Exercises

1. Give a derivation in $K$ of $\vdash (\Diamond \phi \land \Box (\phi \rightarrow \psi)) \rightarrow \Diamond \psi$.

2. Give a derivation in $K$ of $(\Box \phi \lor \Box \psi) \rightarrow \Box (\phi \lor \psi)$.

3. Give a derivation in $T$ of $\Box p \rightarrow \neg \Box \neg p$.

4. Indicate and explain an error in the following derivation in $T$:
   1. $q \rightarrow \Box q$ (necessitation)
   2. $\Box p \rightarrow \Box \Box p$ (substitution, 1)

5. Indicate and explain the error(s) in the following derivation in $T$:
   1. $p$ (assumption)
   2. $\Box p$ (necessitation, 1)
   3. $p \rightarrow \Box p$ (PROP, 1, 3)
   4. $\Box q \rightarrow \Box \Box q$ (substitution, 3)

6. Show that $\neg \Box \neg \Box p \rightarrow p$ is not derivable in $S4$.

7. Give a derivation in $S5$ of $\neg \Box \neg \Box p \rightarrow p$.

8. Prove or disprove the validity of the following formulas in the temporal frame $N = (\mathbb{N}, <)$ of the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ with the usual ordering $<$:
   (a) $\Diamond \Box p \rightarrow \Box \Diamond p$
   (b) $\Box \Diamond p \rightarrow \Diamond \Box p$

9. Show that the formula

   $$\lambda = \Diamond p \land \Diamond q \rightarrow \Diamond (p \land \Diamond q) \lor \Diamond (p \land q) \lor \Diamond (\Diamond p \land q)$$

defines right-linearity, that is, for all (not necessarily temporal) frames $F = (W, R)$:

$$F \vDash \lambda \iff R \text{ is right-linear}$$

A relation $R$ is right-linear if $Rxy \land Rxz$ implies $Ryz \lor y = z \lor Rzy$ for all $x, y, z$. 

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10. Show that right-branching is not modally definable.

   A relation $R$ is right-branching if there exist $x, y, z$ such that $x < y$
   and $x < z$ but $\neg(y < z) \land y \neq z \land \neg(z < y)$

11. Show that $\lozenge p \rightarrow \lozenge \lozenge p$ defines density.

   A relation $R$ is dense if for all $x, z$ we have: if $x < z$ then there is $y$
   such that $x < y$ and $y < z$. 