Exercises

1. Consider the models $M$ and $K$ of MLOM p30: In $M$ we have $W_M = \{a, b, c, d\}$ and $R_M = \{(a, b), (a, c), (c, a), (c, d)\}$. In $K$ we have $W_K = \{1, 2, 3, 4\}$ and $R_K = \{(1, 2), (2, 3), (3, 1), (1, 4)\}$.

Show that $M, a$ and $K, 1$ are not bisimilar using the game approach.

Answer:

Round 0: $(a, 1)$. Local harmony is ok.

Round 1: $S$ takes $M$ and $a \rightarrow c$. $D$ must take $K$ and takes the step $1 \rightarrow 2$. The next pair is $(c, 2)$ for which local harmony is ok.

Round 2: $S$ takes $M$ and $c \rightarrow d$. $D$ must take $K$ and takes the step $2 \rightarrow 3$. The next pair is $(d, 3)$ for which local harmony is ok.


Give a modal formula that distinguishes between $M, a$ and $K, 1$.

Answer: Duplication can win up to round 2. We need a formula of modal depth 3 to distinguish the states $a$ and $1$. We can take the formula $\Box \Diamond \Box \bot$.

2. A frame is said to be connected if $\forall xy (Rxy \vee x = y \vee Ryx)$. Show that the property of ‘connectedness’ is not modally definable.

Answer: Suppose that connectedness is definable by a modal formula $\phi$. We consider two disjoint connected frames $\mathcal{F}_1$ and $\mathcal{F}_2$. (There exist connected frames, and we can make sure two of them are disjoint.) We have $\mathcal{F}_1 \models \phi$ and $\mathcal{F}_2 \models \phi$ by the assumption. Let $\mathcal{F}_1 \uplus \mathcal{F}_2$ be the disjoint union of the two connected frames. We aim to show that $\phi$ is valid in this disjoint union. Let $V$ be a valuation for $\mathcal{F}_1 \uplus \mathcal{F}_2$ and let $x$ be a state in $\mathcal{F}_1 \cup \mathcal{F}_2$. The state $x$ is either a state of $\mathcal{F}_1$ or of $\mathcal{F}_2$. Assume without loss of generality that $x$ is a state in $\mathcal{F}_1$. Because $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, we can (details omitted) split the valuation $V$ into $V_1$ and $V_2$ such that the union of $\mathcal{F}_1, V_1$ and $\mathcal{F}_2, V_2$ is exactly the model $\mathcal{F}_1 \uplus \mathcal{F}_2, V$. We have $\mathcal{F}_1, V_1, x \models \phi$. We have (by the result on disjoint unions) $\mathcal{F}_1, V_1, x$ bisimilar with $\mathcal{F}_1 \uplus \mathcal{F}_2, V, x$. Hence $\mathcal{F}_1 \uplus \mathcal{F}_2, V, x \models \phi$. 

Then by the assumption on \( \phi \) we have that \( F_1 \uplus F_2 \) is connected. This is not the case, so we derive a contradiction.

3. Consider the left model of Question 1(a) in MLOM p35: \( W = \{1, 2, 3\} \) and \( R = \{(1, 2), (1, 3), (2, 3), (3, 1), (3, 2)\} \), and \( V(\rho) = \{2\} \).

Show that the worlds 1 and 3 are bisimilar.

(You do not have to prove that the bisimulations you propose are actually bisimulations.)

\textit{Answer:}

We define \( Z = \{(1, 3), (3, 1), (2, 2)\} \). This is a bisimulation and contains \((1, 3)\).

Show that the worlds 1 and 2 are not bisimilar.

\textit{Answer:} These states are not bisimilar \( 2 \models p \) but \( 1 \not\models p \) so local harmony cannot be satisfied.

Give the bisimulation contraction of the model.

\textit{Answer:}

The bisimulation contraction contains two states: the state \( |1| = |3| \) and the state \( |2| \). There are three arrows: \( |1| \to |1| \) and \( |1| \to |2| \) and \( |2| \to |1| \).

4. Consider the models \( \mathcal{M}_1 = ((W_1, R_1), V_1) \) and \( \mathcal{M}_2 = ((W_2, R_2), V_2) \) in Exercise 1 in MLOM p35. We use the notation \( W_1 = \{1, 2, 3\} \), \( R_1 = \{(1, 2), (1, 3), (2, 3), (3, 1), (3, 2)\} \), \( V_1 \) with \( V_1(\rho) = \{2\} \), and \( W_2 = \{a, b\} \), \( R_2 = \{(a, a), (a, b), (b, a)\} \), \( V_2 \) with \( V_2(\rho) = \{b\} \).

Give the tree unravelling of \( \mathcal{M}_1 \) in 1, and the tree unravelling of \( \mathcal{M}_2 \) in a.

