main benefits of the quantum group approach, the description of excitations on the boundary of the broken phase or interface between two phases.

When $\mathcal{T}$ does not allow for the definition of a universal $R$-matrix there is no consistent braiding of the excitations of the $\mathcal{T}$-theory. This is caused by excitations that do not braid trivially with the condensate and thereby locally destroy it. These excitations pull strings in the condensate with energy proportional to their length. In order to minimize the energy, the excitations that pull strings will be confined, either like

---

Figure 6.1: Topological symmetry breaking - The symmetry breaking scheme has two steps and three stages. The first step is the restriction of the quantum group $\mathcal{A}$ to the intermediate algebra $\cot \mathcal{T} \subset \mathcal{A}$. The second step is the projection onto the Hopf quotient $\mathcal{U}$ of $\mathcal{T}$. Figure from [7].
6.1 Topological symmetry breaking

$\mathcal{T}$-hadrons or on the boundary of the system. The intermediate $\mathcal{T}$-algebra thus provides a natural description of the boundary excitations. If we think of a geometry where a droplet of the system in the broken phase is surrounded by a region in the unbroken phase as in figure 6.2, the $\mathcal{T}$-theory gives a description of the interface between the two phases.

In the second step of the scheme, we project out the confined excitations to obtain a description of the bulk theory. On the level of the quantum group this corresponds to the surjective Hopf map from $\mathcal{T}$ to some Hopf quotient $\text{cat}U$. To describe bulk anyons, $\text{cat}U$ should give rise to consistent braiding and all other features of the tensor categories we described in chapter 4.

![Figure 6.2: Two-phase system](image)

- The figure a system in the unbroken phase I described by $\mathcal{A}$ that contains a droplet in the symmetry broken phase II described by $\mathcal{U}$. The excitations on I/II interface are naturally described by the intermediate algebra $\mathcal{T}$.

6.1.1 Particle spectrum and fusion rules of $\mathcal{T}$

Let us describe the formalism in a bit more detail. To look for a boson $b$ in the spectrum of $\mathcal{A}$, which we label by a Greek index for future convenience, we must first know what we mean by a bosonic excitation in this context. An obvious requirement is trivial spin, expressed by

$$\theta_b = e^{2\pi i h_b} = 1 \quad \leftrightarrow \quad h_b \in \mathbb{Z} \quad (6.1)$$

But in order to form a stable condensate, there must be a multi-particle state of $b$’s that is invariant under braiding. Unlike the higher dimensional case, this is not guaranteed by the trivial spin condition. Suppose a $bb$-pair has total charge $c$. The monodromy of
the two $b$’s leads to a phase $\exp(2\pi i h_c)$ as follows from

\[ e^{2\pi i (h_c - 2h_b)} = e^{2\pi i h_c} \quad \text{(6.2)} \]

Therefore, a second necessary condition is that there is a fusion channel $c \in b \times b$ with $h_c \in \mathbb{Z}$. One might argue that a full definition should guarantee that there is a fusion channel that is completely invariant under monodromies or even braidings for $b^{\otimes n}$ for any $n$. This condition is much more involved to check when only the spins and quantum dimensions of the theory are known/used, as is the case in [7]. In the approach that we will develop later, this is in fact guaranteed by the conditions that we impose. This has the drawback that we need the $F$-symbols and $R$-symbols, which might be harder to obtain. But for now, a working definition of a boson is

If a charge $b \in \mathcal{A}$ has

1. Trivial spin, $h_b \in \mathbb{Z}$,
2. A fusion channel $c \in b \times b$ with $h_c \in \mathbb{Z}$.

then $b$ is a boson.

We will illustrate the theory for $\mathfrak{su}(2)_4$, with fields 0, 1, 2, 3, 4 and non-trivial fusion rules

\[
\begin{align*}
1 \times 1 &= 0 + 2 \\
1 \times 2 &= 1 + 3 \\
1 \times 3 &= 2 + 4 \\
1 \times 4 &= 3 \\
2 \times 2 &= 0 + 2 + 4 \\
2 \times 3 &= 1 + 3 \\
2 \times 4 &= 2 \\
3 \times 3 &= 0 + 2 + 4 \\
3 \times 4 &= 1 \\
4 \times 4 &= 0
\end{align*}
\]

(6.3)

For convenience, we list the spins and quantum dimensions again.

\[
\begin{array}{cc}
\mathfrak{su}(2)_4 \\
0 & d_0 = 1 \\
1 & d_1 = \sqrt{3} \\
2 & d_2 = 2 \\
3 & d_3 = \sqrt{3} \\
4 & d_4 = 1 \\
& h_0 = 0 \\
& h_1 = 1 \\
& h_2 = \frac{1}{2} \\
& h_3 = \frac{1}{2} \\
& h_4 = 1
\end{array}
\]

(6.4)

From this we see that 4 is non-trivial boson, according to the conditions formulated above. It has trivial spin, and because $4 \times 4 = 0$ also the second condition is satisfied.

The next step is to work out what happens after condensation. Restricting the symmetry to $\mathcal{J} \subset \mathcal{A}$ can be noticed on the level of representations by two effects. Different representations of $\mathcal{A}$ might be isomorphic when we restrict to the subalgebra $\mathcal{J}$, which gives identifications. And representations that were irreducible under the action of $\mathcal{A}$ might have invariant subspaces under the transformations of $\mathcal{J}$ which leads...


6.1 Topological symmetry breaking

to decomposition into a direct sum of irreducible $\mathcal{I}$-modules, or splitting. These effects are fully captured by a set of branching rules

$$a \rightarrow \sum_t n^t_a t \quad \text{(restriction)} \quad (6.5)$$

that can be regarded as a linear map of the fusion algebras, which is called the restriction. Here the $t$'s are labels for the $\mathcal{I}$-particle spectrum and the $n^t_a$'s are multiplicities denoting the number of times $t$ occurs in the restriction of $a$.

There are three important, physically motivated conditions on these branching rules.

1. The condensed boson should restrict to the new vacuum (which we also denote by 0), so

$$b \rightarrow 0 + \sum_{t \neq 0} n^t_b t \quad (6.6)$$

2. The restriction should respect the fusion rules

$$\left( \sum_t n^t_a \right) \times \left( \sum_t n^t_b s \right) = \sum_r N^r_{ab} n^r_c \quad (6.7)$$

3. The restriction of particles and antiparticle should be consistent

$$\bar{a} \rightarrow \sum_t n^t_a \bar{t} \quad (6.8)$$

As a consequence of these assumptions and the uniqueness of the vacuum, 0 must restrict to the new vacuum

$$0 \rightarrow 0 \quad (6.9)$$

and the quantum dimensions are preserved

$$d_a = \sum_t n^t_a d_t \quad (6.10)$$

Using these conditions one can in general derive the particle spectrum and fusion rules of the $\mathcal{I}$ theory.

Returning back to the $\mathfrak{su}(2)_4$ example, we see that $4 \rightarrow 0$, since $d_4 = 1$ and therefore no more labels are allowed to occur in the restriction. Because $4 \times 1 = 3$ and $4 \times 3 = 1$, we immediately conclude that 3 and 1 become indistinguishable after symmetry breaking. We will denote the corresponding $\mathcal{I}$-label by 1. Finally, let us look at the fusion of 2 with itself

$$2 \times 2 = 0 + 2 + 4 \rightarrow 0 + \sum_t n^t_2 t + 0 \quad (6.11)$$

Since the vacuum appears twice on the right, the restriction of 2 must have more than one particle. Otherwise, this particle would be able to annihilate with itself in two
different ways which is inconsistent with well defined fusion. Since \( d_2 = 2 \), this is possible and in fact leads us to conclude that \( 2 \rightarrow 2_1 + 2_2 \), with \( d_2 = 1 \). Hence the broken phase has four sectors, 0, 1, 2_1 and 2_2.

The fusion rules of 1 can be straightforwardly induced. Because in the unbroken phase we had \( 1 \times 1 = 0 + 2 \), in the broken phase we have 1 × 1 = 0 + 2_1 + 2_2. This also shows that neither 2_1 nor 2_2 will be identical to the vacuum 0 in the broken phase, since this would imply the splitting of 1 which is impossible because \( d_1 < 2 \). Now turning to the fusion rules for 2_1 and 2_2, we find that they are either self-dual, 2_1 × 2_1 = 2_2 × 2_2 = 0, or are dual to each other, 2_1 × 2_2 = 2_2 × 2_1 = 0. Using the fusion rule 2 × 2 = 0 + 2 + 4 and our current knowledge, we see that

\[
0 + 2_1 + 2_2 + 0 = (2_1 + 2_2) \times (2_1 + 2_2) = 2_1 \times 2_1 + 2_1 \times 2_2 + 2_2 \times 2_1 + 2_2 \times 2_2
\]

(6.12)

So if we assume that 2_1 and 2_2 are self-dual, we find that \( 2_1 \times 2_2 = 2_1 \) and \( 2_2 \times 2_1 = 2_2 \) or \( 2_1 \times 2_2 = 2_2 \) and \( 2_2 \times 2_1 = 2_1 \). Either way, one easily concludes that associativity is violated by evaluating \( 2_1 \times (2_2 \times 2_1) \) and \( (2_1 \times 2_2) \times 2_1 \). Thus the fusion rules of the broken phase are symmetric and given by

\[
\begin{align*}
1 \times 1 &= 0 + 2_1 + 2_2 \\
1 \times 2_1 &= 1 & 2_1 \times 2_1 &= 2_2 \\
1 \times 2_2 &= 1 & 2_1 \times 2_2 &= 0 & 2_2 \times 2_2 &= 2_1
\end{align*}
\]

(6.13)

Note that the symmetry of the fusion rules is in not a requirement in his formalism. Since \( \mathcal{T} \) does not have a consistent braiding, the fusion rules can be non-commutative.

### 6.1.2 Confinement

We now turn to the confinement projection, the second step of the formalism. In order to implement this step, a characterization of the sectors of \( \mathcal{T} \) that get confined is necessary.

In order to get a theory \( \mathcal{U} \) with well defined braiding, we must be able to assign twist factors to the charges of the \( \text{cat} \mathcal{U} \)-theory. It is therefore logical to assume that, for an unconfined excitation \( u \) of the \( \mathcal{U} \)-theory, all charges of \( \mathcal{A} \) that restrict to \( u \) have the same twist factors.

