Remarks

We write \( sRt \) to denote that \((s, t) \in R\); earlier we used the notation \( Rst \), but with all the subscripts to the relations this becomes hard to read. Furthermore, we sometimes write \( xSyTz \) to denote that both \( xSy \) and \( yTz \). We also use \( \mathbb{N}_{<n} := \{0, 1, \ldots, n - 1\} \) (so \( \mathbb{N}_{<0} = \emptyset \)).

We recall the definitions of identity, \((n\text{-fold})\) composition, and reflexive-transitive closure:

\[
\begin{align*}
\text{Id} &= \{(x, y) \mid x = y\} \\
R \circ S &= \{(x, y) \mid \exists u \ (xR u \land uS y)\} \\
R^0 &= \text{Id} \\
R^{n+1} &= R^n \circ R \\
R^* &= \bigcup_{n \in \mathbb{N}} R^n = R^0 \cup R^1 \cup R^2 \cup \cdots
\end{align*}
\]

We also use the notation \( R;S \) instead of \( R \circ S \). We use the following facts of relational algebra:

- relational composition is associative: \((R \circ S) \circ T = R \circ (S \circ T)\), so that we can write \( R \circ S \circ T \) without ambiguity;

- \( \text{Id} \) is a left and right-identity with respect to relational composition: \( \text{Id} \circ R = R = R \circ \text{Id} \);

- composition left- and right-distributes over union: \( R \circ (S \cup T) = (R \circ S) \cup (R \circ T) \) and \((S \cup T) \circ R = (S \circ R) \cup (T \circ R)\);

- the operation \( \cdot^* \) is idempotent: \((R^*)^* = R^*\);

- the operation \( \cdot^* \) is monotone: if \( R \subseteq S \) then \( R^* \subseteq S^* \);

- if \( R^n \subseteq \bigcup_{0 \leq i < n} R^i \) then \( R^* = \bigcup_{0 \leq i < n} R^i \).
Exercises

1. This exercise is concerned with basic modal logic (BML) and arbitrary frames.
   Show that the property of right-branching (slides of lecture 7) is not definable using a formula from basic modal logic.

2. This exercise is concerned with multi-modal logic.
   We consider the set of labels $I = \{a, b, c\}$ and the $I$-frame $F = (W, \{R_a, R_b, R_c\})$ where:
   \[
   W = \{w_1, w_2, w_3, w_4\} \quad R_b = \{(w_2, w_3)\} \quad R_c = \{(w_2, w_4), (w_4, w_1)\}
   \]
   and let $M = (F, V)$ with $V$ the valuation on $F$ defined by:
   \[
   V(p) = \{w_3\} \quad V(q) = \{w_1\}.
   \]
   (a) Draw the model $M$ as a graph.
   (b) Prove or disprove:
      \[
      \begin{align*}
      &i. \; M \models [a]p \\
      &ii. \; M \models [a](p \lor ([b] \top \land [c] \top)) \\
      &iii. \; F \models [a](\top \lor ([b] \top \land [c] \top))
      \end{align*}
      \]

3. This exercise is concerned with multi-modal logic.
   Let $I = \{10c, \text{coffee, tea} \}$ and consider the $I$-models $M_1, M_2, M_3$: 
   \[\begin{array}{c}
   M_1 \\
   \begin{array}{c}
   \text{coffee} \\
   \bullet
   \end{array}
   \begin{array}{c}
   \begin{array}{c}
   10c \\
   \longrightarrow
   \end{array}
   \begin{array}{c}
   \bullet
   \end{array}
   \begin{array}{c}
   \text{tea} \\
   \end{array}
   \end{array}
   \end{array} \quad \begin{array}{c}
   M_2 \\
   \begin{array}{c}
   \text{coffee} \\
   \bullet
   \end{array}
   \begin{array}{c}
   \begin{array}{c}
   10c \\
   \longrightarrow
   \end{array}
   \begin{array}{c}
   \bullet
   \end{array}
   \begin{array}{c}
   \text{tea} \\
   \bullet
   \end{array}
   \end{array}
   \end{array} \quad \begin{array}{c}
   M_3 \\
   \begin{array}{c}
   \text{coffee} \\
   \bullet
   \end{array}
   \begin{array}{c}
   \begin{array}{c}
   10c \\
   \longrightarrow
   \end{array}
   \begin{array}{c}
   \bullet
   \end{array}
   \begin{array}{c}
   \text{tea} \\
   \bullet
   \end{array}
   \end{array}
   \end{array} \]
Give distinguishing formulas (over $I$) for the processes $s_1$, $s_2$ and $s_3$.

4. This exercise is concerned with propositional dynamic logic (PDL).

Let $\text{VAR} = \{p, q\}$ and $A = \{a, b\}$, and consider the following $A$-model $\mathcal{M}$:

(a) Compute $\hat{R}_{b^*}$.
(b) Compute $\hat{R}_{ab^*}$.
(c) Compute $\hat{R}_{ab^*a}$.
(d) Prove that $\mathcal{M} \models p \leftrightarrow [(ab^*a)^*]p$.
(e) Prove that $\mathcal{M} \models q \leftrightarrow [(ba^*b)^*]q$.

5. Let $\alpha$ and $\beta$ be PDL-programs. Which of the following two formulas is valid in PDL, which is not?

(a) $[(\alpha \cup \beta)^*]p \rightarrow [\alpha^*]p \land [\beta^*]p$
(b) $[\alpha^*]p \land [\beta^*]p \rightarrow [(\alpha \cup \beta)^*]p$

Give a counterexample for the invalid one, and prove validity of the other.