1. A frame $\mathcal{F} = (W, R)$ is *partially functional* if $\forall x, y, z \in W\,(Rxy \wedge Rxz \to y = z)$. It is *irreflexive* if $\neg\forall x \in W. R\,x\,x$.

   (a) Show that the formula $\Diamond p \to \Box p$ characterizes the frame propery partial functionality.

   (b) Show that the frame property irreflexivity is not modally definable.

2. Let $\mathcal{N}$ be the frame $(\mathbb{N}, S)$ of the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the successor relation S defined by:

$$Smn \quad \text{if and only if} \quad n = m + 1\,,$$

   and recall the binary tree $\mathcal{B} = (\{0, 1\}^*, R)$ from last week with R defined by:

$$Rst \quad \text{if and only if} \quad t = s0 \text{ or } t = s1$$

   (a) Define a valuation U on $\mathcal{N}$ such that:

$$\mathcal{B}, V, \varepsilon \leftrightarroweq \mathcal{N}, U, 0$$

   where V is a valuation on $\mathcal{B}$ such that:

$$V(p) = \{\, w \in \{0, 1\}^* \mid w \text{ is of even length} \,\}\ .$$

   Show that this is the only possibility for U.

   (b) Let V' be a valuation on $\mathcal{B}$ such that:

$$V'(p) = \{\, 0w \mid w \in \{0, 1\}^* \,\}$$
$$V'(q) = \{\, 1w \mid w \in \{0, 1\}^* \,\}\ .$$

   Show that there exists no valuation U' on $\mathcal{N}$ such that:

$$\mathcal{B}, V', \varepsilon \leftrightarroweq \mathcal{N}, U', 0$$

3. Consider the following two frames $\mathcal{F}$ en $\mathcal{F}'$

$\mathcal{F}$ $\qquad\qquad$ $\mathcal{F}'$

   Here it is to be understood that Frame $\mathcal{F}$ has infinitely many paths of finite length $1, 2, 3, \ldots$. Frame $\mathcal{F}'$ is like $\mathcal{F}$ but additionaly has one infinite path.

   Argue that there cannot be a bisimulation Z between $\mathcal{F}$ and $\mathcal{F}'$ such that wZw'.

4. (a) Give an example of a formula φ and a model $\mathcal{M}$ such that neither $\mathcal{M} \vDash \varphi$ nor $\mathcal{M} \vDash \neg\varphi$.

   (b) Give an example of a formula φ, a frame $\mathcal{F}$ and two models $\mathcal{M}$ and $\mathcal{M}'$ based on $\mathcal{F}$ such that $\mathcal{M} \vDash \varphi$ and $\mathcal{M}' \vDash \neg\varphi$.

5. Consider the frame $\mathcal{F} = (W, R)$ with set of worlds $W = \{1, 2, 3\}$ and $R = \{(1,1), (1,2), (1,3), (2,3)\}$. Give for every $i \in W$ a formula ϕ_i that is for any valuation true in i, but not in any $j \in W \setminus \{i\}$. So $\mathcal{F}, V, i \models \phi_i$ for any V, but for $i \neq j$ we have for all V that $\mathcal{F}, V, j \not\models \phi_i$.

6. Consider the model $\mathcal{M} = (W, R, V)$ with $W = \{1, 2, 3, 4\}$, $R = \{(1,3), (1,4), (2,1), (3,2), (3,3), (4,3)\}$, and V with $V(1) = V(2) = \{p\}$ and $V(3) = V(4) = \varnothing$.

   Use game semantics to show that $\mathcal{M}, 1 \models \Diamond\Box\Diamond p$.