Define the lift as the adjoint of the restriction.

\[
t \rightarrow \sum_a n_{an}^t a, \quad \text{(lift)}
\]

(6.14)

This lifts the charge \( t \) of \( \mathcal{T} \) into the theory \( \mathcal{A} \) as the sum of charges that have \( t \) in their restriction (including multiplicities). The confinement condition can now be formulated as follows

A charge \( t \in \mathcal{T} \) is not confined when \( \theta_a \) is the same for all \( a \) with \( n_{an}^t \neq 0 \). In that case, it belongs to \( \mathcal{U} \).
It is sometimes useful to denote lifts of $t$ as $t^i$. The condition above can then be reformulated as the statement that, for unconfined sectors $u \in \mathcal{U}$, one must have identical twist factors $\theta_{u^i}$ for all lifts $u^i$ of $u$.

Some physical requirements could be made separately on the set of unconfined particles. They should form a closed set under fusion, for instance, and they should of course contain the vacuum. The second requirement rules out condensates that do not have trivial-spin, which was to be expected. These extra requirements seem to be implied by the earlier condition, or are at least true in all worked out examples. We will come back to this when we discuss these kind of phase transitions again in relation to commutative algebra objects in braided tensor categories, exploiting the whole machinery that we have developed in previous chapters.

With the above definition of confinement in place, one may infer an interesting fact about the monodromy of the lifts of non-confined particles with the condensed excitations. Let $u, v, w$ be three unconfined particles of the broken phase, and pick lifts that we denote as $u^i, v^j, w^k$ with $w^k \in u^i \times v^j$. The monodromy of $u^i$ and $v^j$ in fusion channel $w^k$ is by the ribbon equation governed by the combination of spin factors $\theta_{u^i}/(\theta_u \theta_v)$. But since the twist factors of the lifts are the same as the twist factors of the lifted particle, we can write this as $\theta_u/(\theta_u \theta_v)$. When we apply this to $u = 0$, $v = w$, we find that the monodromy of unconfined particles with the lift of the vacuum is trivial. We could also take this latter statement as the definition. In that case, one can derive that the lifts of unconfined particles have identical spins, hence the two definitions are equivalent.

Now let us return to the $su(2)_4$ example. Applying the confinement condition, we see that the restriction 1 of 1 and 3 gets confined. The bulk excitations are therefore the vacuum $0$ and the ones we labelled $2_1$ and $2_2$. These particle labels give the $\mathcal{U}$-theory in this case. It can be shown that the fusion rules for these particles admit precisely one solution to the pentagon and hexagon equations with these spin factors giving a theory known as $Liesu(3)_1$. Usually, the labels are denoted as $1, 3, \bar{3}$ in stead of $0, 2_1, 2_2$.

<table>
<thead>
<tr>
<th>$su(2)_4$ broken</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$d_0 = 1$</td>
</tr>
<tr>
<td>1</td>
<td>$d_1 = \sqrt{3}$</td>
</tr>
<tr>
<td>$2_1$</td>
<td>$d_{2_1} = 1$</td>
</tr>
<tr>
<td>$2_2$</td>
<td>$d_{2_2} = 1$</td>
</tr>
</tbody>
</table>

**Figure 6.3: $su(2)_4$ broken** - The quantum dimensions and spins for the broken phase of $su(2)_4$. The unconfined theory is also known as $su(3)_1$.

This concludes our recap of the topological symmetry breaking formalism. As a last remark, let us point out interesting observations that were done regarding the total quantum dimension and the central charge of the theories involved. For the total
quantum dimensions, the relation
\[
\frac{D_A}{D_T} = \frac{D_T}{D_U}
\]  
was found. (This can actually be understood as a relation concerning the entropy of the different phases, as we will discuss in the next chapter). It was also found that the central charge is conserved under these kind of phase transition,
\[
c_A = c_U
\]
as makes sense. This is also nicely understood by the fact that in a CFT setting, topological symmetry breaking is dual to conformal extensions, as is discussed in [7]. The latter equation seems to limit the applicability of topological symmetry breaking to a restricted range of phase transitions. Phase transitions that change the value of \(c\) seem to lie outside the formalism.

In [?] the formalism was applied in the context of the quantum Hall effect to multilayer systems. This way, a \(c\)-changing phase transition could be accounted for, and the results were to large extend in agreement with a CFT approach [?], up to some subtleties. The multi-layer approach extends the applicability considerably.

### 6.2 Commutative algebra objects as Bose condensates

The motivation to study the mathematical literature on commutative algebras in braided tensor categories, in part, came from the fact that the formulas for the total quantum dimension central charge also appear in this context. In the next section, we will use this mathematics extend the formulation of the topological symmetry breaking scheme and augment it with the diagrammatics developed in chapter 4. A pivotal aspect in this formulation is that we identify the sectors \(t\) of \(T\) with the lift \(\sum_t n_t^q\). Hence diagrams for \(T\) and \(U\) can be interpreted as certain superpositions of diagrams of \(A\), allowing calculation using the \(F\)- and \(R\)-symbols from the \(A\)-theory.

For this tensor category approach to topological symmetry breaking, we take a commutative algebra object in a modular tensor category [?, ?] as the definition of a bosonic condensate. Algebras in modular tensor categories have appeared in relation to physics, for example as boundary conditions in boundary CFT [?], but they have really been related to symmetry breaking phase transitions and particle condensation, it seems.

We believe this is interesting for a couple of reasons. Tensor categories seem provide the appropriate language to discuss topological order, finally more concise and accurate than the language of quantum groups. This becomes apparent, for example, in the formulation of the Levin-Wen models in terms of categories and the fact that it is not straight forward to treat the \(su(2)_k\) models as \(U_q[su(2)]\) representations. Also, the treatment of topological symmetry breaking on the level of general quantum groups is likely to become cumbersome. The topological data captured by the \(F\)- and \(R\)-symbols on the other hand, is physically intuitive. Commutative algebra objects in
modular tensor categories provide a rigorous way to construct the tensor category of the symmetry broken phase. Because all diagrams live in the $\mathcal{A}$ theory, an extra benefit is that a sort of intermediate viewpoint can be attained. One can, so to say, look in between the transition from $\mathcal{A}$ to $\mathcal{T}$. Related work (in progress) by Jesper Romers shows that this might be necessary in when phase transitions in studying a lattice formulation of DGT, numerically. Because diagrammatic calculations can be done, it is also possible to answer different questions. In the next chapter, we apply the formalism below to study the faith of the topological $S$-matrix under topologically symmetry breaking phase transitions.

We would like to point out that in a completely different context, there appeared evidence that condensates in the kind of systems under consideration carry an algebra structure (in the appropriate sense). In [?] the authors study the Kitaev model with boundary and show explicitly that introduction of the boundary leads to condensation of excitations that, collectively, are endowed with the structure of an algebra object in the category of representations of the underlying quantum double symmetry group.

For the discussion below, we made heavy use of [?].

### 6.2.1 The condensate

The intuition we should bare in mind for the tensor category approach to topological symmetry breaking is that charges of the intermediate theory $\mathcal{T}$ are identified with their lift in $\mathcal{A}$. In particular, the new vacuum will be formed out of all particles that condense. This gives rise to a condensate

$$\phi = 0 + \delta_1 + \cdots + \delta_n$$

(6.17)

We will use Greek indices for the condensed excitations from now on. But for a satisfactory alternative description of the symmetry breaking scheme, we actually have to turn this logic around, since we do not want to rely an analysis along the lines of the previous section, or prior knowledge of the broken phase. In stead, we will have to guess the condensate (as we chose a boson to condense) and check that it obeys the right consistency conditions.

Therefore, fix some $\phi = 0 + \delta_1 + \cdots + \delta_n$. Which consistency conditions should be required. We want to consider $\phi$ as the new vacuum and the conditions on $\phi$ can all be related back to this requirement.

As a first condition, in line with (6.8), if $\delta \in \phi$ we should also have $\bar{\delta} \in \phi$. In other words, we require $\bar{\phi} = \phi$.

To keep the discussion transparent, we will often assume that there is only one non-trivial particle that condenses, so $\phi = 0 + \delta$. Note that in this situation, $\bar{\phi} = \phi$ means $\bar{\delta} = \delta$. As a short hand for the charge line labelled by the condensate we introduce a

---

1These should of course not be mistaken with the indices labelling vertices in chapter 4. The theory $\mathcal{A}$ is assumed to be multiplicity free.
6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

Striped line

\[ \phi = 0 + \delta \]  \hspace{1cm} (6.18)

where we can leave out orientation because \( \phi \) is self-conjugate.

Something that should be noted about charge lines labelled by a superposition of charges is that it is always possible to restrict to some subset of the charges appearing in the sum by connecting a specific charge to the outer end. This renders the coefficients of all other charges to zero. For example, connecting the vacuum line to one end of the condensate line kills the \( \delta \). For clarity, we place a dot, since the vacuum line is invisible. So we have

\[ \bullet = \left( 0 + \delta \right) = \downarrow \]  \hspace{1cm} (6.19)

Now we can conveniently express the consistency conditions for \( \phi \) using diagrammatic equations.

For \( \phi \) to be a well-defined condensate, we must be able to define a vertex

\[ \sum_{\alpha, \beta, \gamma \in \phi} M^{\alpha \beta \gamma} \]  \hspace{1cm} (6.20)

where the \( M^{\alpha \beta \gamma} \) are complex coefficients, such that

1. \[ \bullet = 0 \]  \hspace{1cm} (6.21)

2. \[ = \]  \hspace{1cm} (6.22)

3. \[ = \]  \hspace{1cm} (6.23)

Note, again, that we use Greek indices to label general charges in the condensate.

