Exercises

1. Give a derivation in $K$ of $\vdash (\Diamond \phi \land \Box (\phi \rightarrow \psi)) \rightarrow \Diamond \psi$.

   Answer:
   1. $(a \rightarrow b) \rightarrow (\neg b \rightarrow \neg a)$ tauto from prop1
   2. $(\phi \rightarrow \psi) \rightarrow (\neg \psi \rightarrow \neg \psi)$ subst, 1
   3. $\Box (\phi \rightarrow \psi) \rightarrow \Box (\neg \psi \rightarrow \neg \phi)$ DISTR, 2
   4. $\Box (\neg \psi \rightarrow \neg \phi) \rightarrow \Box \neg \psi i \rightarrow \Box \neg \phi$ modal distribution
   5. $\Box (\phi \rightarrow \psi) \rightarrow \Box \neg \psi \rightarrow \Box \neg \phi$ PROP, 3, 4
   6. $\Box (\phi \rightarrow \psi) \rightarrow \Diamond \psi \rightarrow \Diamond \phi$ subst, 4
   7. $\Box (\phi \rightarrow \psi) \rightarrow \Diamond \phi \rightarrow \Diamond \psi$ PROP
   8. $\Diamond \phi \land \Box (\phi \rightarrow \psi) \rightarrow \Diamond \psi$ PROP

   In step 5 we use the prop1 tautology $(a \rightarrow b) \land (b \rightarrow c) \rightarrow (a \rightarrow c)$.

   In step 7 we use the prop1 tautology $(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow \neg c \rightarrow \neg b)$.

   In step 8 we use the prop1 tautology $(a \rightarrow b \rightarrow c) \rightarrow (b \land a \rightarrow c)$.

2. Give a derivation in $K$ of $(\Box \phi \lor \Box \psi) \rightarrow \Box (\phi \lor \psi)$.

   Answer:
   1. $a \rightarrow (a \lor b)$ tautology from prop1
   2. $\phi \rightarrow (\phi \lor \psi)$ subst, 1
   3. $\Box \phi \rightarrow \Box (\phi \lor \psi)$ DISTR, 2
   4. $b \rightarrow (a \lor b)$ tautology from prop1
   5. $\psi \rightarrow (\phi \lor \psi)$ subst, 4
   6. $\Box \psi \rightarrow \Box (\phi \lor \psi)$ DISTR, 5
   7. $(\Box \phi \lor \Box \psi) \rightarrow \Box \phi \lor \Box \psi$ PROP 3, 6

   In step 7 we use $(a \rightarrow c) \land (b \rightarrow c) \rightarrow (a \lor b) \rightarrow c$

3. Give a derivation in $T$ of $\Box p \rightarrow \neg \Box \neg p$.

   1. $\Box p \rightarrow p$ A1
   2. $\Box \neg p \rightarrow \neg p$ subst, 1
   3. $p \rightarrow \neg \Box \neg p$ PROP
   4. $\Box p \rightarrow \neg \Box \neg p$ PROP

   In step 3 we use $(a \rightarrow b) \rightarrow (\neg b \rightarrow \neg a)$.
In step 4 we use \((a \rightarrow b) \land (b \rightarrow c) \rightarrow a \rightarrow c\).

4. Indicate and explain an error in the following derivation in \(T\):
   1. \(q \rightarrow \square q\) (necessitation)
   2. \(\square p \rightarrow \square \square p\) (substitution, 1)

   \textit{Answer:}
   Step 1 is incorrect: necessitation is: if \(\phi\) derivable then \(\square \phi\) derivable. Not \(\phi \rightarrow \square \phi\) derivable.

5. Indicate and explain the error(s) in the following derivation in \(T\):
   1. \(p\) (assumption)
   2. \(\square p\) (necessitation, 1)
   3. \(p \rightarrow \square p\) (PROP, 1, 3)
   4. \(\square q \rightarrow \square \square q\) (substitution, 3)

   \textit{Answer:}
   Step 3 is incorrect: we do not have an introduction rule for implication available.

