1. (a) We have to show: $\mathcal{F} \models \Diamond p \to \Box p$ if and only if $\mathcal{F}$ is partially functional.

   We start by proving the implication: if $\mathcal{F}$ is partially functional, then $\mathcal{F} \vDash \Diamond p \to \Box p$. Let $\mathcal{M} = (W, R, V)$ be an arbitrary model with R a partial function. Consider an arbitrary point x of $\mathcal{M}$ and assume $x \vDash \Diamond p$. This means that $y \vDash p$ for some y with Rxy. In order to show that also $x \vDash \Box p$, we consider an arbitrary R-successor z of x. Goal is $z \vDash p$. Because R is functional we know it has at most one R-successor. This means that $y = z$ is forced in our situation. Hence $z \vDash p$. We conclude that $\Diamond p \to \Box p$ is valid in all frames (W, R) with R a partial function.

   For the other direction, assume, contrapositively, that (W, R) is not partially functional, i.e., there is a point x that has two R-successors, say y and z. So we have points x, y, z such that Rxy and Rxz and $y \neq z$. In order to falsify the formula in x we choose a valuation V that makes p true in y but not in z. For example $V(p) = \{y\}$ (alternatively, we can take $V(p) = W \setminus \{z\}$). Then, in the model (W, R, V), we have $x \vDash \Diamond p$ (due to Rxy and $y \vDash p$), but $x \nvDash \Box p$ (due to Rxz and $z \nvDash p$). Hence $x \nvDash \Diamond p \to \Box p$, and thus we have shown that $\Diamond p \to \Box p$ is not valid in frames whose transition relation is not partially functional.

   (b)

2. Given are $\mathcal{N} = (\mathbb{N}, S)$ with $S = \{(n, n+1) \mid n \in \mathbb{N}\}$, and $\mathcal{B} = (\{0,1\}^*, R)$ with $R = \{(s, sa) \mid s \in \{0,1\}^*, a \in \{0,1\}\}$.

   (a) The valuation V on $\mathcal{B}$ is $V(p) = \{s \in \{0,1\}^* \mid s \text{ is of even length}\}$. We have to define a valuation U on $\mathcal{N}$ such that $\mathcal{B}, V, \varepsilon$ and $\mathcal{N}, U, 0$ are bisimilar. We define

   $$U(p) = \{n \in \mathbb{N} \mid n \text{ is even}\}.$$

   Next we show that $\mathcal{B}, V, \varepsilon \leftrightarrow \mathcal{N}, U, 0$. We do this by defining a relation $E \subseteq \{0,1\}^* \times \mathbb{N}$, show that E is a bisimulation indeed, and that $(\varepsilon, 0) \in E$. Define

$$E = \{(s, n) \mid n = |s|\}$$

where by $|s|$ we denote the length of a string $s \in \{0,1\}^*$, for example $|\varepsilon| = 0$ and $|011| = 3$.

We fix an arbitrary pair $(s, n) \in E$ and show that it fulfills the three bisimulation conditions. First of all we have $s \in V(p)$ iff $|s|$ is even iff n is even iff $n \in V(p)$, so atomic harmony is satisfied. To verify the *zig* clause, consider a step from s in the $\mathcal{B}$ frame, Rst. Then $t = sa$ with $a \in \{0,1\}$. In both cases ($a = 0$ or $a = 1$) we find $n+1$ as the witnessing point in the $\mathcal{N}$ frame: clearly $(sa, n+1) \in E$ ($|sa| = |s| + 1 = n + 1$ since we started with $(s, n) \in E$) and $(n, n+1) \in S$. For the *zag* clause: consider the step $(n, n+1) \in S$ in the model $(\mathcal{N}, U)$. A corresponding step in the model $(\mathcal{B}, V)$ is $(s, s0) \in R$ as we have that $(s0, n+1) \in E$. (Taking $s1$ would of course be equally good.)

So we have shown that E is bisimulation between the models $(\mathcal{B}, V)$ and $(\mathcal{N}, U)$. Moreover we clearly have $(\varepsilon, 0) \in E$. Hence we have shown $\mathcal{B}, V, \varepsilon \leftrightarrows \mathcal{N}, U, 0$.