We can of course write these conditions purely in terms of the coefficients \( M^{\alpha \beta \gamma}_\gamma \) appearing in the condensate vertex. This would give

1. \[ M^{0 \beta \gamma}_\gamma = \delta_{\beta \gamma} \]  \hspace{1cm} (6.24)

2. \[ M^{\beta \alpha \gamma}_\gamma = R^{\beta \alpha}_{\gamma \gamma} M^{\alpha \beta}_\gamma \]  \hspace{1cm} (6.25)
6.2 Commutative algebra objects as Bose condensates

3. \[ \sum_{\mu \in \phi} M_\mu^\alpha M_\alpha^\beta \xi [F^\alpha_{\beta \gamma}]_{\mu \nu} = M_\mu^\alpha M_\nu^\beta \xi \] (6.26)

These expressions are obtained by expanding the diagrams and comparing coefficients in front of basis vectors in the according splitting spaces, applying \( F \)-moves first when necessary. For example, the left hand site of the third consistency condition (6.27) expands in \( \mathcal{A} \) as

\[ \sum_{\alpha, \beta, \gamma, \xi, \mu \in \phi} M_\mu^\alpha M_\alpha^\beta \xi [F^\alpha_{\beta \gamma}]_{\mu \nu} = \sum_{\alpha, \beta, \gamma, \xi, \mu, \nu \in \phi} M_\mu^\alpha M_\nu^\beta \xi [F^\alpha_{\beta \gamma}]_{\mu \nu} \] (6.27)

while the right hand site gives

\[ \sum_{\alpha, \beta, \gamma, \xi, \nu \in \phi} M_\xi^\alpha M_\nu^\beta \xi \] (6.28)

Comparing the coefficients in front of the diagrams with corresponding charges leads to equation (6.26).

Before we discuss the physical meaning of these conditions, let us simply use them to simplify the expression for the vertex when \( \phi = 0 + \delta \), i.e. in the case when there is only one condensed boson. Using (6.24) and (6.25) we see that the expression (6.20) is of the form

\[ \sum_{\alpha, \beta, \gamma, \xi, \nu \in \phi} \tilde{M}_\alpha^\beta \xi \nu \] (6.29)

We have the freedom to put \( M_\xi^{\delta \delta} = 1 \) as well. To see this, suppose we have found \( M_\gamma^{\alpha \beta} \) such that above conditions are all satisfied. Now choose arbitrary non-zero numbers \( c_\alpha \) for \( \alpha \in \phi \) with the restriction that \( c_0 = 1 \). Then define

\[ \tilde{M}_\gamma^{\alpha \beta} = \frac{c_\alpha c_\beta}{c_\gamma} M_\gamma^{\alpha \beta} \] (6.30)

It is easy to see that if the \( M_\gamma^{\alpha \beta} \) satisfy the conditions, so do the \( \tilde{M}_\gamma^{\alpha \beta} \), therefore we can switch to these just as well.\(^1\) By choosing \( c_\delta = (M_0^{\delta \delta})^{-1/2} \) we get \( M_0^{\delta \delta} = 1 \).

So the one non-trivial coefficient we should find is \( M \equiv M_0^{\delta \delta} \). By using the third relation, we can express this in terms of the \( F \)-symbols by

\[ M^2 = \frac{-\delta_{c \xi} - [F_\delta^{\xi \delta \delta}]_{c \xi}}{\delta_{c \xi} - [F_\delta^{\xi \delta \delta}]_{c \xi}}, \quad \forall c \in \mathcal{A} \] (6.31)

\(^1\)Mathematically speaking, the \( c_\alpha \) define an isomorphism of (co)algebras.
6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

Here $\delta_{xy}$ is the Kronecker $\delta$-function that tests if the arguments are equal, while the other $\delta$’s denote the condensed particle. This equation is most easily derived by attaching $\delta$ on the outer lines of (6.23). Then this equation becomes

$$\delta \delta \delta \delta \delta \delta \delta \delta \delta \delta = \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta + M^2 \delta \delta \delta \delta \delta \delta \delta \delta \delta \delta$$

(6.32)

Applying the $F$-move on the left, and solving for $M^2$ we find (6.31). This equation actually puts a strong constrained on the particle $\delta$ since it must hold for all charges $c$. But this is not the only constrained on $\delta$.

Let us turn to the physical interpretation of the conditions on the condensate. Remember that we are looking to describe phase transitions induced by a bosonic condensate. The second condition, involving the braiding, is most clearly connected to this physical statement. It gives a more intricate condition on the particles in the condensate that just having trivial spin, but it makes sense that a stable condensate should have a state invariant under braiding. This condition therefore seems logical.

Let us take a closer look at the trivial braiding condition for condensate lines and what it implies for monodromy. By virtue of (6.22) we have

$$\theta_{\alpha} \theta_{\alpha} = 1$$

(6.33)

Expanding both sides and using the monodromy equation this gives

$$\frac{\theta_{\gamma}}{\theta_{\alpha} \theta_{\beta}} = 1$$

(6.34)

for all $\alpha, \beta, \gamma \in \phi$ that give a vertex allowed by fusion. By taking $\gamma = 0$, we see that $\theta_{\alpha} \theta_{\alpha} = \theta_{\alpha}^2 = 1$ which leaves $\theta_{\alpha} = \pm 1$ as the only possible solution for any $\alpha \in \phi$. Of course, $\theta_{\alpha} = 1$, is precisely what we expect from a bosonic condensate. Note that when we start with some $\alpha$ with $\theta_{\alpha} = 1$ that we require to condense, this condition generalizes the condition for a boson given before. Not only should there be a fusion channel $\gamma \in \alpha \times \alpha$ with trivial spin, this fusion channel should be part of the condensate.

The consistency conditions give here make sure that for arbitrary particle number $n$, there is a fusion channel in $\alpha^n$ that is invariant under monodromy (and even braiding).

The exotic case, where some $\alpha \in \phi$ has $\theta_{\alpha} = -1$, is truly rare. When $\alpha \in \alpha \times \alpha$, the monodromy equation gives $\theta_{\alpha} / (\theta_{\alpha} \theta_{\alpha}) = 1$, such that $\theta_{\alpha} = -1$ is impossible. Furthermore, equation (6.22) implies that $R_{0}^{\alpha \alpha} = \theta_{\alpha}^{*} \text{fs}_{\alpha}^{*} = 1$, so this can only happen for self-dual $\alpha$ with Frobenius-Schur indicator equal to $-1$. We see that it possibly gives rise to fermionic condensates which might be interesting, but we have investigated this case yet. We will exclude this exotic case from the discussion for convenience.
6.2 Commutative algebra objects as Bose condensates

The logic of the argument shows that when $\theta_\alpha = 1$ for all $\alpha$ in the condensate, also the Frobenius-Schur indicators equal unity. This means that conveniently, we do not have to keep track of the flags introduced earlier when working with the condensate.

The physical necessity of the other two consistency conditions is in a sense less clear, but have a clear interpretation when we agree that we want the condensate to be like the vacuum of a new theory. The third condition (6.21) says that we can freely reconnect the condensate lines; a sensible property for vacuum lines and in fact necessary when we look at the axioms of tensor categories. The first condition gives a rule to remove outer legs labelled by the condensate from diagrams by attaching a dot, which gives a way to pass from a tensor product state of two copies of the vacuum to a single copy, also a necessary condition for the vacuum to obey.

For concreteness, let us look again at the $\mathfrak{su}(2)_4$ example. From the earlier discussion, we expect that the condensate $\phi = 0 + 4$ should fit the new scheme. Since $4 \notin 4 \times 4$, the vertex simply becomes

\[
\begin{array}{c}
\text{Condensate vertex}
\end{array}
\]

The Frobenius-Schur indicator for general $\mathfrak{su}(2)_k$ theories is $+1$ for particles with even labels and $-1$ for particles with odd labels (corresponding to bosonic and fermionic representations for $\mathfrak{su}(2)$ in the non-$q$-deformed case). Hence, $R_{44}^{04} = \mathfrak{fs}_{4\theta_4 = 1}$ and (6.21) is satisfied. Condition (6.21) and (6.21) are also trivially satisfied in this case. So the condensate $\phi = 0 + 4$ indeed fits the scheme.

An interesting observation is that the rules for the condensate allow us to construct a family of states for arbitrary particle number with remarkable invariance properties. In fact, if we allow superposition of anyonic charges, we can conveniently write down a state formed by stacking condensate vertices together, for example

\[
\begin{array}{c}
\text{Stacked condensate vertices}
\end{array}
\]

This expands in $\mathcal{A}$ as a sum of states with mixed particle number. By connecting the outer legs to lines labelled by single charges from the condensate we can project to a state with fixed particle number and fixed charge on the outer legs. By using equations (6.22) and (6.23) we can show that is invariant under the full braid operator

\[
\begin{array}{c}
\text{Braid operator acting on stacked condensate vertices}
\end{array}
\]

This is a hint that we indeed have the right definition of boson in condensate. Note that the most general conditions discussed in [7] are satisfied if this definition holds in a particularly elegant way, and we have a constructive way to build states that are fully invariant under braiding and not just monodromy.
The invariance properties of states of the form (6.36) suggest that they might be connected to the ground state of the broken phase. In lattice models with a finite size lattice, this state can be concretely realized by operators acting on the vacuum. In the Kitaev model with a boundary for example, an important role is played by so called ribbon operators and (6.36) can essentially be interpreted as an expression in terms of these operators. It would be interesting to investigate the ground state of the symmetry broken phase in these kind of lattice models and see if they relate to the diagram in (6.36). Examples could be the Kitaev model with boundary [?] where the boundary conditions implement an explicit symmetry breaking mechanism, or lattice formulations of discrete gauge theories as in for example [?]. In any case, the conditions 1–3 have interesting physical consequences and interpretations. Let us proceed with the construction of a consistent tensor category based on these. The next step is the construction of the particle spectrum of the $\mathcal{T}$-theory.

### 6.2.2 The particle spectrum of $\mathcal{T}$

Now that we have defined the vacuum of the broken theory, i.e. the condensate $\phi$, it is time to introduce particles. In order to create a diagrammatic formalism, we will not only need to know the particle labels that occur in the sum, but also how to construct diagrams. A minimal requirement is then, that we define how the particles couple to the vacuum. This leads to the definition of a vertex obeying certain consistency conditions, much like for the condensate.