6. Show that \(\neg \square \neg \square p \rightarrow p\) is not derivable in \(S4\).

   \textit{Answer:}
   We use the soundness and completeness theorem: \(\phi\) is derivable in \(S4\) if and only if \(\phi\) is valid in all reflexive-transitive frames.

   We give a reflexive and transitive frame in which \(\neg \square \neg \square p \rightarrow p\) is not valid/ Take \(F = (W, R)\) with \(W = \{x, y\}\) and \(R = \{(x, x), (y, y), (x, y)\}\).

   This is a reflexive and transitive frame. We use the valuation \(V\) with \(V(p) = \{y\}\). Now we have: \(y \models \square p\), so \(y \not\models \neg \square p\). so \(x \not\models \square \neg \square p\), so \(x \models \neg \square \neg \square p\). But \(x \not\models p\). Because we have given a valuation \(V\) and a state \(x \in W\) wuch that \(F, V, x \not\models \neg \square \neg \square p \rightarrow p\), we conclude that \(F \not\models \neg \square \neg \square p \rightarrow p\).

   We conclude (using soundness) that \(\phi\) is not derivable in \(S4\).

7. Give a derivation in \(S5\) of \(\neg \square \neg \square p \rightarrow p\).

8. Prove or disprove the validity of the following formulas in the temporal frame \(N = (N, <)\) of the natural numbers \(N = \{0, 1, \ldots\}\) with the usual ordering \(<\):

   (a) \(\diamond \square p \rightarrow \square \diamond p\)

   \textit{Answer:}
We prove that $\Diamond \Box p \rightarrow \Box \Diamond p$ is valid in the frame $\mathcal{N} = (\mathbb{N}, <)$.
Let $V$ be an arbitrary valuation on $\mathcal{N}$, let $n \in \mathbb{N}$ be an arbitrary point of the model $(\mathcal{N}, V)$ and assume $n \models \Diamond \Box p$. We must prove $n \models \Box \Diamond p$. By the assumption there is $m > n$ such that $m \models \Box p$.
So, for all $k > m$ we have $k \models \Diamond p$. In order to show that $n \models \Box \Diamond p$, we let $x > n$ and prove $x \models \Diamond p$. In the point $y = x + m$ we have $y \models p$ because $y > m$. Hence $x \models \Diamond p$ as $y > x$.

(b) $\Box \Diamond p \rightarrow \Diamond \Box p$

9. Show that the formula

$$\lambda = \Diamond p \land \Diamond q \rightarrow \Diamond (p \land \Diamond q) \lor \Diamond (p \land q) \lor \Diamond (\Diamond p \land q)$$

defines right-linearity, that is, for all (not necessarily temporal) frames $\mathcal{F} = (W, R)$:

$$\mathcal{F} \models \lambda \quad \text{if and only if} \quad R \text{ is right-linear}$$

A relation $R$ is right-linear if $Rxy \land Rxz$ implies $Ryz \lor y = z \lor Rzy$ for all $x, y, z$.

Answer:
We show that, for all frames $\mathcal{F} = (W, R)$,

$$\mathcal{F} \models \lambda \iff R \text{ is right-linear},$$

where $\lambda = \Diamond p \land \Diamond q \rightarrow \Diamond (p \land \Diamond q) \lor \Diamond (p \land q) \lor \Diamond (\Diamond p \land q)$.

$(\Leftarrow)$ Let $\mathcal{F}$ be right-linear. Let $V$ be an arbitrary valuation on $\mathcal{F}$, let $x$ be an arbitrary point of the model $(\mathcal{F}, V)$, and assume $x \models \Diamond p \land \Diamond q$. We have to prove $x \models \Diamond (p \land \Diamond q) \lor \Diamond (p \land q) \lor \Diamond (\Diamond p \land q)$. From $x \models \Diamond p$ we get a point $y$ with $Rxy$ such that $y \models p$, and from $x \models \Diamond q$ we get a point $z$ with $Rxz$ such that $z \models q$. By right-linearity of the frame there are three possible situations, namely $Ryz$ or $y = z$ or $Rzy$. In all three cases, one of the disjuncts of the right-hand side of the implication $\lambda$ holds in $x$:

- if $Ryz$, then $y \models p \land \Diamond q$ and so $x \models \Diamond (p \land \Diamond q);$ 
- if $y = z$, then $y \models p \land q$ and so $x \models \Diamond (p \land q);$ 
- if $Rzy$, then $z \models p \land q$ and so $x \models \Diamond (\Diamond p \land q);$ 

and so we have shown that $x \models \Diamond (p \land \Diamond q) \lor \Diamond (p \land q) \lor \Diamond (\Diamond p \land q)$.  

