Resit Advanced Logic 2017-2018

Wednesday June 27, 2018, 15.15–18.00

6 exercises

**Motivate your answers**

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**Exercise 1.** (*6+6+6 points*)
This exercise is concerned with basic modal logic (BML).

(a) Prove or disprove $\Box(\Box p \to p) \to \Box p$.

(b) Give an example of a formula $\varphi$ and a model $\mathcal{M}$ such that neither $\mathcal{M} \vDash \varphi$ nor $\mathcal{M} \vDash \neg\varphi$.

(c) Show that the formula $\mathcal{F} \vDash \Box p \to \Diamond p$ characterizes seriality. A frame is serial if $\forall x\,(\exists y\,(Rxy))$.

**Exercise 2.** (*6+6 points*)
This exercise is concerned with (non-)definablility.

(a) The operator $\mathsf{A}$ called *global box* is defined as follows:
$\mathcal{M}, w \vDash \mathsf{A}\,\phi$ if and only if $\mathcal{M}, v \vDash \phi$ for all worlds $v$.

Show that the global box is not definable in basic modal logic.

(b) Show that irreflexivity (which is: $\forall x \neg Rxx$) is not modally definable.

**Exercise 3.** (*6+6 points*)
The *binary tree* is the frame $\mathcal{B} = (W, R)$ where the domain $W$ is the set of all finite strings over the alphabet {0,1}:

$$W = \{0, 1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}\,.$$

($\varepsilon$ denotes the empty string) and where the transition relation $R \subseteq W \times W$ is given by:

$$Rst \quad \text{if and only if} \quad t = s0 \text{ or } t = s1$$

Let $\mathcal{N}$ be the frame $(\mathbb{N}, S)$ of the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ with the successor relation $S$ defined by:

$$Smn \quad \text{if and only if} \quad n = m + 1\,,$$

(a) Let $V$ be the valuation on $\mathcal{B}$ such that:

$$V(p) = \{\, w \in \{0,1\}^* \mid w \text{ is of even length} \,\} \ .$$

Define a valuation $U$ on $\mathcal{N}$ such that:

$$\mathcal{B}, V, \varepsilon \;\underline{\leftrightarrow}\; \mathcal{N}, U, 0$$

(You only have to define such $U$; no proof needed.)

Further, show that this is the only possibility for $U$.

(b) Now let $V'$ be a valuation on $\mathcal{B}$ such that:

$$V'(p) = \{\, 0w \mid w \in \{0,1\}^* \,\}$$
$$V'(q) = \{\, 1w \mid w \in \{0,1\}^* \,\} \ .$$

Show that there exists no valuation $U'$ on $\mathcal{N}$ such that:

$$\mathcal{B}, V', \varepsilon \;\underline{\leftrightarrow}\; \mathcal{N}, U', 0$$

**Exercise 4.** *(6+6 points)*
This exercise is concerned with decidability.
Use semantic tableaux or sequents to investigate the validity of the following propositions. If the proposition is not valid, give explicitly the countermodel that you found.

(a) $(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$

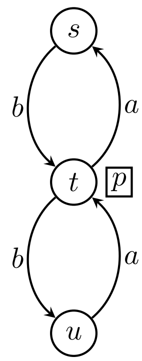
(b) $\Box(p \to q) \to \Box p \to \Box q$

**Exercise 5.** *(6+6+6+6 points)*
This exercise is concerned with Propositional Dynamic Logic (PDL).

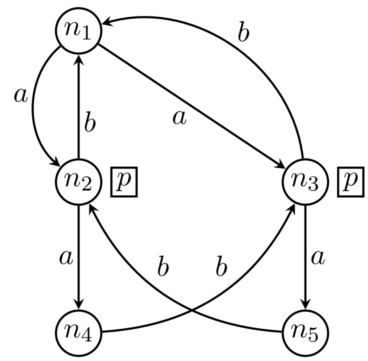
(a) Show that $[\alpha(\beta \cup \gamma)]p \to [\alpha\beta \cup \alpha\gamma]p$ is a PDL-tautology;

here $\alpha, \beta, \gamma$ are unspecified arbitry PDL-programs.

Now consider the $\{a, b\}$-models $\mathcal{M}$ and $\mathcal{N}$ defined by:




$\mathcal{M}$




$\mathcal{N}$

(b) Is there a modal formula that distinguishes state $n_3$ in model $\mathcal{N}$ from state $t$ in model $\mathcal{M}$?

(c) Let $\widehat{\mathcal{N}}$ be the PDL-extension of model $\mathcal{N}$. Compute the transition relation $\widehat{R}_\pi$ corresponding to the PDL-program $\pi = \textsf{if}\, p \,\textsf{then}\, ba \,\textsf{else}\, ab$.

(d) Is there a state in $\widehat{\mathcal{N}}$ where $([\pi]p \to \bot)$ holds? Shortly motivate your answer.

**Exercise 6.** *(6+6 points)*
This exercise is concerned with epistemic logic.
We recall the definitions of the Hilbert systems:

– system $K$ is the most basic system with the tautologies of first-order propositional logic as axioms, the axiom for modal distribution ($K(p \to q) \to (Kp \to Kq)$), the rule for modus ponens, the rule for necessitation (if $\vdash \phi$ then $\vdash K\phi$), and the rule for substitution;

– system $T$ is system $K$ plus the truth axiom (or veridicality):
$A1 : Kp \to p$;

– system $S4$ is system $T$ plus the axiom of positive introspection:
$A2 : Kp \to KKp$;

– system $S5$ is system $S4$ plus the axiom of negative introspection:
$A3 : \neg Kp \to K\neg Kp$ .

(a) Show that $A2$ is not derivable in $T$.

(b) Indicate and explain the error(s) in the following derivation in $T$:

1. $p$ (assumption)
2. $Kp$ (necessitation, 1)
3. $p \to Kp$ (PROP, 1, 3)
4. $Kq \to KKq$ (substitution, 3)

*The exam grade is (the total number of points plus 10) divided by 10.*
*In addition, the bonus is added, and the maximum final grade is 10.*