Charges of $\mathcal{T}$ will be denoted with labels $t, s, r, \ldots$, and we write

\[
\frac{t}{a} = \sum_{a \in \ell} N_{ab}^c \frac{s}{c}
\]  

(6.38)

Now, how do we find what $a \in \ell$, i.e. which particles $a$ restrict to $t$, if we do not want to repeat the analysis along the lines of the previous section? To give an idea, we will momentarily switch back to the notation of the previous section. Recall the symmetries of the fusion multiplicities

\[
N_{ab}^c = N_{ba}^c = N_{bc}^a = N_{ac}^b
\]  

(6.39)

Now note that condition 2 and 3 on the branching rules, i.e. equation (6.7) and (3), can be summarized by the equalities

\[
\sum_{t,s} n_a^t n_b^s N_{ts}^r = \sum_{t,s} N_{ab}^c n_c^r
\]  

(6.40)

and

\[
n_a^t = n_a^r
\]  

(6.41)

Using these, we find

\[
\sum_b N_{ab}^c n_b^0 = \sum_{t,s} n_a^t n_c^s N_{t0}^s
\]  

(6.42)

\[
= \sum_t n_a^t n_c^t
\]  

(6.43)
Writing the condensate as the lift of the vacuum, \( \phi = \sum_b n_b^0 \), this gives

\[
\phi \times a = \sum_{b,c} n_b^0 N_{bc}^a = \sum_t n_t^a n_t^c \quad (6.44)
\]

In words, we see that \( \phi \) falls apart as the sum of lifts of the charges \( t \) in the restriction of \( a \). Identifying \( t \) with it’s lift, we might say that we find all \( t \) in the restriction of \( a \) as parts of \( \phi \times a \). In fact, a lemma 3.4 in [?] assures that the particles of the broken phase can all be found this way, in the tensor category approach to topological symmetry breaking.

In the \( \mathfrak{su}(2)_4 \) example, we indeed find back the particle spectrum of \( \mathcal{J} \) in a particularly quick way. With \( \phi = 0 + 4 \) the fusion rules (6.3) give

\[
\begin{align*}
\phi \times 0 &= 0 + 4 \\
\phi \times 1 &= 1 + 3 \\
\phi \times 2 &= 2 + 2 \\
\phi \times 3 &= 1 + 3 \\
\phi \times 4 &= 0 + 4
\end{align*}
\]

This shows 4 gets identified with the vacuum, 1 and 3 get identified with each other in the broken phase, in agreement with the earlier discussion. It also shows that the particle 2 will split into two components as we found before.

One more important observation we can make, is that the multiplicity of \( a \) in the lift of \( t \) is equal to the multiplicity of \( t \) in the restriction of \( a \), since they are both given by \( n_t^a \).\(^1\) Multiplicities greater \( n_t^a > 1 \) do not seem to appear, although they might in exotic examples. To keep the discussion streamlined, we will assume that they do not occur, such that \( a \in t \) occurs at most ones. This implies that splitting can be seen directly from \( \phi \times a \) as the occurrence of multiple copies of \( a \). For \( \phi = 0 + \delta \), we see that \( a \) splits iff \( a \in \delta \times a \).

Now, we will look at the coupling of the \( \mathcal{J} \) particles with the condensate, for which we need to define the vertex

\[
\bar{\omega} \gamma_t^t = \sum_{\alpha \in \phi} \sum_{a,b \in t} \theta_b \bar{A}_a^{\alpha a} \gamma_b^a \quad (6.51)
\]

The coefficients \( \theta_b \bar{A}_a^{\alpha a} \) must be such that some consistency conditions are satisfied, namely

1. \[
\bar{\omega} \gamma_t^t = \gamma_t^t \quad (6.52)
\]

\(^1\)In mathematical language: The corresponding functors are each others adjoint (left and right). This is morally the same as Frobenius reciprocity for group representations.
This gives the following equations

1. \[ \theta^{\alpha a}_{\beta b} = \delta_{ab}, \quad \forall a, b \in s \] (6.54)

2. \[ \sum_c M_{\alpha \beta}^{\gamma a} \theta^{\alpha a}_{\gamma c} [F^{\gamma \beta a}_b] = \theta^{\alpha a}_{\beta b} \theta^{\alpha a}_{\gamma c} \] (6.55)

Formally, we might view a charge \( t \) as the combination of a sum of fields of \( A \) together with the set of coefficients

\[ t = (a_1 + \cdots + a_n, \theta^{\alpha a}_{\beta b}) \] (6.56)

Indeed, if either one is different for some \( s \) and \( t \), we must regard \( s \) and \( t \) to be different particles of \( T \). This means that some particles of \( T \) can consist of the same sum of \( A \)-charge but only differ in the coefficients \( \theta^{\alpha a}_{\beta b} / \theta^{\alpha a'}_{\beta b'} \), corresponding to different \( T \)-sectors with identical lifts. This raises a subtle issue. Because there is a certain freedom in the choice of \( \theta^{\alpha a}_{\beta b} \) and a notion of isomorphism of charges of \( T \), when do we know that \( T \) charges are different or the same. We will come back to this.

Let us work out the case \( \phi = 0 + \delta \). The first conditions (6.52) says that only the \( \theta^{\delta a}_{\beta b} \) can be non-trivial, so we leave out the index \( \delta \) in this case. The vertex then becomes

\[ \sum_{a, b \in t} \delta_{ab} \theta^{\alpha a}_{\beta b} + \theta^{\alpha a}_{\beta b} \theta^{\alpha a}_{\beta b} \] (6.57)

The second condition gives

\[ \delta_{ab} [F^{\delta a}_a]_{0c} + M \theta^{\alpha a}_{\beta b} [F^{\delta a}_b]_{\delta c} = \theta^{\alpha a}_{\beta b} \theta^{\alpha a}_{\gamma c} \] (6.58)

From here, we can deduce some useful expressions to facilitate the calculation of the \( \theta^{\alpha a}_{\beta b} \) in concrete examples.

- By choosing \( a \neq b \) and \( a = c \) we find
  \[ \theta^{\alpha a}_{\beta b} = M [F^{\delta a}_a]_{\delta a}, \quad a \neq b, a, b \in t \] (6.59)

- By choosing \( a = b = c \) we get the quadratic equation
  \[ (\theta^{\alpha a}_{\beta b})^2 = [F^{\delta a}_a]_{0a} + M \theta^{\alpha a}_{\beta b} [F^{\delta a}_a]_{\delta a} \] (6.60)

  which has solution
  \[ \theta^{\alpha a}_{\beta b} = \frac{1}{2} \left( M [F^{\delta a}_a]_{\delta a} \pm \sqrt{M^2 [F^{\delta a}_a]_{\delta a}^2 + 4 [F^{\delta a}_a]_{0a}} \right) \] (6.61)
6.2 Commutative algebra objects as Bose condensates

For the $^{\alpha}A_b^a$ with $a \neq b$ we will have to solve the general equation (6.58).

We need to make an important remark about the freedom in the choice of the $^{\alpha}A_b^a$. We can pick non-zero numbers $c_a$ for $a \in t$ and change to

$$^{\alpha}A_b^a = \frac{c_a}{c_b}^{\alpha}A_b^a$$  \hspace{1cm} (6.62)

If the coefficients differ by this relation, the two structures must be thought of as defining the same particle.\footnote{Mathematically, this change of coefficients constitutes an isomorphism of modules over the algebra defined here as the condensate. It might be interesting to think about the meaning of the coefficients $^{\alpha}A_b^a$ and how to explain the freedom, etc. The remarks here are just a translation of the mathematics.} Note that the $^{\alpha}A_b^a$ cannot be changed by this relation and are gauge invariant, so to say.

It can occur that there are multiple solutions to the equations for the $^{\alpha}A_b^a$ that are not related by a transformation of the form (6.62). In this case, one particle of the original theory splits into two particles in the broken theory, carrying one of the two possible structures each. This happens, for example for the two charges in the restriction of 2 in the $\text{su}(2)_4$ example.

Let us discuss the definitions of the vertices for the particles of the broken theory of $\text{su}(2)_4$. The confined particle, that we denoted before by 1, is now treated as the sum $1 + 3$. Since $4 \times 1 = 3$ and $4 \times 3 = 1$ the coefficients we need to find are $^{0+3}A_1^1$ and $^{0+3}A_1^3$. From equation (6.58) we find that

$$^{0+3}A_3^{1}A_1^3 = [F_{1}^{441}]_{03} = -1$$  \hspace{1cm} (6.63)

Taking the freedom from equation (6.62) into account, we see that we can put the vertex

$$\begin{align*}
(1+3) & = 0 \begin{array}{c} 1 \\
1+3 & \end{array} + 0 \begin{array}{c} 3 \\
1 & \end{array} + 4 \begin{array}{c} 1 \\
3 & \end{array} - 4 \begin{array}{c} 3 \\
1 & \end{array}
\end{align*}$$  \hspace{1cm} (6.64)

Next, we look at the label 2. According to $\phi \times 2 = 2 + 2$ and our earlier considerations, 2 should give rise to two $\tilde{T}$-particles. Equation 6.61 reduces to

$$\left(^{2}A_2^2\right)^2 = \sqrt{[F_{2}^{442}]_{02}} = 1$$  \hspace{1cm} (6.65)

There are indeed two solutions $^{2}A_2^2 = \pm 1$ and we will denote the corresponding particles by $2_\pm$, so $^{2}A_2^2 = +1$ and $^{2}A_2^2 = -1$. The corresponding vertices are thus

$$\begin{align*}
2_+ & = 0 \begin{array}{c} 2 \\
2 & \end{array} + 4 \begin{array}{c} 2 \\
2 & \end{array}
\end{align*}$$  \hspace{1cm} (6.66)

and

$$\begin{align*}
2_- & = 0 \begin{array}{c} 2 \\
2 & \end{array} - 4 \begin{array}{c} 2 \\
2 & \end{array}
\end{align*}$$  \hspace{1cm} (6.67)
6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

It is now easy to verify the conditions (6.53) and (6.53) when we plug in these definitions of the vertices using the $F$-moves for the $\mathfrak{su}(2)_4$-theory.