Is the state 1 in the tree unravelling of \( \mathcal{M}_1 \) bisimilar with the state a in the tree unravelling of \( \mathcal{M}_2 \)?

What can we conclude about those states in their original models?

5. For \( n = 1, 2, \ldots \), let the ‘looping frame’ \( \mathcal{L}_n = (W_n, R_n) \) be defined by

\[
W_n = \{0, \ldots, n-1\} \\
R_n = \{(k, k')| k' = k + 1 \text{ if } k + 1 < n \text{ and } k' = 0 \text{ otherwise}\}
\]

(a) Draw the frames \( \mathcal{L}_2 \) and \( \mathcal{L}_4 \).

\textit{Answer:}
(b) Give a modal formula that distinguishes frame $\mathcal{L}_2$ from $\mathcal{L}_4$, that is, a formula $\phi$ such that $\mathcal{L}_2 \models \phi$ and $\mathcal{L}_4 \not\models \phi$. Prove your answer.

**Answer:**

An example of a formula that is valid in $\mathcal{L}_2$ but not in $\mathcal{L}_4$ is $p \leftrightarrow \Diamond \Diamond p$.\(^1\) To see that $\mathcal{L}_2 \models p \leftrightarrow \Diamond \Diamond p$ (more generally $\mathcal{L}_n \models p \leftrightarrow \Diamond^n p$ for all $n > 0$), observe that $R_2^2 = (R_2; R_2) = \text{Id}$ (in general $R_n^n = \text{Id}$), where $\text{Id}$ denotes the identity relation on $W_2$, $\text{Id} = \{(x, x) \mid x \in W_2\}$. Let $x \in W_2 = \{0, 1\}$, and $V$ an arbitrary valuation on $\mathcal{L}_2$. Then in the model $M = (\mathcal{L}_2, V)$, it clearly holds that $x \models p$ if and only if $x \models \Diamond \Diamond p$.

To see that $\mathcal{L}_4 \not\models p \leftrightarrow \Diamond \Diamond p$ (more generally $\mathcal{L}_n \not\models p \leftrightarrow \Diamond^n p$ for all $n > 0$), consider for example the valuation $V$ on $\mathcal{L}_4$ defined by $V(p) = \{0\}$. In the model $(\mathcal{L}_4, V)$ we then have $0 \models p$, but $0 \not\models \Diamond \Diamond p$ since $2 \models p$ (for $n < m$, in the model $(\mathcal{L}_m, V)$, with $V(p) = \{0\}$, we similarly get $0 \not\models \Diamond^n p$ since $n \not\models p$).

For questions (c) and (d) you have to define a bisimulation, but reporting on the verification of the bisimulation conditions is not required.

(c) Let $\mathcal{M}_2$ be some model based on $\mathcal{L}_2$. Define a model $\mathcal{M}_4$ based on $\mathcal{L}_4$ such that $\mathcal{M}_2, 0$ is bisimilar with $\mathcal{M}_4, 0$.

**Answer:**

Let $\mathcal{M}_2 = (\mathcal{L}_2, V_2)$ for some valuation $V_2$. We define a valuation $V_4$ on $\mathcal{L}_4$ by

$$V_4(p) = \{x \mid (x \text{ mod } 2) \in V_2\}$$

(So $V_4(p) = \emptyset$ if $V_2(p) = \emptyset$, $V_4(p) = \{0, 2\}$ if $V_2(p) = \{0\}$, $V_4(p) = \{1, 3\}$ if $V_2(p) = \{1\}$, and $V_4(p) = W_4$ if $V_2(p) = W_2$.) Then the relation $E \subseteq W_2 \times W_4$ given by $E = \{(0, 0), (0, 2), (1, 1), (1, 3)\}$ is a bisimulation between the models $\mathcal{M}_2$ and $\mathcal{M}_4 = (\mathcal{L}_4, V_4)$. As $(0, 0) \in E$, we conclude $\mathcal{M}_2, 0 \sqsubseteq \mathcal{M}_4, 0$.

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\(^1\)Other examples are $p \rightarrow \Diamond \Diamond p$, $\Diamond \Diamond p \rightarrow p$, or any variation of them obtained by replacing $\Diamond$’s with $\Box$’s (whose meanings coincide on these functional frames).
(d) Let $\mathcal{M}_3$ be some model based on $L_3$. Define an acyclic model $N$ bisimilar to $\mathcal{M}_3$.

