1. (a) We have to show: $F \models \Diamond p \to \Box p$ if and only if $F$ is partially functional.

We start by proving the implication: if $F$ is partially functional, then $F \not\models \Diamond p \to \Box p$. Let $M = (W, R, V)$ be an arbitrary model with $R$ a partial function. Consider an arbitrary point $x$ of $M$ and assume $x \not\models \Diamond p$. This means that $y \not\models p$ for some $y$ with $Rxy$. In order to show that also $x \not\models \Box p$, we consider an arbitrary $R$-successor $z$ of $x$. Goal is $z \models p$. Because $R$ is functional we know it has at most one $R$-successor. This means that $y = z$ is forced in our situation. Hence $z \models p$. We conclude that $\Diamond p \to \Box p$ is valid in all frames $(W, R)$ with $R$ a partial function.

For the other direction, assume, contrapositively, that $(W, R)$ is not partially functional, i.e., there is a point $x$ that has two $R$-successors, say $y$ and $z$. So we have points $x, y, z$ such that $Rxy$ and $Rxz$ and $y \neq z$. In order to falsify the formula in $x$ we choose a valuation $V$ that makes $p$ true in $y$ but not in $z$. For example $V(p) = \{y\}$ (alternatively, we can take $V(p) = W \setminus \{z\}$). Then, in the model $(W, R, V)$, we have $x \models \Diamond p$ (due to $Rxy$ and $y \models p$), but $x \not\models \Box p$ (due to $Rxz$ and $z \not\models p$). Hence $x \not\models \Diamond p \to \Box p$, and thus we have shown that $\Diamond p \to \Box p$ is not valid in frames whose transition relation is not partially functional.

(b)

2. Given are $\mathcal{N} = (\mathbb{N}, S)$ with $S = \{(n, n + 1) \mid n \in \mathbb{N}\}$, and $\mathcal{B} = (\{0, 1\}^*, R)$ with $R = \{(s, sa) \mid s \in \{0, 1\}^*, a \in \{0, 1\}\}$.

(a) The valuation $V$ on $\mathcal{B}$ is $V(p) = \{s \in \{0, 1\}^* \mid s \text{ is of even length}\}$. We have to define a valuation $U$ on $\mathcal{N}$ such that $\mathcal{B}, V, \varepsilon$ and $\mathcal{N}, U, 0$ are bisimilar. We define

$$U(p) = \{n \in \mathbb{N} \mid n \text{ is even}\}.$$ 

Next we show that $\mathcal{B}, V, \varepsilon \leftrightarrow \mathcal{N}, U, 0$. We do this by defining a relation $E \subseteq \{0, 1\}^* \times \mathbb{N}$, show that $E$ is a bisimulation indeed,
and that \((\varepsilon, 0) \in E\). Define

\[ E = \{(s, n) \mid n = |s|\} \]

where by \(|s|\) we denote the length of a string \(s \in \{0, 1\}^*\), for example \(|\varepsilon| = 0\) and \(|011| = 3\).

We fix an arbitrary pair \((s, n) \in E\) and show that it fulfills the three bisimulation conditions. First of all we have \(s \in V(p)\) iff \(|s|\) is even iff \(n\) is even iff \(n \in V(p)\), so atomic harmony is satisfied.

To verify the zig clause, consider a step from \(s\) in the \(B\) frame, \(Rst\). Then \(t = sa\) with \(a \in \{0, 1\}\). In both cases \((a = 0\) or \(a = 1)\) we find \(n + 1\) as the witnessing point in the \(N\) frame: clearly \((sa, n + 1) \in E\) (\(|sa| = |s| + 1 = n + 1\) since we started with \((s, n) \in E\)) and \((n, n + 1) \in S\). For the zag clause: consider the step \((n, n + 1) \in S\) in the model \((N, U)\). A corresponding step in the model \((B, V)\) is \((s, s0) \in R\) as we have that \((s0, n + 1) \in E\). (Taking \(s1\) would of course be equally good.)

So we have shown that \(E\) is bisimulation between the models \((B, V)\) and \((N, U)\). Moreover we clearly have \((\varepsilon, 0) \in E\). Hence we have shown \(B, V, \varepsilon \leftrightarrow N, U, 0\).