As occurred above, some combinations of charges do not occur for the $\theta_{t_A}^{\alpha a}$, namely when $N^b_{\alpha a} = 0$ and the vertex is not allowed by fusion. In this case we put $\theta_{t_A}^{\alpha a} = 0$ by convention. Just as well, we can take $\theta_{t_B}^{\alpha a} = 0$ when $a, b \notin t$. However, when $\alpha \in \phi$, $a, b \in t$ and the vertex is allowed by fusion, $\theta_{t_B}^{\alpha a}$ will always be non-zero for a particle $t$. This is a kind of irreducibility condition, distinguishing true particles $t$ from a superposition of particles $t_1 + \ldots + t_n$, for which solutions to the consistency conditions may exist for which zero coefficients do occur. These basically correspond to the projection onto a subset of the particles that occur in the superposition.

Above, we mentioned that the particles of the broken phase can all be found in the fusion products $\phi \times a$. Let us give a graphical illustration of why the defining conditions have to hold for $\phi \times a$, hence either this corresponds to a particle of the broken theory or is a superposition of particles of the broken theory. The charge line labelled $\phi \times a$ can also be written as two parallel lines, one labelled $\phi$ the other $a$. Hence we have

$$\phi \times a = a$$ (6.68)

We can use the condensate vertex to define a vertex for $\phi \times a$.

$$\phi \times a = a$$ (6.69)

The conditions on the condensate now give the requirements stated above.

$$a = a$$ and $$a = a$$ (6.70)

At this point we have constructed a description of all particles of the new theory, which includes the definition of certain vertices involving the new vacuum. Note that we have not derived the fusion rules, nor did we need to. They can be inferred by the methods of section 6.1, but it would be nice to derive them in a more straightforward manner in the present context or even have a closed formula for the $N_{t_B}$ in terms of the $N_{c_{ab}}$.

The issue is subtle, because the definition of the tensor product in the category $\mathcal{T}$ is non-trivial. This deserves more attention, but we have no results on the subject yet.

6.2.3 Derived vertices

The definition of the vertices as in (6.20) and (6.51) allow us to derive several vertices and useful relations. The condensate vertex with two legs down is defined as

$$= \sum_{\alpha, \beta, \gamma \in \phi} (M^{\alpha \beta \gamma})$$
6.3 Confinement

6.2 Commutative algebra objects as Bose condensates — i.e.  
\[
\left( \begin{array}{c}
\gamma \\
\phi
\end{array} \right) \dagger
\]  
(6.72)

This definition ensures that  
\[
\left( \begin{array}{c}
\gamma \\
\phi
\end{array} \right) \dagger = \left( \begin{array}{c}
\gamma \\
\phi
\end{array} \right)
\]  
(6.73)

since this is just the adjoint relation of (6.23). Similarly, the adjoint version of the other conditions hold.

6.3 Confinement

The confinement in the present context is in fact the same as it was in the earlier discussion. Recall that unconfined particles were characterized by the condition that all charges in the lift have identical twist factors, or equivalently, that the lift has trivial monodromy with the lift of the vacuum, i.e. the condensate. The second condition, which is in fact slightly more general as it can be used for condensates with \( \theta_\alpha \neq 1 \), is nicely expressed diagrammatically as  
\[
\begin{array}{c}
\begin{array}{c}
\gamma \\
\phi
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\gamma \\
\phi
\end{array}
\end{array}
\]
(6.74)

Using the monodromy equation, we find that this is equivalent to  
\[
\theta_a / (\theta_a \theta_b) = 1, \quad \forall a, b \in t, \alpha \in \phi
\]  
(6.75)

Because \( \theta_\alpha = 1 \) for all particles \( \alpha \) of the condensate, we indeed recover the condition that  
\[
\theta_a = \theta_b, \quad \forall a, b \in t
\]  
(6.76)

which is precisely the statement that all lifts have equal topological spin.

Up until now, the extension of the symmetry breaking formalism to include the definitions of certain diagrams has not provided new information. We can recovered the particle spectrum of the broken phase, but this was also possible an analysis along the lines of the previous chapter. Now, however, we will actually use the diagrammatic formalism to define an operator that is able to project out confined particles from the spectrum.

Using the definitions of the vertices, we can write down the operator  
\[
P_t \equiv \frac{1}{q} \begin{array}{c}
\begin{array}{c}
\gamma \\
\phi
\end{array}
\end{array}
\]  
(6.77)
where \( q = d_\phi \), the quantum dimension of the condensate. By stacking two of the diagrams on top of each other one can prove that \( P_t \) is a projector, i.e. \( P_t \circ P_t = P_t \). The sequence of topological manipulations of the diagram that is needed to show this is illustrated in figure 6.4.

\[
\begin{align*}
\text{Figure 6.4: Confinement projector - The figure shows a series of diagrammatic manipulations that can be used to show that } & P_t \circ P_t = P_t. \text{ The left hand side is equal to } q^2 P_t \circ P_t. \text{ On the right hand side, the ‘inner product’ of the condensate vertex with itself leads to a factor } d_\phi = q \text{ when eliminated. Hence, the right hand side is equal to } q^2 P_t. \\
\end{align*}
\]

Now let us look at \( P_u \) for an unconfined charge \( u \). Using equations (6.53) and (6.74), we get for unconfined charges \( u \) that

\[
\begin{align*}
\frac{1}{q} \left< u \right| u \rangle & = \frac{1}{q} \left< u \right| u \rangle = \frac{1}{q} \left< u \right| u \rangle = \frac{1}{q} \left< u \right| u \rangle \\
\text{which is simply the identity operator } & \\
\end{align*}
\]

\[
\begin{align*}
\left(6.79\right)
\end{align*}
\]

The fact that the projector evaluates to zero on confined particles \( t \) is a bit subtle. Imagine that we have some operator \( X \) that has the property that for particles \( t \in T \) we have

\[
\begin{align*}
X & = \\
\left(6.80\right)
\end{align*}
\]

Since \( X \) is a single anyon operator, we know that on anyons of \( A \), it acts by multiplication by a complex number

\[
\begin{align*}
X & = \sum_a X_a a \\
\left(6.81\right)
\end{align*}
\]
Now, by expanding both sides of (6.3) as diagrams in $\mathcal{A}$, we find that

$$A^a_b X_a = A^a_b X_b, \quad \forall a, b \in t$$

(6.82)

This implies $X_a = X_b$ for all $a, b \in s$, unless $A^a_b = 0$ for all $\alpha \in \phi$ for some $a$ and $b$. The latter case can actually not occur, since this would mean that $a$ and $b$ do not occur in the same particle $t$. Or in other words, what we thought was a particle $t$ is actually not a single particle of $\mathcal{T}$ but splits as a sum of particles.

Consider now a situation, where we apply the projector $P_t$ on a $t$ in the broken phase, followed by a clock-wise $2\pi$ rotation of the system. The corresponding operator is

$$t_{\uparrow \uparrow} = \sum_{a \in t} q_{\theta_a}$$

(6.83)

where the equality is shown by applying (6.53) and the monodromy equation, and using that the twist of $\phi$ is trivial). From the presentation on the right of equation (6.83) we see that this operator obeys ($\cdot$). On the other hand,

$$t_{\uparrow \uparrow} = \sum_{a \in t} q_{\theta_a}$$

(6.84)

So

### 6.4 Breaking $\text{su}(2)_{10}$

The $\text{su}(2)_{10}$ theory is an interesting example to consider. It has a bosonic charge that can condense with more exceptional properties than the $\text{su}(2)_{4}$ theory. The quantum
6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

Dimensions and topological spins of the charges of $\mathfrak{su}(2)_{10}$ are

<table>
<thead>
<tr>
<th>$\mathfrak{su}(2)_{10}$</th>
<th>$d_0 = 1$</th>
<th>$h_0 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1 = \sqrt{1 + \sqrt{3}}$</td>
<td>$h_1 = \frac{1}{16}$</td>
<td></td>
</tr>
<tr>
<td>$d_2 = 1 + \sqrt{3}$</td>
<td>$h_2 = \frac{1}{6}$</td>
<td></td>
</tr>
<tr>
<td>$d_3 = \sqrt{2 + \sqrt{2 + \sqrt{3}}}$</td>
<td>$h_3 = \frac{5}{16}$</td>
<td></td>
</tr>
<tr>
<td>$d_4 = 2 + \sqrt{3}$</td>
<td>$h_4 = \frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>$d_5 = 2\sqrt{2 + \sqrt{3}}$</td>
<td>$h_5 = \frac{35}{48}$</td>
<td></td>
</tr>
<tr>
<td>$d_6 = 2 + \sqrt{3}$</td>
<td>$h_6 = 1$</td>
<td></td>
</tr>
<tr>
<td>$d_7 = \sqrt{2 + \sqrt{2 + \sqrt{3}}}$</td>
<td>$h_7 = \frac{21}{16}$</td>
<td></td>
</tr>
<tr>
<td>$d_8 = 1 + \sqrt{3}$</td>
<td>$h_8 = \frac{5}{3}$</td>
<td></td>
</tr>
<tr>
<td>$d_9 = \sqrt{2 + \sqrt{3}}$</td>
<td>$h_9 = \frac{33}{16}$</td>
<td></td>
</tr>
<tr>
<td>$d_{10} = 1$</td>
<td>$h_{10} = \frac{1}{2}$</td>
<td></td>
</tr>
</tbody>
</table>

The particle label 6 has trivial twist so it makes sense to try the condensate $\phi = 0 + 6$.

Fusion following the general rule (4.1) gives

\[
\begin{align*}
\phi \times 0 &= 0 + 6 = (0 + 6) \\
\phi \times 1 &= 1 + 5 + 7 = (1 + 5 + 7) \\
\phi \times 2 &= 2 + 4 + 6 + 8 = (2 + 4 + 6 + 8) \\
\phi \times 3 &= 3 + 3 + 5 + 7 + 9 = (3 + 5 + 9) + (3 + 7) \\
\phi \times 4 &= 2 + 4 + 4 + 6 + 8 + 10 = (2 + 4 + 6 + 8) + (4 + 10) \\
\phi \times 5 &= 1 + 3 + 5 + 5 + 7 + 9 = (1 + 5 + 7) + (3 + 5 + 9) \\
\phi \times 6 &= 0 + 2 + 4 + 6 + 6 + 8 = (0 + 6) + (2 + 4 + 6 + 8) \\
\phi \times 7 &= 1 + 3 + 5 + 7 + 7 = (1 + 5 + 7) + (3 + 7) \\
\phi \times 8 &= 2 + 4 + 6 + 8 = (2 + 4 + 6 + 8) \\
\phi \times 9 &= 3 + 5 + 9 = (3 + 5 + 9) \\
\phi \times 10 &= 4 + 10 = (4 + 10)
\end{align*}
\]

On the outer right hand side we grouped the labels in the unique way, such that they nicely split up all the fusion products. These are the sums corresponding to the particles of $\mathcal{J}$. So, for example, we see that 8 and 2 get identified in the broken phase. The fields that split are 3, 4, 5, 6 and 7. An interesting feature is that the condensed particle 6 splits.

