VU University Amsterdam — exercise class 4: some answers

advanced logic 2018-2019 — Friday 2018 03 01

Exercises

1. Give a derivation in K of $\vdash (\Diamond\phi \wedge \Box(\phi \to \psi)) \to \Diamond\psi$.

   Answer:

1.	$(a \to b) \to (\neg b \to \neg a)$	tauto from prop1
2.	$(\phi \to \psi) \to (\neg\psi \to\neq \psi)$	subst, 1
3.	$\Box(\phi \to \psi) \to \Box(\neg\psi \to \neg\phi)$	DISTR, 2
4.	$\Box(\neg\psi \to \neg\phi) \to \Box\neg psi \to \Box\neg\phi)$	modal distribution
5.	$\Box(\phi \to \psi) \to \Box\neg\psi \to \Box\neg\phi)$	PROP, 3, 4
6.	$\Box(\phi \to \psi) \to \neg\Diamond\psi \to \neg\Diamond\phi$	rewriting formula
7.	$\Box(\phi \to \psi) \to \Diamond\phi \to \Diamond\psi$	PROP
8.	$\Diamond\phi \wedge \Box(\phi \to \psi) \to \Diamond\psi$	PROP

   In step 5 we use the prop1 tautology $(a \to b) \wedge (b \to c) \to (a \to c)$.

   In step 7 we use the prop1 tautology $(a \to b \to c) \to (a \to \neg c \to \neg b)$.

   In step 8 we use the prop1 tautology $(a \to b \to c) \to (b \wedge a \to c)$.

2. Give a derivation in K of $(\Box\phi \vee \Box\psi) \to \Box(\phi \vee \psi)$.

   Answer:

1 .	$a \to (a \vee b)$	tautology from prop1
2.	$\phi \to (\phi \vee \psi)$	subst, 1
3.	$\Box\phi \to \Box(\phi \vee \psi)$	DISTR, 2
4.	$b \to (a \vee b)$	tautology from prop1
5.	$\psi \to (\phi \vee \psi)$	subst, 4
6.	$\Box\psi \to \Box(\phi \vee \psi)$	DISTR, 5
7.	$(\Box\phi \vee \Box\psi) \to \Box\phi \vee \Box\psi$	PROP 3, 6

   In step 7 we use $(a \to c) \wedge (b \to c) \to (a \vee b) \to c$

3. Give a derivation in T of $\Box p \to \neg\Box\neg p$.

1.	$\Box p \to p$	A1
2.	$\Box\neg p \to \neg p$	subst, 1
3.	$p \to \neg\Box\neg p$	PROP
4.	$\Box p \to \neg\Box\neg p$	PROP

   In step 3 we use $(a \to b) \to (\neg b \to \neg a)$.

In step 4 we use $(a \to b) \wedge (b \to c) \to a \to c$.

4. Indicate and explain an error in the following derivation in T:

1.	$q \to \Box q$	(necessitation)
2.	$\Box p \to \Box\Box p$	(substitution, 1)

   Answer:

   Step 1 is incorrect: necessitation is: if ϕ derivable then $\Box\phi$ derivable. Not $\phi \to \Box\phi$ derivable.

5. Indicate and explain the error(s) in the following derivation in T:

1.	p	(assumption)
2.	$\Box p$	(necessitation, 1)
3.	$p \to \Box p$	(PROP, 1, 3)
4.	$\Box q \to \Box\Box q$	(substitution, 3)

   Answer:

   Step 3 is incorrect: we do not have an introduction rule for implication available.

6. Show that $\neg\Box\neg\Box p \to p$ is not derivable in $S4$.

   Answer:

   We use the soundness and completeness theorem: ϕ is derivable in $S4$ if and only if ϕ is valid in all reflexive-transitive frames.

   We give a reflexive and transitive frame in which $\neg\Box\neg\Box p \to p$ is not valid/ Take $\mathcal{F} = (W, R)$ with $W = \{x, y\}$ and $R = \{(x, x), (y, y), (x, y)\}$. This is a reflexive and transitive frame. We use the valuation V with $V(p) = \{y\}$. Now we have: $y \models \Box p$, so $y \not\models \neg\Box p$. so $x \not\models \Box\neg\Box p$, so $x \models \neg\Box\neg\Box p$. But $x \not\models p$. Because we have given a valuation V and a state $x \in W$ wuch that $\mathcal{F}, V, x \not\models \neg\Box\neg\Box p \to p$, we conclude that $\mathcal{F} \not\models \neg\Box\neg\Box p \to p$.

   We conclude (using soundness) that ϕ is not derivable in $S4$.

7. Give a derivation in $S5$ of $\neg\Box\neg\Box p \to p$.

8. Prove or disprove the validity of the following formulas in the temporal frame $\mathcal{N} = (\mathbb{N}, <)$ of the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ with the usual ordering $<$:

   (a) $\Diamond\Box p \to \Box\Diamond p$

   Answer:

We prove that $\Diamond\Box p \to \Box\Diamond p$ is valid in the frame $\mathcal{N} = (\mathbb{N}, <)$. Let V be an arbitrary valuation on $\mathcal{N}$, let $n \in \mathbb{N}$ be an arbitrary point of the model $(\mathcal{N}, V)$ and assume $n \models \Diamond\Box p$. We must prove $n \models \Box\Diamond p$. By the assumption there is $m > n$ such that $m \models \Box p$. So, for all $k > m$ we have $k \models p$. In order to show that $n \models \Box\Diamond p$, we let $x > n$ and prove $x \models \Diamond p$. In the point $y = x + m$ we have $y \models p$ because $y > m$. Hence $x \models \Diamond p$ as $y > x$.