To see that \(U\) is the only valuation on \(N\) such that there exists a bisimulation between \(B, V, \varepsilon\) and \(N, U, 0\), let \(U'\) be some valuation on \(N\) and assume that \(B, V, \varepsilon \leftrightarrow N', U', 0\). Let \(E'\) be the witnessing bisimulation, so with \((\varepsilon, 0) \in E'\). We show that for all \(n \in \mathbb{N}\), \((0^n, n) \in E'\) by induction on \(n\). Then it follows that (as \(E'\) satisfies the condition of atomic harmony), for all \(n \in \mathbb{N}\), \((0^n, n) \in U'(p)\) iff \(0^n \in V(p)\) iff \(|0^n|\) is even iff \(n\) is even, so that \(U' = U\). The base case \(n = 0\) we get for free. Let \(n = n' + 1\) and assume (induction hypothesis) we have \((0^n, n') \in E'\). By \(E'\) satisfying the zig-condition, and \((0^{n'}, 0^{n'+1}) \in R\), there is \(m\) such that \((n', m) \in S\) and \((0^{n'+1}, m) \in E\). By definition of \(S\) we get that \(m = n' + 1\), as desired.

(b) Let \(V'\) be the valuation on \(B\) defined by

\[ V'(p) = \{0w \mid w \in \{0, 1\}^*\} \quad V'(q) = \{1w \mid w \in \{0, 1\}^*\} \]

We will show that there exists no valuation \(U'\) on \(N\) such that \(B, V', \varepsilon \leftrightarrow N', U', 0\). Assume, to the contrary, such a valuation \(U'\) would exist. Let \(Z \subseteq \{0, 1\}^* \times \mathbb{N}\) be the witnessing bisimulation. Since \((\varepsilon, 0) \in Z\) and \((\varepsilon, 0) \in R\) we obtain that \((0, 1) \in Z\) by the
zig condition (1 is the only $S$-successor of 0). Likewise we obtain $(1, 1) \in Z$. Since $Z$ fulfills the property of atomic harmony we get $0 \in V'(p)$ iff $1 \in U'(p)$, and at the same time $1 \in V'(p)$ iff $1 \in U'(p)$. Since $0 \in V'(p)$ and $1 \not\in V'(p)$ this leads to a contradiction.

3. Assume there would be a bisimulation $Z$ linking the root nodes $w$ and $w'$ of the frames $F$ and $F'$ as depicted. We show this leads to a contradiction.

We denote the transition relations of both models by $\to$. Now let $w_{k,1}, \ldots, w_{k,k}$ be the (non-root) nodes on a maximal length-$k$ path in $F$, so that $w \to w_{k,1}$ and $w_{k,i} \to w_{k,i+1}$ for all $1 \leq i < k$, and $w_{k,k}$ blind. Similarly, the points of $F'$ are named $w'_{k,i}$ ($1 \leq i \leq k$), and those on the infinite path: $w'_{\infty,1}, w'_{\infty,2}, \ldots$.

As we assumed $wZw'$ and $Z$ satisfies the zag clause of bisimulations, there must be a $k \geq 1$ such that the step $w' \to w'_{\infty,1}$ in $F'$ (the first step on the infinite path) is matched by the step $w \to w_{k,1}$ in $F$, i.e., so that $w_{k,1}Zw'_{\infty,1}$. Using the zig (or zag) clauses $k - 1$ more times, we arrive at the situation $w_{k,k}Zw'_{\infty,k}$. But this cannot be: $w_{k,k}$ is blind whereas $w'_{\infty,k}$ has the successor $w'_{\infty,k+1}$.

4. (a) An example of a formula $\varphi$ and a model $M$ such that neither $M \models \varphi$ nor $M \models \neg \varphi$ is $\varphi = p$ with $M = (\{a, b\}, \emptyset, V)$ and $V(p) = \{a\}$. Then $M \not\models p$ because $M, b \not\models p$, but also $M \not\models \neg p$ due to $M, a \not\models \neg p$.

(b) An example of a formula $\varphi$, a frame $F$ and two models $M$ and $M'$ based on $F$ such that $M \models \varphi$ and $M' \models \neg \varphi$ is as follows: Let $F = (\{a\}, \emptyset)$ and let $V$ and $V'$ be such that $V(p) = \{a\}$ and $V'(p) = \emptyset$. Then we have $(F, V) \models p$ and $(F, V') \models \neg p$.

5. For example, $\Diamond \Diamond \top$ is true only in state 3.

6.