Exercises

1. We consider basic modal logic (BML) and aribitrary frames.

   Show that the property of right-branching (slides of lecture 8) is not definable using a formula from basic modal logic.

2. We consider basic modal logic (BML) and temporal fames.

   Show that the operator until is not definable.

   Show that the operator next is not definable.

3. We consider basic temporal logic.

   Prove or disprove the validity of the following formulas in the temporal frame $\mathcal{N} = (\mathbb{N}, <)$ of the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ with the usual ordering $<$:

   (a) $\langle F \rangle [F] \bot$
   (b) $\langle P \rangle \top \to \langle P \rangle [P] \top$
   (c) $\langle P \rangle \langle F \rangle q \to (\langle P \rangle q \vee q \vee \langle F \rangle q)$

4. We consider basic temporal logic.

   Let τ and γ abbreviate the following formulas:

$$\tau = (\langle F \rangle [F] q \wedge \langle F \rangle \neg q) \to \langle F \rangle (\neg q \wedge [F] q)$$
$$\gamma = (\langle F \rangle [F] q \wedge \langle F \rangle \neg q) \to \langle F \rangle ([F] q \wedge [P] \langle F \rangle \neg q)$$

   Consider the temporal frames $\mathcal{Z} = (\mathbb{Z}, <)$, $\mathcal{Q} = (\mathbb{Q}, <)$, and $\mathcal{R} = (\mathbb{R}, <)$ of the integers, rational and real numbers, respectively, with their usual orderings.

   (a) Show that τ is valid in $\mathcal{Z}$.
   (b) Show that τ is not valid in $\mathcal{Q}$.
   (c) Show that γ is not valid in $\mathcal{Q}$.
   (d) Is γ valid in $\mathcal{R}$?

5. A temporal order $<$ is *dense* if between any two distinct points x and y we can find a third point z. It is *discrete* if to each non-final point x we can associate an *immediate successor* z.

$$\begin{aligned} \textit{density}: &\quad \forall xy\,(x < y \to \exists z\,(x < z \land z < y)) \\ \textit{discreteness}: &\quad \forall xy\,(x < y \to \exists z\,(x < z \land \neg\exists u\,(x < u \land u < z))) \end{aligned}$$

   (a) Define an temporal order which is neither dense nor discrete.

   (b) Define a temporal order which is both dense and discrete.

6. We consider basic temporal logic and temporal frames.

   Show that the operator until is not definable.

7. We consider multi-modal logic.

   Let I be an arbitrary index set, and let $i, j \in I$.

   (a) Prove that the formula $p \to [i]\langle j\rangle p$ characterizes the class of I-frames $\mathcal{F} = (W, \{R_k \mid k \in I\})$ that satisfy the property $R_i \subseteq R_j^{-1}$.

   (b) Use the result of the previous question to show that the formula $\langle i\rangle[j]p \to p$ also characterizes the frame property $R_i \subseteq R_j^{-1}$.

   (c) Are the formulas $p \to [i]\langle j\rangle p$ and $\langle i\rangle[j]p \to p$ equivalent? Prove your answer.

8. We consider Hoare logic for proving correctness of the gcd-program.

9. We consider the set of labels $I = \{a, b, c\}$ and the I-frame $\mathcal{F} = (W, \{R_a, R_b, R_c\})$ where:

$$W = \{w_1, w_2, w_3, w_4\} \qquad R_b = \{(w_2, w_3)\}$$
$$R_a = \{(w_1, w_2), (w_3, w_3)\} \qquad R_c = \{(w_2, w_4), (w_4, w_1)\}$$

   and let $\mathcal{M} = (\mathcal{F}, V)$ with V the valuation on $\mathcal{F}$ defined by:

$$V(p) = \{w_3\} \quad \text{and} \quad V(q) = \{w_1\}$$

   (a) Draw the model $\mathcal{M}$ as a graph.

   (b) Prove or disprove:

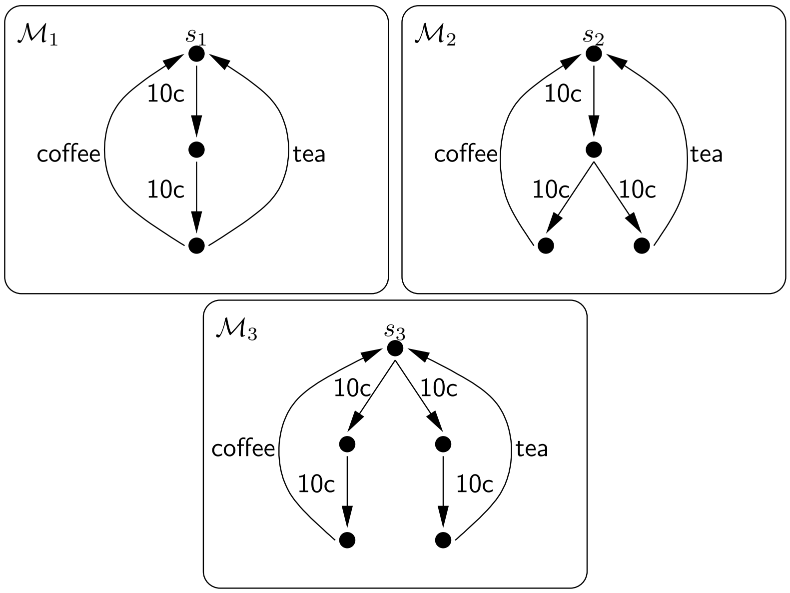
      i. $\mathcal{M} \models [a]p$

      ii. $\mathcal{M} \models [a](p \lor (\langle b\rangle\top \land \langle c\rangle\top))$

      iii. $\mathcal{F} \models [a](\langle a\rangle\top \lor (\langle b\rangle\top \land \langle c\rangle\top))$

10. We consider multi-modal logic.

Let $I = \{\mathsf{10c}, \mathsf{coffee}, \mathsf{tea}\}$ and consider the I-models $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$:



Give distinguishing formulas (over I) for the processes s_1, s_2 and s_3.