Goal

Getting further familiarity with the notion of bisimilation and the use of the disjoint union, the bisimulation contraction, the tree unravelling. A first introduction to the use of sequents for decidability, first for prop1 with the aim to extend this to modal prop1.

Exercises

1. Consider the models M and K of MLOM p30: In M we have $W_M = \{a, b, c, d\}$ and $R_M = \{(a, b), (a, c), (c, a), (c, d)\}$. In K we have $W_K = \{1, 2, 3, 4\}$ and $R_K = \{(1, 2), (2, 3), (3, 1), (1, 4)\}$.

   Show that M, a and $K, 1$ are not bisimilar using the game approach.

   Give a modal formula that distinguishes between M, a and $K, 1$.

2. A frame is said to be connected if $\forall xy\,(Rxy \vee x = y \vee Ryx)$. Show that the property of 'connectedness' is not modally definable.

3. Consider the left model of Question 1(a) in MLOM p35: $W = \{1, 2, 3\}$ and $R = \{(1, 2), (1, 3), (2, 3), (3, 1), (3, 2)\}$, and $V(p) = \{2\}$.

   Show that the worlds 1 and 3 are bisimilar.

   Show that the worlds 1 and 2 are not bisimilar.

   (You do not have to prove that the bisimulations you propose are actually bisimulations.)

   Give the bisimulation contraction of the model.

4. Consider the models $\mathcal{M}_1 = ((W_1, R_1), V_1)$ and $\mathcal{M}_2 = ((W_2, R_2), V_2)$ in Exercise 1 in MLOM p35. We use the notation $W_1 = \{1, 2, 3\}$, $R_1 = \{(1, 2), (1, 3), (2, 3), (3, 1), (3, 2)\}$, V_1 with $V_1(p) = \{2\}$, and $W_2 = \{a, b\}$, $R_2 = \{(a, a), (a, b), (b, a)\}$, V_2 with $V_2(p) = \{b\}$.

   Give the tree unravelling of $\mathcal{M}_1$ in 1, and the tree unravelling of $\mathcal{M}_2$ in a.

   Is the state 1 in the tree unravelling of $\mathcal{M}_1$ bisimilar with the state a in the treee unravelling of $\mathcal{M}_2$?

What can we conclude about those states in their original models?

5. For $n = 1, 2, \ldots$, let the 'looping frame' $\mathcal{L}_n = (W_n, R_n)$ be defined by

$$W_n = \{0, \ldots, n-1\}$$
$$R_n = \{(k, k') | k' = k + 1 \text{ if } k + 1 < n \text{ and } k' = 0 \text{ otherwise}\}$$

   (a) Draw the frames $\mathcal{L}_2$ and $\mathcal{L}_4$.

   (b) Give a modal formula that distinguishes frame $\mathcal{L}_2$ from $\mathcal{L}_4$, that is, a formula ϕ such that $\mathcal{L}_2 \models \phi$ and $\mathcal{L}_4 \not\models \phi$. Prove your answer.

   For questions **(c)** and **(d)** you have to define a bisimulation, but reporting on the verification of the bisimulation conditions is not required.

   (c) Let $\mathcal{M}_2$ be some model based on $\mathcal{L}_2$. Define a model $\mathcal{M}_4$ based on $\mathcal{L}_4$ such that $\mathcal{M}_2, 0$ is bisimilar with $\mathcal{M}_4, 0$.

   (d) Let $\mathcal{M}_3$ be some model based on $\mathcal{L}_3$. Define an acyclic model $\mathcal{N}$ bisimilar to $\mathcal{M}_3$.

6. Formulate the box version of "Modal Decomposition" (MLOM, p. 41), i.e., give necessary and sufficient conditions for validity of a modal sequent of the form

$$\vec{p}, \Box\varphi_1, \ldots, \Box\varphi_k \implies \Box\psi_1, \ldots, \Box\psi_m, \vec{q}$$

7. Use sequents to investigate the validity of the following prop1 formulas:

   (a) $(p \to q) \vee (q \to p)$

   (b) $p \to q \to p$

   (c) $(p \vee q) \to (p \wedge q)$

   (d) $((p \to q) \to p) \to p$

   (e) $\Diamond p \to \Box p$

   (f) $\Box p \to \Diamond p$

   (g) $(p \to q) \vee (\Box p \vee \Diamond p)$