The equation for $M$ is

\[
M^2 = -\frac{\delta_{0c} - [F_{666}^{[6]}]_{0c}}{\delta_{6c} - [F_{666}^{[6]}]_{6c}} = -\sqrt{2} \quad (6.85)
\]

which is indeed true for all charges $c$. We choose one root and define the condensate vertex as

\[
\chi \left[ \begin{array}{c}
0 \\
0 \\
0 \\
6 \\
6 \\
0 \\
6 \\
6 \\
6 \\
6
\end{array} \right] = 0 + 6 + 6 + 6 + 6 + 6 + 6 + 2^{1/4}i 6 + 6 \quad (6.86)
\]
Using the $F$-symbols, one can check that equation (6.23) is satisfied.

We will not try to obtain all the $\check{\mathbf{1}}^a$ for this theory. But in later calculations we will need the $\check{\mathbf{1}}^a(a)$. These are most easily obtained by equation (6.59). To calculate $\check{\mathbf{1}}^a(a)$ we just choose a $b \in t$ that is not equal to $a$, something that is always possible in this case, and apply

$$\check{\mathbf{1}}^a(a) = 2^{1/4}_i [F^6]^{6a}_{6a}$$

This way we obtain the following table

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\check{\mathbf{1}}^a(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 + 5 + 7)$</td>
<td>$\check{\mathbf{1}}^1_1 = 0$</td>
</tr>
<tr>
<td></td>
<td>$\check{\mathbf{1}}^7_7 = 2^{1/4}_i \cdot \sqrt{2 - \sqrt{3}}$</td>
</tr>
<tr>
<td>$(2 + 4 + 6 + 8)$</td>
<td>$\check{\mathbf{1}}^2_2 = 0$</td>
</tr>
<tr>
<td></td>
<td>$\check{\mathbf{1}}^6_6 = 2^{1/4}_i \cdot \frac{1}{2}(1 - \sqrt{3})$</td>
</tr>
<tr>
<td>$(3 + 5 + 9)$</td>
<td>$\check{\mathbf{1}}^3_3 = -2^{1/4}_i \cdot \sqrt{2 - \sqrt{3}}$</td>
</tr>
<tr>
<td></td>
<td>$\check{\mathbf{1}}^9_9 = 0$</td>
</tr>
<tr>
<td>$(3 + 7)$</td>
<td>$\check{\mathbf{1}}^3_3 = 2^{1/4}_i \cdot \frac{1}{2}\sqrt{2}$</td>
</tr>
<tr>
<td>$(4 + 10)$</td>
<td>$\check{\mathbf{1}}^4_4 = -2^{1/4}_i \cdot \frac{1}{2}\sqrt{2}$</td>
</tr>
</tbody>
</table>
6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES
Indicators for topological order

Because topologically ordered phases defy the Landau symmetry breaking scheme of characterization by local order parameters, other indicators not based on symmetry have appeared in the literature. These are ground state degeneracy on surfaces of non-zero genus \([? , ?]\), fractional of quantum numbers (charge \([23]\), spin \([?]\)), gapless edge excitations \([?]\), and topological entanglement entropy \([? , 28]\). It is clearly of interest to know the behaviour of these indicators under a topological symmetry breaking phase transition. The topological entanglement entropy, that has been related to the total quantum dimension, shows an interesting relation with the quantum dimension of the condensate. This condensate quantum dimension \(q = d_\phi\) seems an interesting number characterizing the phase transition. It can be computed as the ratio between the quantum dimensions of the lift of an arbitrary sector \(t\) of a the broken phase and the quantum dimension of \(t\) itself. One can show that the increase in topological entanglement entropy in each step of the phase transition is \(\log \sqrt{q}\) for a disc-like geometry.

Another interesting problem is to construct ‘topological order parameters’, operators that can be used to ‘measurement’ the kind of topological order and signal phase transitions, in simulations or experiments. The topological \(S\)-matrix is a particularly interesting object. Through interferometry measurements it might become possible soon to obtain matrix elements of the topological \(S\)-matrix in, for example, FQH states. On the other hand, it decodes a lot of information about the underlying theory. In this chapter, we will also show how to obtain the topological \(S\)-matrix using the tensor category approach to topological symmetry breaking.
7. INDICATORS FOR TOPOLOGICAL ORDER

7.1 Topological entanglement entropy

For a quantum system described by a density matrix $\rho$, the von Neumann entropy $S_E$ is defined as

$$S_E = -\text{Tr} \rho \log \rho$$  \hspace{1cm} (7.1)$$

For a gapped system in a planar region with boundary of length $L$, it takes the form

$$S_E = \alpha L - \gamma + \ldots$$  \hspace{1cm} (7.2)$$

The ellipses represent terms that vanish in the limit $L \to \infty$. The universal additive constant $-\gamma$ characterizes global features of the the entanglement in the ground state and has been called the topological entanglement entropy. It signals non-trivial topological order and in [?, 28] it was shown that for a disc-like geometry it relates to the total quantum dimension of the theory underlying the topological order as

$$\gamma = \log D$$  \hspace{1cm} (7.3)$$

In a topological symmetry breaking phase transition $\mathcal{A} \to \mathcal{T} \to \mathcal{U}$, the relation between the total quantum dimensions $D_A, D_T$ and $D_U$ therefore relates the topological entropy of the different phases. In particular the topological entanglement entropy increases by an amount

$$\gamma_A - \gamma_U = \log \frac{D_A}{D_U}$$  \hspace{1cm} (7.4)$$

We will show that this ration is directly related to $q = d_\phi$, the quantum dimension of the condensate. This is in fact equal to

$$q = \sum_a \frac{n^t_a a}{d_t}$$  \hspace{1cm} (7.5)$$

for any $t$ in the $\mathcal{T}$-theory.

We will prove that the right hand side of equation (7.5) in indeed not depending on the choice of $t$. But let us first make a remark on notational conventions and assumptions for this section. Throughout this section, we use the notation and assumptions and terminology of the original approach to topological quantum group symmetry breaking. In the literature on commutative algebras in tensor categories the same result is present (see [?]). But it is interesting that one can derive this from less involved assumptions.

Now, to show (7.5), we will just need the description of the theories on the level of the fusion algebras which we will also denote with $\mathcal{A}, \mathcal{T}$ and $\mathcal{U}$. The fusion product is written $|a\rangle \times |b\rangle = \sum_c N_{ab}^c |c\rangle$ for $\mathcal{A}$. This defines the fusion matrices $N_a$ by $(N_a)^b_c = N_{ab}^c$. For $\mathcal{T}$ we use the same notations, but with $\{a, b, c\} \to \{t, r, s\}$. Recall that the vector

$$|\omega_A\rangle = \sum_a d_a |a\rangle$$  \hspace{1cm} (7.6)$$
7.1 Topological entanglement entropy

is a common Perron-Frobenius eigenvector for the fusion matrices $N_a$ with eigenvalue equations

$$N_a \ket{\omega_A} = d_a \ket{\omega_A}$$  \hspace{1cm} (7.7)

where $d_a$ is the quantum dimension of the charge $a$. This can be checked explicitly from the relation

$$d_ad_b = N_{ab}d_c$$  \hspace{1cm} (7.8)

Similarly,

$$\ket{\omega_T} = \sum_t d_t \ket{t}$$  \hspace{1cm} (7.9)

is a common Perron-Frobenius eigenvector for the fusion matrices $N_t$, even though these do not necessarily commute.

The phase transition is described by the restriction map, as discussed in the previous chapter. This is conveniently written as $B: A \rightarrow T$. The lift is regarded as the transpose map $B^\dagger: T \rightarrow A$. Written out explicitly, we have

$$B \ket{a} = \sum_t n_t^a \ket{t}, \quad B^\dagger \ket{t} = \sum_a n_a^t \ket{a}$$  \hspace{1cm} (7.10)

The restriction is required to commute with fusion. This can be de written as $B \ket{a} \times B \ket{b} = B(\ket{a} \times \ket{b})$ or more explicitly

$$n_{\bar{a}}^t = n_a^t, \quad \sum_{t,s} n_t^a n_b^s N_{ts} = \sum_c N_{ab}^c n_c^r$$  \hspace{1cm} (7.11)

We will study the behaviour of the Perron-Frobenius eigenvectors $\ket{\omega_A}$ and $\ket{\omega_T}$ under the restriction and lift. We have

$$B^\dagger \ket{\omega_T} = \sum_{a,t} n_{\bar{a}}^t d_t \ket{a}$$  \hspace{1cm} (7.12)

$$= \sum_a d_a \ket{a}$$  \hspace{1cm} (7.13)

$$= \ket{\omega_T}$$  \hspace{1cm} (7.14)

This holds, because quantum dimensions are conserved under the restriction.

