Notation: For a frame (W, R) and a point $x \in W$, we will use $R[x]$ to denote the set of R-successors of x as follows: $R[x] = \{y \mid Rxy\}$.

Answers:

1. (a)

$$W = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$$
$$R = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_5), (a_3, a_5), (a_3, a_6), (a_4, a_6), (a_5, a_7), (a_6, a_7)\}$$
$$V(p) = \{a_1, a_2, a_5, a_7\}$$
$$V(q) = \{a_2, a_3, a_4, a_6\}$$

(b) (ii) $a_2 \vDash \Diamond q \to \Diamond\Diamond q$ holds because $a_2 \nvDash \Diamond q$, and this is because $a_5 \notin V(q)$ and a_5 is the only R-successor of a_2.

(iii) To see that $a_3 \vDash \Diamond p \to \Box(q \to \Box(p \to \Box p))$ we have to check that $a_3 \vDash \Box(q \to \Box(p \to \Box p))$ (because $a_3 \vDash \Diamond p$ holds). So we check $x \vDash q \to \Box(p \to \Box p)$ for every $x \in R[a_3] = \{a_5, a_6\}$:

- $a_5 \nvDash q$, whence $a_5 \vDash q \to \Box(p \to \Box p)$;
- $a_6 \vDash q$, so we need to check $a_6 \vDash \Box(p \to \Box p)$, which means, as $R[a_6] = \{a_7\}$, to check that $a_7 \vDash p \to \Box p$. This indeed holds since $R[a_7] = \varnothing$, i.e., a_7 is a blind world, and so, by the truth definition of $\Box$, we get $a_7 \vDash \Box p$, and therefore $a_7 \vDash p \to \Box p$.

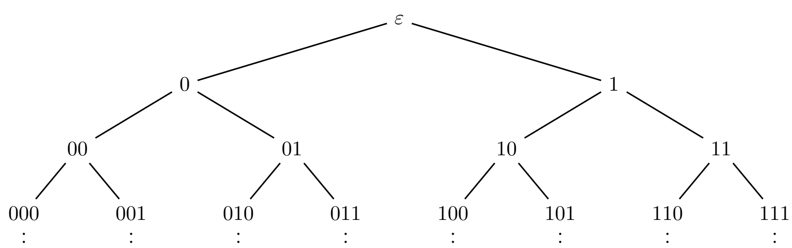
(c) (i) The formula $p \to \Diamond p$ is not globally true (= true in all states) in $\mathcal{M}$, as it is false in the dead-end a_7: we have $a_7 \vDash p$ but $a_7 \nvDash \Diamond p$ (as it has no R-successor), and hence $\mathcal{M}, a_7 \nvDash p \to \Diamond p$.

(iii) We have to find a point a_i such that $\mathcal{M}, a_i \nvDash q \to (\Diamond q \to \Box(q \to \Diamond q))$, i.e., such that $a_i \vDash q$ and $a_i \vDash \Diamond q$ but $a_i \nvDash \Box(q \to \Diamond q)$. The only candidates meeting the first two, are a_3 and a_4. In fact both will do, as the share a_6 as their R-successor, and indeed in a_6 the implication $q \to \Diamond q$ is false (we have $a_6 \vDash q$ and $a_6 \nvDash \Diamond q$). Hence we have $a_4 \nvDash \Box(q \to \Diamond q)$.

(d) Just let p be true in all worlds. Then $\Box p \to p$ holds everywhere.

In fact, this valuation ($V'(p) = W$) is the only option to make $\Box p \to p$ globally true. The reason is that $\Box p$ is true in a_7 independent of the valuation we choose. So, in order for the implication $\Box p \to p$ to become true in a_7, we have to make p true in a_7. Reasoning backwards, we see that then also a_5 and a_6 need to have p, and, in turn, also a_2, a_3, and a_4. At last, because of that, also a_1 must be in the valuation of p.

2. (a) The first four levels of the complete binary tree $\mathcal{B}$:



(b) The given valuation V is that p holds at all and only the strings of even length. In order to show that the formula $\Box\Diamond p \to \Diamond\Box p$ is true throughout the model $(\mathcal{B}, V)$, we consider an arbitrary point $s \in \{0,1\}^*$ in this model, and assume $s \vDash \Box\Diamond p$. Our goal is to show $s \vDash \Diamond\Box p$. From $s \vDash \Box\Diamond p$ we obtain $s0 \vDash \Diamond p$ (and also $s1 \vDash \Diamond p$). In turn, $s0 \vDash \Diamond p$ means that $s00 \vDash p$ or $s01 \vDash p$. In both cases we see that s is of even length. This entails that $s0 \vDash \Box p$ because both children of $s0$ have even length. We conclude $s \vDash \Diamond\Box p$.