(b) $\Box\Diamond p \to \Diamond\Box p$

9. Show that the formula

$$\lambda = \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$$

defines right-linearity, that is, for all (not necessarily temporal) frames $\mathcal{F} = (W, R)$:

$$\mathcal{F} \models \lambda \quad \text{if and only if} \quad R \text{ is right-linear}$$

A relation R is right-linear if $Rxy \wedge Rxz$ implies $Ryz \vee y = z \vee Rzy$ for all x, y, z.

Answer:

We show that, for all frames $\mathcal{F} = (W, R)$,

$$\mathcal{F} \models \lambda \iff R \text{ is right-linear},$$

where $\lambda = \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$.

(⇐) Let $\mathcal{F}$ be right-linear. Let V be an arbitrary valuation on $\mathcal{F}$, let x be an arbitrary point of the model $(\mathcal{F}, V)$, and assume $x \models \Diamond p \wedge \Diamond q$. We have to prove $x \models \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$. From $x \models \Diamond p$ we get a point y with Rxy such that $y \models p$, and from $x \models \Diamond q$ we get a point z with Rxz such that $z \models q$. By right-linearity of the frame there are three possible situations, namely Ryz or $y = z$ or Rzy. In all three cases, one of the disjuncts of the right-hand side of the implication λ holds in x:

- if Ryz, then $y \models p \wedge \Diamond q$ and so $x \models \Diamond(p \wedge \Diamond q)$;
- if $y = z$, then $y \models p \wedge q$ and so $x \models \Diamond(p \wedge q)$;
- if Rzy, then $z \models \Diamond p \wedge q$ and so $x \models \Diamond(\Diamond p \wedge q)$;

and so we have shown that $x \models \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$.

(⇒) We prove this direction by contraposition, i.e., assuming that $\mathcal{F}$ is *not* right-linear, we show that $\mathcal{F} \not\models \lambda$. $\mathcal{F}$ not right-linear means there exist points a, b, c such that Rab, Rac, $\neg Rbc$, $b \neq c$, and $\neg Rcb$. Our task is to find a valuation V on $\mathcal{F}$, and a point x such that $\mathcal{F}, V, x \not\models \lambda$. This means that, in the model we are after, we want $x \models \Diamond p$, $x \models \Diamond q$, $x \not\models \Diamond(p \wedge \Diamond q)$, $x \not\models \Diamond(p \wedge q)$, and $x \not\models \Diamond(\Diamond p \wedge q)$. The right candidate for x seems to be a. The first two requirements ($a \models \Diamond p$, $a \models \Diamond q$) we can then fulfill by making p true in b and q true in c. The other requirements are met by forcing that p is *only* true in b and that q is *only* true in c. So we put $V(p) = \{b\}$ and $V(q) = \{c\}$. Then we have $b \not\models \Diamond q$ (due to $\neg Rbc$) and $c \not\models \Diamond p$ (due to $\neg Rcb$). Hence a has no R-successor that satisfies $p \wedge \Diamond q$ (the only successor that has p is b, but $b \not\models \Diamond q$), a has no R-successor that satisfies $p \wedge q$ (simply because $b \neq c$), and a has no R-successor that satisfies $\Diamond p \wedge q$ (the only successor that has q is c, but $c \not\models \Diamond q$). We conclude $a \not\models \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$, and so $a \not\models \lambda$. Hence λ is not valid outside the class of right-linear frames. (And so if it is valid, we are inside the class.)

10. Show that right-branching is not modally definable.

    A relation R is right-branching if there exist x, y, z such that $x < y$ and $x < z$ but $\neg(y < z) \wedge y \neq z \wedge \neg(z < y)$.

    Answer:

    Done in exercise class 5.

11. Show that $\Diamond p \to \Diamond\Diamond p$ defines density.

    A relation R is dense if for all x, z we have: if $x < z$ then there is y such that $x < y$ and $y < z$.

    Answer:

    Let $\mathcal{T} = (T, <)$ be a (not necessarily temporal) frame We show that

$$\mathcal{T} \text{ is dense} \quad \text{if and only if} \quad \mathcal{T} \models \Diamond p \to \Diamond\Diamond p\,,$$

in two parts:

(⇐) Assume that $\mathcal{T} = (T, <)$ is a dense frame, that is, for all $x, y \in T$ with $x < y$ there is a $z \in T$ such that $x < z < y$. In order to show that $\mathcal{F} \models \Diamond p \to \Diamond\Diamond p$, we consider an arbitrary valuation V on $\mathcal{T}$, and an arbitrary point $t \in T$, and assume $\mathcal{T}, V, t \models \Diamond p$. We have to prove $\mathcal{T}, V, t \models \Diamond\Diamond p$. By the assumption $t \models \Diamond p$ there is $v > t$ such that $v \models p$. By the frame being dense, there is a a point u between t and v, so $t < u < v$. Now it holds that $u \models \Diamond p$ (due to $u < v$ and $v \models p$) and so $t \models \Diamond\Diamond p$ (due to $t < u$ and $u \models \Diamond p$).

($\Rightarrow$) Assume $\mathcal{T} = (T, <)$ is *not* dense. We prove that $\mathcal{T} \not\models \Diamond p \to \Diamond\Diamond p$. By the assumption, there are points t, v such that $t < v$ with no point in between, i.e., there is no point u such that $t < u < v$. We falsify the formula in t by defining a valuation V such that $t \models \Diamond p$ and $t \not\models \Diamond\Diamond p$. We take $V(p) = \{v\}$. Then indeed $t \models \Diamond p$ (as $t < v$ and $v \models p$). Moreover, $t \not\models \Diamond\Diamond p$ since there is no point u with $t < u$ and $u \models \Diamond p$, because that would require $u < v$ (since by the definition of V, p only holds in v), and the existence of a point u with $t < u < v$ was excluded.