We will now show that

$$B \ket{\omega_A} = q \ket{\omega_T}$$  \hspace{1cm} (7.15)

Let us define the map $M: T \rightarrow T$ by

$$M \equiv \sum_a M_a = \sum_{a,t} n_{\bar{a}}^t N_t$$  \hspace{1cm} (7.16)

i.e. $M_a$ is the fusion matrix of the restriction of $a$ and $M$ is obtained by taking the sum over all charges $a$ of $catA$. The matrix elements of $M$ are all strictly positive, since for any $s,r \in T$ there is some $t \in T$ with $N_{ts}^r > 0$ and also any $t \in T$ occurs in the
7. INDICATORS FOR TOPOLOGICAL ORDER

decomposition of some \( a \in A \). By the Perron-Frobenius theorem, the eigenspace of \( M \) corresponding to the Perron-Frobenius eigenvalue \( \sum_a d_a \) is non-degenerate and spanned by \( |\omega_T\rangle \). If we can show that

\[
MB |\omega_A\rangle = (\sum_a d_a) B |\omega_A\rangle \tag{7.17}
\]

it therefore follows that \( B |\omega_A\rangle = \lambda |\omega_T\rangle \) for some \( \lambda \in \mathbb{C} \). In components, equation (7.17) reads

\[
\sum_{a,c,s,r} d cn_{a}n_{c}N_{rt} = \sum_{a,b} d ad_{b}n_{t} \tag{7.18}
\]

To prove this equation, we massage the left hand side using \( N_{nt} = N_{\bar{r}s} \) and \( n_{r}n_{\bar{a}} = n_{\bar{b}}n_{t} \), apply (7.11), use that \( N_{\bar{b}ac} = N_{c\bar{a}b} = N_{c\bar{b}a} \) and finally apply (??). Some intermediate results are shown in the following computation

\[
\sum_{a,c,s,r} d cn_{a}n_{c}N_{rt} = \sum_{a,c,s,r} d cn_{a}n_{c}N_{rt} \tag{7.19}
\]

\[
= \sum_{a,b,c} d bN_{\bar{a}c}n_{t} \tag{7.20}
\]

\[
= \sum_{a,b,c} d cN_{\bar{a}c}n_{t} \tag{7.21}
\]

\[
= \sum_{a,b} d ad_{b}n_{t} \tag{7.22}
\]

Now compare components on the left and right hand side of \( B |\omega_A\rangle = \lambda |\omega_T\rangle \). This shows that

\[
\lambda = \sum_{a} n_{a}d_{a} = q \tag{7.23}
\]

This shows that the fraction is indeed independent of the choice of \( t \). The equation of the number \( q \) resembles the embedding index introduced by Dynkin in his work on the classification of subalgebras of compact Lie algebras, hence is was named the quantum embedding index.

From \( B |\omega_A\rangle = Q |\omega_T\rangle \) and \( B^\dagger |\omega_T\rangle = |\omega_A\rangle \) it follows that the integer square matrices \( B^\dagger B \) and \( BB^\dagger \) satisfy

\[
B^\dagger B |\omega_A\rangle = q |\omega_A\rangle \quad \text{and} \quad BB^\dagger |\omega_T\rangle = q |\omega_T\rangle \tag{7.24, 7.25}
\]

From this we see that \( q = D_A^2 / D_T^2 \) because

\[
D_A^2 = \langle \omega_A | \omega_A \rangle = \langle \omega_T | BB^\dagger |\omega_T\rangle = q \langle \omega_T | \omega_T \rangle = q D_T^2 \tag{7.26}
\]
7.2 Topological $S$-matrix as an order parameter

Much of the information about the topological order of a system is encoded in the topological $S$-matrix. Knowing the $S$-matrix gives direct access to the quantum dimensions of the charges, as well as the total quantum dimension, and the fusion rules can be derived via the Verlinde formula. In addition, it is important in experimental verification of topological order as it is closely related to the monodromy matrix that often appears in the probabilities for interferometry experiments.

Using the extended formalism for topological symmetry breaking from last chapter, we can study the effect of the formation of a Bose condensate on the $S$-matrix of the theory diagrammatically. We write down an explicit formula and illustrate this by calculations for the $\mathfrak{su}(2)_4$ and $\mathfrak{su}(2)_{10}$ example.

It will be convenient to leave out the normalization factor $D^{-1}$ from the definition of the $S$-matrix in this section. Hence, we define the $S$-matrix of $A$ simply as the evaluation of the well known Hopf link

$$S_{ab} = \begin{array}{c}
\hline
a & b \\
\hline
0 & \sqrt{3} & 2 & \sqrt{3} & 1 \\
1 & \sqrt{3} & 0 & -\sqrt{3} & -\sqrt{3} \\
2 & 2 & -2 & 0 & 2 \\
3 & \sqrt{3} & -\sqrt{3} & 0 & \sqrt{3} & -\sqrt{3} \\
4 & 1 & -\sqrt{3} & 2 & -\sqrt{3} & 1 \\
\end{array}
$$

The $S$-matrix describes a process where two particle-anti-particle-pairs are drawn out of the vacuum, the two particles undergo a full monodromy and the pair annihilates again by fusing to the vacuum. Now imagine the same process in the phase after condensation. To make sense, the charges labelling the two loops of the Hopf-link should lie in the unconfined theory $\mathcal{U}$. However, we can calculate the diagram for charges in $\mathcal{T}$ just as well, when we expand the operator as a diagram in $A$. So we will write $S_{bst}$ for the $S$-matrix elements of the broken theory and allow $s, t \in \mathcal{T}$. But writing down the diagram for $S_{bst}$ is not as straightforward as labelling the Hopf-link. Since the condensate is like the vacuum for the new theory, we can connect the two loops of the Hopf-link by a condensate line for free. At least, this makes sense physically. From the mathematics, this also appears as the right definition definition of the $S$-matrix for the broken theory. Furthermore, we need a normalization factor of $q^2$. 

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7. INDICATORS FOR TOPOLOGICAL ORDER

So we define $\tilde{S}_{Tst}$ diagrammatically as

$$\tilde{S}_{Tst} = \frac{1}{q^2}$$

(7.30)

where $q = d_\phi$ is the quantum embedding index. The vertices are defined following the previous chapter. The vertex involving $t$ expands as

$$t = \sum_{\alpha \in \phi} \sum_{a,b \in t} (a^b_a)^{\alpha} a$$

(7.31)

For the vertex involving $s$, we need to take a complex conjugation into account

$$s = \sum_{\alpha \in \phi} \sum_{a,b \in s} (a^b_a)^* a$$

(7.32)

Note that because the diagram for $\tilde{S}_{st}$ closes on the top and bottom, actually no vertices with $a \neq b$ enter the equation and we only need the coefficients $A^{ab}_{a} \cdot A^{aa}_{\alpha}$.

Specialize to the case $\phi = 0 + \delta$, as is applicable to $su(2)_4$ and $su(2)_{10}$. Write $\tilde{A}^{ab}_{b} = \tilde{A}^{aa}_{a}$, as we did before. Now define

$$\tilde{S}_{ab} =$$

(7.33)

Then we can write $\tilde{S}_{st}$ explicitly as

$$\tilde{S}_{st} = \sum_{a \in s, b \in t} \left( S_{ab} + (\tilde{A}_{a}^{ab})^{*} \tilde{A}_{b}^{aa} \tilde{S}_{ab} \right)$$

(7.34)

By using the diagrammatic formalism, we can calculate $\tilde{S}_{ab}$ in terms of $F$-symbols, quantum dimensions an topological spins, as follows.

$$\tilde{S}_{ab} = \sum_{e} \left[ F^{ab}_{e} \right] \delta e$$

(7.35)
7.2 Topological S-matrix as an order parameter

This gives

\[
\bar{S}_{ab} = \sum_c [F_{ab}]_{dc} \theta_c \sqrt{d_a d_b d_c} \quad (7.36)
\]

In the case of \( su(2)_4 \), this gives the matrix

\[
\begin{array}{c|cccc}
   & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 2\sqrt{3}i & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\quad (7.37)
\]

Note that we need the splitting of fields in the phase transition for \( \bar{S}_{ab} \) to be non-trivial, because only fields \( a, b \) with \( a \in \delta \times a \) and \( b \in \delta \times b \) can give non-zero contributions, and we have seen earlier that this is equivalent to splitting. Indeed, in above example we find only one non-zero matrix coefficient, \( \bar{S}_{22} \), corresponding to the single splitting field 2.

Let us now calculate \( \tilde{S}_{st} \) when \( \mathcal{A} = su(2)_4 \). The quantum embedding index in this case is \( q = 2 \). We get

\[
\tilde{S}_{T,s} = \begin{array}{c|cccc}
   s, t & (0 + 4) & (1 + 3) & 2_+ & 2_- \\
\hline
(0 + 4) & 1 & 0 & 1 & 1 \\
(1 + 3) & 0 & 0 & e^{2\pi i/3} & e^{-2\pi i/3} \\
2_+ & 1 & 0 & e^{-2\pi i/3} & e^{2\pi i/3} \\
2_- & 1 & 0 & e^{2\pi i/3} & e^{-2\pi i/3} \\
\end{array}
\quad (7.38)
\]

since \( e^{\pm 2\pi i/3} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i \). We recognize \( (1 + 3) \) as the confined charge of \( T \). The residual matrix

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & e^{2\pi i/3} & e^{-2\pi i/3} \\
1 & e^{-2\pi i/3} & e^{2\pi i/3}
\end{pmatrix}
\quad (7.39)
\]

is indeed the unique unnormalized \( S \)-matrix describing topological order with the fusion rules found in the previous chapter [30] and belongs to the \( su(3)_1 \) model consistent with the analysis of [7].

Let us turn to the more intricate example of \( su(2)_{10} \). Because of the size of the matrices, it is a bit inconvenient to give \( S_{ab} \) and \( \bar{S}_{ab} \), but the calculations are straight
7. INDICATORS FOR TOPOLOGICAL ORDER

forward. They are printed in the appendix. The result for $\tilde{S}_{st}$ is

$$\tilde{S}_{st} =$$

<table>
<thead>
<tr>
<th>$s,t$</th>
<th>$(0 + 6)$</th>
<th>$(1 + 5 + 7)$</th>
<th>$(2 + 4 + 6 + 8)$</th>
<th>$(3 + 5 + 9)$</th>
<th>$(3 + 7)$</th>
<th>$(4 + 10)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0 + 6)$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\sqrt{2}$</td>
<td>1</td>
</tr>
<tr>
<td>$(1 + 5 + 7)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$(2 + 4 + 6 + 8)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$(3 + 5 + 9)$</td>
<td>$\sqrt{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$(3 + 7)$</td>
<td>$\sqrt{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\sqrt{2}$</td>
<td>$-\sqrt{2}$</td>
</tr>
<tr>
<td>$(4 + 10)$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\sqrt{2}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Again, the confined excitations show as rows and columns of zeros. The $S$-matrix for the bulk theory is

$$\begin{pmatrix}
1 & $\sqrt{2}$ & 1 \\
$\sqrt{2}$ & 0 & $-\sqrt{2}$ \\
1 & $-\sqrt{2}$ & 1
\end{pmatrix}$$

which is indeed the $S$-matrix for $\mathfrak{so}(5)_1$, in correspondence with the results of [7].