(c) Now we consider the valuation V' on $\mathcal{B}$, which makes p true at all strings that start with a 0, and make q true at all points that start with a 1. So p holds everywhere in the left immediate subtree of the root ε, whereas q holds in the entire right immediate subtree of ε. Moreover, $\varepsilon \notin V'(p)$ and $\varepsilon \notin V'(q)$.
We have to show that the formula

$$\Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$$

is not true in all points of the model $(\mathcal{B}, V')$. The only candidate to falsify this formula is ε, as it is the only one that sees a point with p (namely 0) and a point with q (namely 1), so that $\varepsilon \vDash \Diamond p \wedge \Diamond q$. All three disjuncts of the right-hand side of the implication are false in ε:

- $0 \not\models \Diamond q$ (by $00 \not\models q$ and $01 \not\models q$), so $0 \not\models p \wedge \Diamond q$. As $1 \not\models p$, also $1 \not\models p \wedge \Diamond q$. Hence $\varepsilon \not\models \Diamond(p \wedge \Diamond q)$.
- $0 \not\models q$, so $0 \not\models p \wedge q$, and $1 \not\models p$, so $1 \not\models p \wedge q$. Hence $\varepsilon \not\models \Diamond(p \wedge q)$.
- $0 \not\models q$, so $0 \not\models \Diamond p \wedge q$. $1 \not\models \Diamond p$ (by $10 \not\models p$ and $11 \not\models p$), so $1 \not\models \Diamond p \wedge q$. Hence $\varepsilon \not\models \Diamond(\Diamond p \wedge q)$.

(d) To show that $\Diamond\Diamond p \to \Diamond p$ is not valid in $\mathcal{B}$ we have to come up with a valuation V'' and a point x, such that $\mathcal{B}, V'', x \models \Diamond\Diamond p$ and $\mathcal{B}, V'', x \not\models \Diamond p$. For example, we can take $V''(p) = \{00\}$. Then $\mathcal{B}, V'', \varepsilon \models \Diamond\Diamond p$ (since $\mathcal{B}, V'', 0 \models \Diamond p$), but $\mathcal{B}, V'', \varepsilon \not\models \Diamond p$ (since $0 \not\models p$ and $1 \not\models p$). Another example is the model from (b).

3. (a) We have $\models \Box(p \to q) \to (\Diamond p \to \Diamond q)$.

(b) The formula $\Box(p \wedge q) \to (\Box p \wedge \Box q)$ is valid in all frames. Let $\mathcal{M} = (W, R, V)$ be an arbitrary model, x an arbitrary point of $\mathcal{M}$, and assume $x \models \Box(p \wedge q)$. In order to show $x \models \Box p$ we consider an arbitrary y with Rxy (so the goal is to show $y \models p$). From the assumption $x \models \Box(p \wedge q)$ we know that $y \models p \wedge q$, and so $y \models p$. The argument for $x \models \Box q$ is analogous (pick an arbitrary R-successor z of $x \ldots$). So we conclude $x \models \Box(p \wedge q) \to (\Box p \wedge \Box q)$. As $\mathcal{M}$ and x were arbitrary we thus have shown universal validity of the formula.

(c) The formula $\Box p \to \Diamond p$ is valid precisely in the serial frames, that is frames where every point has at least one successor. The one-point model where we have one world $\bullet$, and the empty accessibility relation forms a counterexample against universal validity: $\bullet \models \Box p$ but $\bullet \not\models \Diamond p$.

(e) $\Box(\Box p \to p) \to \Box p$. This is called Löb's formula, important in provability logic. It is not universally valid, as can be seen from the model $(\{a, b, c\}, \{(a, b), (b, c)\}, V)$ with $V(p) = \varnothing$. Here $b \models \Box p \to p$ since $b \not\models \Box p$ (as $c \not\models p$). Hence $a \models \Box(\Box p \to p)$ while $a \not\models \Box p$.