Exercises

1. Consider the models M and K of MLOM p30: In M we have $W_M = \{a, b, c, d\}$ and $R_M = \{(a, b), (a, c), (c, a), (c, d)\}$. In K we have $W_K = \{1, 2, 3, 4\}$ and $R_K = \{(1, 2), (2, 3), (3, 1), (1, 4)\}$.

   Show that M, a and $K, 1$ are not bisimilar using the game approach.

   Answer:

   Round 0: $(a, 1)$. Local harmony is ok.

   Round 1: S takes $\mathcal{M}$ and $a \to c$. D must take $\mathcal{K}$ and takes the step $1 \to 2$. The next pair is $(c, 2)$ for which local harmony is ok.

   Round 2: S takes $\mathcal{M}$ and $c \to d$. D must take $\mathcal{K}$ and takes the step $2 \to 3$. The next pair is $(d, 3)$ for which local harmony is ok.

   Round 3: S takes $\mathcal{K}$ and $3 \to 1$. D is stuck. So S wins.

   Give a modal formula that distinguishes between M, a and $K, 1$.

   Answer: Duplication can win up to round 2. We need a formula of modal depth 3 to distinguish the states a and 1. We can take the formule $\Diamond\Diamond\Box\bot$.

2. A frame is said to be connected if $\forall xy\,(Rxy \vee x = y \vee Ryx)$. Show that the property of 'connectedness' is not modally definable.

   Answer: Suppose that connectedness is definable by a modal formula ϕ. We consider two disjoint connected frames $\mathcal{F}_1$ and $\mathcal{F}_2$. (There exist connected frames, and we can make sure two of them are disjoint.) We have $\mathcal{F}_1 \models \phi$ and $\mathcal{F}_2 \models \phi$ by the assumption. Let $\mathcal{F}_1 \uplus \mathcal{F}_2$ be the disjoint union of the two connected frames. We aim to show that ϕ is valid in this disjoint union. Let V be a valuation for $\mathcal{F}_1 \uplus \mathcal{F}_2$ and let x be a state in $\mathcal{F}_1 \cup \mathcal{F}_2$. The state x is either a state of $\mathcal{F}_1$ or of $\mathcal{F}_2$. Assume without loss of generality that x is a state in $\mathcal{F}_1$., Because $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, we can (details omitted) split the valuation V into V_1 and V_2 such that the union of $\mathcal{F}_1, V_1$ and $\mathcal{F}_2, V_2$ is exactly the model $\mathcal{F}_1 \uplus \mathcal{F}_2, V$. We have $\mathcal{F}_1, V_1, x \models \phi$. We have (by the result on disjoint unions) $\mathcal{F}_1, V_1, x$ bisimular with $\mathcal{F}_1 \uplus \mathcal{F}_2, V, x$. Hence $\mathcal{F}_1 \uplus \mathcal{F}_2, V, x \models \phi$.

Then by the assumption on ϕ we have that $\mathcal{F}_1 \uplus \mathcal{F}_2$ is connected. This is not the case, so we derive a contradiction.

3. Consider the left model of Question 1(a) in MLOM p35: $W = \{1, 2, 3\}$ and $R = \{(1, 2), (1, 3), (2, 3), (3, 1), (3, 2)\}$, and $V(p) = \{2\}$.

   Show that the worlds 1 and 3 are bisimilar.

   (You do not have to prove that the bisimulations you propose are actually bisimulations.)

   Answer:

   We define $Z = \{(1, 3), (3, 1), (2, 2)\}$. This is a bisimulation and contains $(1, 3)$.

   Show that the worlds 1 and 2 are not bisimilar.

   Answer: These states are not bisimilar $2 \models p$ but $1 \not\models p$ so local harmony cannot be satisfied.

   Give the bisimulation contraction of the model.

   Answer:

   The bisimulation contraction contains two states: the state $|1| = |3|$ and the state $|2|$. There are three arrows: $|1| \to |1|$ and $|1| \to |2|$ and $|2| \to |1|$.

4. Consider the models $\mathcal{M}_1 = ((W_1, R_1), V_1)$ and $\mathcal{M}_2 = ((W_2, R_2), V_2)$ in Exercise 1 in MLOM p35. We use the notation $W_1 = \{1, 2, 3\}$, $R_1 = \{(1, 2), (1, 3), (2, 3), (3, 1), (3, 2)\}$, V_1 with $V_1(p) = \{2\}$, and $W_2 = \{a, b\}$, $R_2 = \{(a, a), (a, b), (b, a)\}$, V_2 with $V_2(p) = \{b\}$.

   Give the tree unravelling of $\mathcal{M}_1$ in 1, and the tree unravelling of $\mathcal{M}_2$ in a.

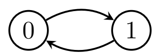
   Is the state 1 in the tree unravelling of $\mathcal{M}_1$ bisimilar with the state a in the treee unravelling of $\mathcal{M}_2$?

