Exercises

1. Give a derivation in K of $\vdash (\Diamond\phi \wedge \Box(\phi \to \psi)) \to \Diamond\psi$.
2. Give a derivation in K of $(\Box\phi \vee \Box\psi) \to \Box(\phi \vee \psi)$.
3. Give a derivation in T of $\Box p \to \neg\Box\neg p$.
4. Indicate and explain an error in the following derivation in T:

1.	$q \to \Box q$	(necessitation)
2.	$\Box p \to \Box\Box p$	(substitution, 1)

5. Indicate and explain the error(s) in the following derivation in T:

1.	p	(assumption)
2.	$\Box p$	(necessitation, 1)
3.	$p \to \Box p$	(PROP, 1, 3)
4.	$\Box q \to \Box\Box q$	(substitution, 3)

6. Show that $\neg\Box\neg\Box p \to p$ is not derivable in $S4$.
7. Give a derivation in $S5$ of $\neg\Box\neg\Box p \to p$.
8. Prove or disprove the validity of the following formulas in the temporal frame $\mathcal{N} = (\mathbb{N}, <)$ of the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ with the usual ordering $<$:

   (a) $\Diamond\Box p \to \Box\Diamond p$

   (b) $\Box\Diamond p \to \Diamond\Box p$

9. Show that the formula

   $$\lambda = \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$$

   defines right-linearity, that is, for all (not necessarily temporal) frames $\mathcal{F} = (W, R)$:

   $$\mathcal{F} \vDash \lambda \quad \text{if and only if} \quad R \text{ is right-linear}$$

   A relation R is right-linear if $Rxy \wedge Rxz$ implies $Ryz \vee y = z \vee Rzy$ for all x, y, z.

10. Show that right-branching is not modally definable.

    A relation R is right-branching if there exist x, y, z such that $x < y$ and $x < z$ but $\neg(y < z) \wedge y \neq z \wedge \neg(z < y)$

11. Show that $\Diamond p \to \Diamond\Diamond p$ defines density.

    A relation R is dense if for all x, z we have: if $x < z$ then there is y such that $x < y$ and $y < z$.