We see that the formalism developed in the previous chapter allows us to treat the phase transition directly on the level of the $S$-matrix. We find the confined charges as rows and columns of zeros. Since the $S$-matrix encodes most of the interesting topological order data, it is perfectly fit as an order parameter for topological order.

An interesting feature of the present approach is that it allows for an intermediate treatment. We might calculate several variations of equation (7.34), for instance, we can calculate

$$S_{ab} + \bar{S}_{ab}$$

possibly with the insertion of the $A$’s. It appears that in lattice simulations of DGTs, where the parameters are varied, phase transitions occur that can be signalled by string-like operators (see [?]) essentially measuring the condensed excitations. Also, $S$-matrix elements can be measured in these simulations. It appears that $\bar{S}$-diagrams start to contribute when a non-trivial condensate is formed, but since the operators that are measured are labelled by the sectors of the original phase one seems to find (7.41) or a closely related quantity.

Further development of both the numeric and analytic approaches to these systems is necessary to obtain a complete understanding.
CHAPTER 8

Conclusion and outlook
8. CONCLUSION AND OUTLOOK
In this appendix, the axioms of quasi-triangular Hopf algebras or quantum groups are gathered with the corresponding mathematical nomenclature.

A Hopf algebra $H$ is, in short, a bialgebra with an antipode. A bit more elaborate, this means that we have an associative algebra with multiplication $m: H \otimes H \rightarrow H$ and unit $1: \mathbb{C} \rightarrow H$. Associativity is given by the condition

$$m(m \otimes \text{id}) = m(\text{id} \otimes m) \quad (A.1)$$

The unit axiom is

$$m(1 \otimes \text{id}) = m(\text{id} \otimes 1) = \text{id} \quad (A.2)$$

It also has to be a coalgebra, which requires the definition of a comultiplication

$$\Delta: H \rightarrow H \otimes H \quad (A.3)$$

and a counit

$$\epsilon: H \rightarrow \mathbb{C} \quad (A.4)$$

They have to satisfy coassociativity

$$(\text{id} \otimes \Delta) \Delta = (\Delta \otimes \text{id}) \Delta \quad (A.5)$$

and the counit property

$$(\epsilon \otimes \text{id}) \Delta = \text{id} = (\text{id} \otimes \epsilon) \Delta \quad (A.6)$$

There is a natural multiplication, also denoted $m$, on $H \otimes H$. For elements $a_1 \otimes a_2$ and $b_1 \otimes b_2$ of $H \otimes H$, this is $m(a_1 \otimes a_2, b_1 \otimes b_2) = m(a_1, b_1) \otimes m(a_2, b_2)$. An algebra that is also a coalgebra, is called a bialgebra if algebra and coalgebra structure are compatible, which is the case if comultiplication and counit are algebra homomorphisms.
One ingredient that is still lacking to make \( \mathcal{H} \) a Hopf algebra, is the existence of an antipode. This is a linear map \( S : \mathcal{H} \to \mathcal{H} \) satisfying
\[
m(id \otimes S)\Delta = 1 = m(S \otimes id)\Delta \quad (A.7)
\]
It can be shown that, if a bialgebra admits an antipode, it is unique. So there is at most one Hopf algebra structure when we have a bialgebra.

Now what is the significance of Hopf algebras to physics? Symmetries are often of course often specified by algebras. The intuition behind the definition of the coalgebra structure and the antipode is the following.

We will be interested in the representations of \( H \), as these correspond to particles in some quantum theory. To consider multiparticle states, a consistent definition of tensor product representations is needed. This is done by using the comultiplication, as seen for discrete gauge theory. Suppose \( \alpha, \beta \) are two representations of \( \mathcal{H} \) with carrier spaces \( V_\alpha, V_\beta \) given by \( \pi^{\alpha,\beta} : H \to \text{End}(V_\alpha, V_\beta) \). Then we define the tensor product representation \( \alpha \otimes \beta \), acting on \( V_\alpha \otimes V_\beta \), by
\[
\pi^{\alpha \otimes \beta}(a) \equiv \pi^\alpha \otimes \pi^\beta(\Delta(a)) \quad (A.8)
\]
Coassociativity ensures that multiparticle states defined by successive application of the coproduct do not depend on the chosen order for the application of the comultiplication. The tensor product representations will generally fall apart as direct sums of irreducible representations. These are the fusion rules of the theory.

The counit gives the definition of the trivial representation \( \pi^0 = \epsilon \). The counit property precisely provides the triviality of fusion on the level of representations, \( 0 \otimes \alpha = \alpha = \alpha \otimes 0 \), because
\[
\pi^{0 \otimes \alpha}(a) = (\epsilon \otimes \pi^\alpha)\Delta(a) = \pi^\alpha(\epsilon \otimes \text{id})\Delta(a) = \pi^\alpha(a) \quad (A.9)
\]
Now suppose we have some representation \( \alpha \). There should be a conjugate (or dual) representation \( \bar{\alpha} \) corresponding to the antiparticle. We can define such a representation on the dual vector space \( V_\alpha^* \) by means of the antipode as
\[
\pi^{\bar{\alpha}}(a) = \pi^\alpha(S(a))^T \quad (A.10)
\]
where the \( T \) denotes matrix transposition.

Our particular interest lies in physics in the plane with anyonic excitations that have interesting, and possibly non-abelian braid statistics. The ingredient needed to describe the braiding representations of a hopf algebra consistently is the universal \( R \)-matrix. This is an invertible element \( R \in \mathcal{H} \otimes \mathcal{H} \) which has the properties
\[
\Delta^{op}(a)R = R\Delta(a) \quad (\forall a \in \mathcal{H}) \\
(\Delta \otimes \text{id})(R) = R_{13}R_{23} \\
(id \otimes \Delta)(R) = R_{13}R_{12}
\]
Here $\Delta^{\text{op}} = \sigma \Delta$ is the composition of the comultiplication with the exchange $\sigma$ of tensor factors in $\mathcal{H} \otimes \mathcal{H}$. The notation $R_{ij}$ is an abbreviation for the action of $R$ on the $i^{th}$ and $j^{th}$ tensor leg of $\mathcal{H} \otimes \mathcal{H}$, so for example $R_{12} = R \otimes 1$. To implement the clockwise exchange two adjacent particles, we act by $R$ on the tensor product representation, and then exchange the tensor legs. So for example, when we have a system of three particles all carrying representation $\alpha$, the exchange of the left two particles is effectuated by

$$(\sigma \otimes \text{id})(\pi^\alpha \otimes \pi^\alpha)(R \otimes 1) \quad (A.14)$$

The defining properties of $R$ ensure that exchanging particles by application of $\sigma R$ makes physical sense. Using either 6.6 and 6.7 or 6.6 and 6.8 one may prove the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad (A.15)$$

It implies that in any representation of $\mathcal{H}$ we have the equality

$$(\sigma R \otimes 1)(1 \otimes \sigma R)(\sigma R \otimes 1) = (1 \otimes \sigma R)(\sigma R \otimes 1)(1 \otimes \sigma R) \quad (A.16)$$

from which one may deduce that in a system of $n$ identical particles, the exchange of adjacent particle satisfies the defining relations for the braid group. Since $R$ is invertible, this generates a representation of the braid group $B_n$. Because the exchange commutes with the action of $\mathcal{H}$, it follows that the system carries a representation of $\mathcal{H} \times B_n$.

A Hopf algebra $\mathcal{H}$ with universal $R$-matrix is called \textit{quasitriangular}, so this is what we understand by the term quantum group.

### A.0.1 Algebra axioms

In relation to the graphical formalism of anyon theories and anticipating the next chapter, it is interesting to put the axioms in a graphical form. We will represent the identity by a vertical line, the multiplication by an unlabelled fusion vertex and the comultiplication by a similar splitting vertex. Thus maps are displayed by lines from bottom to top, endpoints corresponding to a copy of $\mathcal{H}$ and the tensor product is given by juxtaposition. The unit and counit will be represented by a line starting or ending at a dot, which is consistent with the interpretation of $\mathcal{C}$ as the vacuum module by means of $\epsilon$. Thus

$$\text{id} \quad \begin{array}{c} m \\ \Delta \end{array} \quad 1 \quad \epsilon \quad (A.17)$$

The unit ant counit axiom then become

$$\begin{array}{c} \vdash \quad = \quad \begin{array}{c} \\
\end{array} \\
\begin{array}{c} \\
\end{array} \quad \text{and} \\
= \quad = \quad \begin{array}{c} \\
\end{array} \\
\begin{array}{c} \\
\end{array} \quad (A.18)$$

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respectively. Similarly we can depict (co)associativity graphically. The associativity axiom becomes

\[ \begin{array}{c}
\text{associativity axiom becomes}
\end{array} \]

while coassociativity reads

\[ \begin{array}{c}
\text{coassociativity reads}
\end{array} \]

It is very tempting to interpret these in terms of fusion and splitting of particles. This is in fact not entirely crazy. By using the multiplication, we can let \( H \) act on itself. It will probably not be irreducible, but it does fall apart as a direct sum of irreducible representations, just like any other representation and as such has a relation to the particles of the theory. It is not our intention to pursue this point in any way, but in the next chapter similar diagrammatic equations will occur. This little discussion shows why these are studied by mathematicians under the name of ‘algebras in tensor categories’.

Important examples of quasi-triangular Hopf algebras are given by the quantum double of finite groups, which was discussed in the last chapter. In this case, above description indeed works. The particles correspond to the irreducible representations and we can work out all kinds of details by using the Hopf algebra structures and the universal \( R \)-matrix.
APPENDIX B

Tensor categories: from math to physics
B. TENSOR CATEGORIES: FROM MATH TO PHYSICS
Data for $\mathfrak{su}(2)_{10}$

C.0.2 Algebra axioms
C. DATA FOR $\mathfrak{su}(2)_{10}$
References


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