   What can we conclude about those states in their original models?

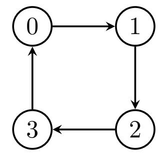
5. For $n = 1, 2, \ldots$, let the 'looping frame' $\mathcal{L}_n = (W_n, R_n)$ be defined by

$$W_n = \{0, \ldots, n-1\}$$
$$R_n = \{(k, k') | k' = k + 1 \text{ if } k + 1 < n \text{ and } k' = 0 \text{ otherwise}\}$$

   (a) Draw the frames $\mathcal{L}_2$ and $\mathcal{L}_4$.
   Answer:



$\mathcal{L}_2$



$\mathcal{L}_4$

(b) Give a modal formula that distinguishes frame $\mathcal{L}_2$ from $\mathcal{L}_4$, that is, a formula ϕ such that $\mathcal{L}_2 \models \phi$ and $\mathcal{L}_4 \not\models \phi$. Prove your answer.

Answer:

An example of a formula that is valid in $\mathcal{L}_2$ but not in $\mathcal{L}_4$ is $p \leftrightarrow \Diamond\Diamond p$.[1] To see that $\mathcal{L}_2 \vDash p \leftrightarrow \Diamond\Diamond p$ (more generally $\mathcal{L}_n \vDash p \leftrightarrow \Diamond^n p$ for all $n > 0$), observe that $R_2^2 = (R_2; R_2) = \mathrm{Id}$ (in general $R_n^n = \mathrm{Id}$), where Id denotes the identity relation on W_2, $\mathrm{Id} = \{(x, x) \mid x \in W_2\}$. Let $x \in W_2 = \{0, 1\}$, and V an arbitrary valuation on $\mathcal{L}_2$. Then in the model $\mathcal{M} = (\mathcal{L}_2, V)$, it clearly holds that $x \vDash p$ if and only if $x \vDash \Diamond\Diamond p$.

To see that $\mathcal{L}_4 \nvDash p \leftrightarrow \Diamond\Diamond p$ (more generally $\mathcal{L}_m \nvDash p \leftrightarrow \Diamond^n p$ for all $n, m > 0$ with $n < m$), consider for example the valuation V on $\mathcal{L}_4$ defined by $V(p) = \{0\}$. In the model $(\mathcal{L}_4, V)$ we then have $0 \vDash p$, but $0 \nvDash \Diamond\Diamond p$ since $2 \nvDash p$ (for $n < m$, in the model $(\mathcal{L}_m, V)$, with $V(p) = \{0\}$, we similarly get $0 \nvDash \Diamond^n p$ since $n \nvDash p$).

For questions **(c)** and **(d)** you have to define a bisimulation, but reporting on the verification of the bisimulation conditions is not required.

(c) Let $\mathcal{M}_2$ be some model based on $\mathcal{L}_2$. Define a model $\mathcal{M}_4$ based on $\mathcal{L}_4$ such that $\mathcal{M}_2, 0$ is bisimilar with $\mathcal{M}_4, 0$.

Answer:

Let $\mathcal{M}_2 = (\mathcal{L}_2, V_2)$ for some valuation V_2. We define a valuation V_4 on $\mathcal{L}_4$ by

$$V_4(p) = \{x \mid (x \bmod 2) \in V_2\}$$

(So $V_4(p) = \varnothing$ if $V_2(p) = \varnothing$, $V_4(p) = \{0, 2\}$ if $V_2(p) = \{0\}$, $V_4(p) = \{1, 3\}$ if $V_2(p) = \{1\}$, and $V_4(p) = W_4$ if $V_2(p) = W_2$.) Then the relation $E \subseteq W_2 \times W_4$ given by $E = \{(0, 0), (0, 2), (1, 1), (1, 3)\}$ is a bisimulation between the models $\mathcal{M}_2$ and $\mathcal{M}_4 = (\mathcal{L}_4, V_4)$. As $(0, 0) \in E$, we conclude $\mathcal{M}_2, 0 \leftrightarroweq \mathcal{M}_4, 0$.

[1] Other examples are $p \rightarrow \Diamond\Diamond p$, $\Diamond\Diamond p \rightarrow p$, or any variation of them obtained by replacing $\Diamond$'s with $\Box$'s (whose meanings coincide on these functional frames).

(d) Let $\mathcal{M}_3$ be some model based on $\mathcal{L}_3$. Define an acyclic model $\mathcal{N}$ bisimilar to $\mathcal{M}_3$.