To see that U is the *only* valuation on $\mathcal{N}$ such that there exists a bisimulation between $\mathcal{B}, V, \varepsilon$ and $\mathcal{N}, U, 0$, let U' be some valuation on $\mathcal{N}$ and assume that $\mathcal{B}, V, \varepsilon \leftrightarrows \mathcal{N}, U', 0$. Let E' be the witnessing bisimulation, so with $(\varepsilon, 0) \in E'$. We show that for all $n \in \mathbb{N}$, $(0^n, n) \in E'$ by induction on n. Then it follows that (as E' satisfies the condition of atomic harmony), for all $n \in \mathbb{N}$, $n \in U'(p)$ iff $0^n \in V(p)$ iff $|0^n|$ is even iff n is even, so that $U' = U$. The base case $n = 0$ we get for free. Let $n = n' + 1$ and assume (induction hypothesis) we have $(0^{n'}, n') \in E'$. By E' satisfying the zig-condition, and $(0^{n'}, 0^{n'+1}) \in R$, there is m such that $(n', m) \in S$ and $(0^{n'+1}, m) \in E$. By definition of S we get that $m = n' + 1$, as desired.

(b) Let V' be the valuation on $\mathcal{B}$ defined by

$$V'(p) = \{\, 0w \mid w \in \{0,1\}^* \,\} \qquad V'(q) = \{\, 1w \mid w \in \{0,1\}^* \,\} \ .$$

We will show that there exists no valuation U' on $\mathcal{N}$ such that $\mathcal{B}, V', \varepsilon \leftrightarrows \mathcal{N}, U', 0$. Assume, to the contrary, such a valuation U' would exist. Let $Z \subseteq \{0,1\}^* \times \mathbb{N}$ be the witnessing bisimulation. Since $(\varepsilon, 0) \in Z$ and $(\varepsilon, 0) \in R$ we obtain that $(0, 1) \in Z$ by the *zig* condition (1 is the only S-successor of 0). Likewise we obtain $(1,1) \in Z$. Since Z fulfills the property of atomic harmony we get $0 \in V'(p)$ iff $1 \in U'(p)$, and at the same time $1 \in V'(p)$ iff $1 \in U'(p)$. Since $0 \in V'(p)$ and $1 \notin V'(p)$ this leads to a contradiction.

3. Assume there would be a bisimulation Z linking the root nodes w and w' of the frames $\mathcal{F}$ and $\mathcal{F}'$ as depicted. We show this leads to a contradiction.

   We denote the transition relations of both models by $\rightarrow$ Now let $w_{k,1}, \ldots, w_{k,k}$ be the (non-root) nodes on a maximal length-k path in $\mathcal{F}$, so that $w \rightarrow w_{k,1}$ and $w_{k,i} \rightarrow w_{k,i+1}$ for all $1 \leq i < k$, and $w_{k,k}$ blind. Similarly, the points of $\mathcal{F}'$ are named $w'_{k,i}$ $(1 \leq i \leq k)$, and those on the infinite path: $w'_{\infty,1}, w'_{\infty,2}, \ldots$.

   As we assumed wZw' and Z satisfies the *zag* clause of bisimulations, there must be a $k \geq 1$ such that the step $w' \rightarrow w'_{\infty,1}$ in $\mathcal{F}'$ (the first step on the infinite path) is matched by the step $w \rightarrow w_{k,1}$ in $\mathcal{F}$, i.e., so that $w_{k,1}Zw'_{\infty,1}$. Using the *zig* (or *zag*) clauses $k-1$ more times, we arrive at the situation $w_{k,k}Zw'_{\infty,k}$. But this cannot be: $w_{k,k}$ is blind whereas $w'_{\infty,k}$ has the successor $w'_{\infty,k+1}$.

4. (a) An example of a formula φ and a model $\mathcal{M}$ such that neither $\mathcal{M} \vDash \varphi$ nor $\mathcal{M} \vDash \neg\varphi$ is $\varphi = p$ with $\mathcal{M} = (\{a,b\}, \varnothing, V)$ and $V(p) = \{a\}$. Then $\mathcal{M} \nvDash p$ because $\mathcal{M}, b \nvDash p$, but also $\mathcal{M} \nvDash \neg p$ due to $\mathcal{M}, a \nvDash \neg p$.

   (b) An example of a formula φ, a frame $\mathcal{F}$ and two models $\mathcal{M}$ and $\mathcal{M}'$ based on $\mathcal{F}$ such that $\mathcal{M} \vDash \varphi$ and $\mathcal{M}' \vDash \neg\varphi$ is as follows: Let $\mathcal{F} = (\{a\}, \varnothing)$ and let V and V' be such that $V(p) = \{a\}$ and $V'(p) = \varnothing$. Then we have $(\mathcal{F}, V) \vDash p$ and $(\mathcal{F}, V') \vDash \neg p$.

5. For example, $\Diamond\Diamond\top$ is true only in state 3.

6.