Answer:

Every model is bisimilar to its tree unfolding (and trees are acyclic). Let $\mathcal{M}_3 = (L_3, V)$. The tree unfolding of $\mathcal{M}_3$ is the model $N = (\mathbb{N}, S, U)$ where $S = \{(n, n+1) \mid n \in \mathbb{N}\}$, and with valuation $U$ defined for all variables $p$, and all $n \in \mathbb{N}$ by

\[
3n \in U(p) \iff 0 \in V(p),
3n + 1 \in U(p) \iff 1 \in V(p),
3n + 2 \in U(p) \iff 2 \in V(p).
\]

Then the relation $\{(0, 3n), (1, 3n+1), (2, 3n+2)\}$ is a bisimulation between the models $\mathcal{M}_3$ and $N$.

6. Formulate the box version of “Modal Decomposition” (MLOM, p. 41), i.e., give necessary and sufficient conditions for validity of a modal sequent of the form

\[
\vec{p}, \Box \varphi_1, \ldots, \Box \varphi_k \Rightarrow \Box \psi_1, \ldots, \Box \psi_m, \vec{q}
\]

Answer:

The modal sequent

\[
\vec{p}, \Box \varphi_1, \ldots, \Box \varphi_k \Rightarrow \Box \psi_1, \ldots, \Box \psi_m, \vec{q}
\]

is valid iff either

(a) $\vec{p}$ and $\vec{q}$ overlap, or

(b) the sequent $\varphi_1, \ldots, \varphi_k \Rightarrow \psi_j$ is valid, for some $j \in \{1, \ldots, m\}$.

7. Use sequents to investigate the validity of the following prop1 formulas:

(a) $(p \rightarrow q) \lor (q \rightarrow p)$:

Answer:

\[
\Rightarrow (p \rightarrow q) \lor (q \rightarrow p)
\Rightarrow (\neg p \lor q) \lor (\neg q \lor p)
\Rightarrow (\neg p \lor q), (\neg q \lor p)
\Rightarrow \neg p, q, (\neg q \lor p)
\Rightarrow \neg p, q, \neg q, p
\]

$p, q \Rightarrow p, q$

This sequent is valid, hence the formula $(p \rightarrow q) \lor (q \rightarrow p)$ is valid.
(b) \( p \to q \to p \)

*Answer:*  
We have \( p \to q \to p \equiv \neg p \lor (q \to p) \equiv \neg p \lor (\neg q \lor p) \).
We transform sequents:

\[
\begin{align*}
\Rightarrow & \quad \neg p \lor (\neg q \lor p) \\
\Rightarrow & \quad \neg p, (\neg q \lor p) \\
p \Rightarrow & \quad \neg q \lor p \\
p \Rightarrow & \quad \neg q, p \\
p, q \Rightarrow & \quad p
\end{align*}
\]

The sequent is valid, hence the formula \( p \to q \to p \) is valid.

(c) \((p \lor q) \to (p \land q)\)

*Answer:*  
\[
\begin{align*}
\Rightarrow & \quad \neg (p \lor q) \lor (p \land q) \\
\Rightarrow & \quad \neg (p \lor q), (p \land q) \\
p \lor q \Rightarrow & \quad p \land q
\end{align*}
\]

We get two sequents. The first one is \( p \Rightarrow p \land q \) which has as successor two sequents: \( p \Rightarrow p \) and \( p \Rightarrow q \). The first one is valid, the second one is not valid. This already gives that the sequent is not valid.

The second one is \( q \Rightarrow p \land q \) which has as successor two sequents: \( q \Rightarrow p \) and \( q \Rightarrow q \). This first one is not valid.

(d) \(((p \to q) \to p) \to p\)

*Answer:*  
We have \(((p \to q) \to p) \to p \equiv \neg((p \to q) \to p) \lor p \equiv \neg(\neg(p \to q) \lor p) \lor p \equiv \neg(\neg(p \lor q) \lor p) \lor p \).
We transform sequents:

\[
\begin{align*}
\Rightarrow & \quad \neg(\neg(p \lor q) \lor p) \lor p \\
\Rightarrow & \quad \neg(\neg(p \lor q) \lor p), p \\
\neg(p \lor q) \lor p \Rightarrow & \quad p
\end{align*}
\]

We get two new sequents to consider. The first one:  
\[
\begin{align*}
\neg(p \lor q) \Rightarrow & \quad p \\
\Rightarrow & \quad p, \neg p \lor q \\
\Rightarrow & \quad p, \neg p, q \\
p \Rightarrow & \quad p, q
\end{align*}
\]

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This sequent is valid. We continue with the second one:

\[ p \Rightarrow p \]

This sequent is also valid. We conclude that the sequent \( \Rightarrow \neg(\neg(p \lor q) \lor p) \lor p \) is valid. Hence the initial formula \(((p \rightarrow q) \rightarrow p) \rightarrow p\) (the law of Peirce) is valid.

(e) \( \diamond p \rightarrow \Box p \)

(f) \( \Box p \rightarrow \diamond p \)

(g) \( (p \rightarrow q) \lor (\Box p \lor \diamond p) \)