Answer:

Every model is bisimilar to its tree unfolding (and trees are acyclic). Let $\mathcal{M}_3 = (\mathcal{L}_3, V)$. The tree unfolding of $\mathcal{M}_3$ is the model $\mathcal{N} = (\mathbb{N}, S, U)$ where $S = \{(n, n+1) \mid n \in \mathbb{N}\}$, and with valuation U defined for all variables p, and all $n \in \mathbb{N}$ by

$$\begin{aligned} 3n \in U(p) &\iff 0 \in V(p) \\ 3n+1 \in U(p) &\iff 1 \in V(p) \\ 3n+2 \in U(p) &\iff 2 \in V(p) \end{aligned}$$

Then the relation $\{(0, 3n), (1, 3n+1), (2, 3n+2)\}$ is a bisimulation between the models $\mathcal{M}_3$ and $\mathcal{N}$.

6. Formulate the box version of "Modal Decomposition" (MLOM, p. 41), i.e., give necessary and sufficient conditions for validity of a modal sequent of the form

$$\vec{p}, \Box\varphi_1, \ldots, \Box\varphi_k \implies \Box\psi_1, \ldots, \Box\psi_m, \vec{q}$$

Answer:

The modal sequent

$$\vec{p}, \Box\varphi_1, \ldots, \Box\varphi_k \implies \Box\psi_1, \ldots, \Box\psi_m, \vec{q}$$

is valid iff either

(a) $\vec{p}$ and $\vec{q}$ overlap, or

(b) the sequent $\varphi_1, \ldots, \varphi_k \implies \psi_j$ is valid, for some $j \in \{1, \ldots, m\}$.

7. Use sequents to investigate the validity of the following prop1 formulas:

(a) $(p \to q) \vee (q \to p)$:

Answer:

$$\begin{aligned} &\Rightarrow (p \to q) \vee (q \to p) \\ &\Rightarrow (\neg p \vee q) \vee (\neg q \vee p) \\ &\Rightarrow (\neg p \vee q), (\neg q \vee p) \\ &\Rightarrow \neg p, q, (\neg q \vee p) \\ &\Rightarrow \neg p, q, \neg q, p \\ p, q &\Rightarrow p, q \end{aligned}$$

This sequent is valid, hence the formula $(p \to q) \vee (q \to p)$ is valid.

(b) $p \to q \to p$

Answer:

We have $p \to q \to p \equiv \neg p \vee (q \to p) \equiv \neg p \vee (\neg q \vee p)$.

We transform sequents:

$$\begin{array}{rcl} & \Rightarrow & \neg p \vee (\neg q \vee p) \\ & \Rightarrow & \neg p, (\neg q \vee p) \\ p & \Rightarrow & \neg q \vee p \\ p & \Rightarrow & \neg q, p \\ p, q & \Rightarrow & p \end{array}$$

The sequent is valid, hence the formula $p \to q \to p$ is valid.

(c) $(p \vee q) \to (p \wedge q)$

Answer:

$$\begin{array}{rcl} & \Rightarrow & \neg(p \vee q) \vee (p \wedge q) \\ & \Rightarrow & \neg(p \vee q), (p \wedge q) \\ p \vee q & \Rightarrow & p \wedge q \end{array}$$

We get two sequents. The first one is $p \Rightarrow p \wedge q$ which has as successor two sequents: $p \Rightarrow p$ and $p \to q$. The first one is valid, the second one is not valid. This already gives that the sequent is not valid.

The second one is $q \Rightarrow p \wedge q$ which has as successor two sequents: $q \Rightarrow p$ and $q \Rightarrow q$. This first one is not valid.

(d) $((p \to q) \to p) \to p$

Answer:

We have $((p \to q) \to p) \to p \equiv \neg((p \to q) \to p) \vee p \equiv \neg(\neg(p \to q) \vee p) \vee p \equiv \neg(\neg(\neg p \vee q) \vee p) \vee p$.

We transform sequents:

$$\begin{array}{rcl} & \Rightarrow & \neg(\neg(\neg p \vee q) \vee p) \vee p \\ & \Rightarrow & \neg(\neg(\neg p \vee q) \vee p), p \\ \neg(\neg p \vee q) \vee p & \Rightarrow & p \end{array}$$

We get two new sequents to consider. The first one:

$$\begin{array}{rcl} \neg(\neg p \vee q) & \Rightarrow & p \\ & \Rightarrow & p, \neg p \vee q \\ & \Rightarrow & p, \neg p, q \\ p & \Rightarrow & p, q \end{array}$$

This sequent is valid. We continue with the second one:

$$p \quad \Rightarrow \quad p$$

This sequent is also valid. We conclude that the sequent $\Rightarrow \neg(\neg(\neg p \vee q) \vee p) \vee p$ is valid. Hence the initial formula $((p \rightarrow q) \rightarrow p) \rightarrow p$ (the law of Peirce) is valid.

(e) $\Diamond p \rightarrow \Box p$

(f) $\Box p \rightarrow \Diamond p$

(g) $(p \rightarrow q) \vee (\Box p \vee \Diamond p)$