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We prove this direction by contraposition, i.e., assuming that \( F \) is not right-linear, we show that \( F \not\vDash \lambda \). \( F \) not right-linear means there exist points \( a, b, c \) such that \( R_{ab}, R_{ac}, \neg R_{bc}, b \neq c, \) and \( \neg R_{cb} \). Our task is to find a valuation \( V \) on \( F \), and a point \( x \) such that \( F, V, x \not\vDash \lambda \). This means that, in the model we are after, we want \( x \vDash \Box p \), \( x \vDash \Box q \), \( x \not\vDash (p \land \Box q) \), and \( x \not\vDash (\Box p \land q) \). The right candidate for \( x \) seems to be \( a \). The first two requirements (\( a \vDash \Box p, a \vDash \Box q \)) we can then fulfill by making \( p \) true in \( b \) and \( q \) true in \( c \). The other requirements are met by forcing that \( p \) is only true in \( b \) and that \( q \) is only true in \( c \). So we put \( V(p) = \{ b \} \) and \( V(q) = \{ c \} \). Then we have \( b \not\vDash \Box q \) (due to \( \neg R_{bc} \)) and \( c \not\vDash \Box p \) (due to \( \neg R_{cb} \)). Hence \( a \) has no \( R \)-successor that satisfies \( p \land \Box q \) (the only successor that has \( p \) is \( b \), but \( b \not\vDash \Box q \)), \( a \) has no \( R \)-successor that satisfies \( p \land q \) (simply because \( b \neq c \)), and \( a \) has no \( R \)-successor that satisfies \( \Box p \land q \) (the only successor that has \( q \) is \( c \), but \( c \not\vDash \Box q \)). We conclude \( a \not\vDash \Box(p \land q) \lor \Box(p \land q) \lor \Box(p \land q) \), and so \( a \not\vDash \lambda \). Hence \( \lambda \) is not valid outside the class of right-linear frames. (And so if it is valid, we are inside the class.)

10. Show that right-branching is not modally definable.

A relation \( R \) is right-branching if there exist \( x, y, z \) such that \( x < y \) and \( x < z \) but \( \neg(y < z) \land y \neq z \land \neg(z < y) \).

Answer:
Done in exercise class 5.

11. Show that \( \Box p \rightarrow \Box \Box p \) defines density.

A relation \( R \) is dense if for all \( x, z \) we have: if \( x < z \) then there is \( y \) such that \( x < y \) and \( y < z \).

Answer:
Let \( T = (T, <) \) be a (not necessarily temporal) frame. We show that

\[ T \text{ is dense } \text{ if and only if } T \vDash \Box p \rightarrow \Box \Box p, \]

in two parts:

\( \Rightarrow \) Assume that \( T = (T, <) \) is a dense frame, that is, for all \( x, y \in T \) with \( x < y \) there is a \( z \in T \) such that \( x < z < y \). In order to show that \( F \vDash \Box p \rightarrow \Box \Box p \), we consider an arbitrary valuation \( V \) on \( \mathcal{T} \),
and an arbitrary point \( t \in T \), and assume \( \mathcal{T}, V, t \models \lozenge p \). We have to prove \( \mathcal{T}, V, t \models \lozenge \lozenge p \). By the assumption \( t \models \lozenge p \) there is \( v > t \) such that \( v \models p \). By the frame being dense, there is a point \( u \) between \( t \) and \( v \), so \( t < u < v \). Now it holds that \( u \models \lozenge p \) (due to \( u < v \) and \( v \models p \)) and so \( t \not\models \lozenge \lozenge p \) (due to \( t < u \) and \( u \not\models \lozenge p \)).

\( \Rightarrow \) Assume \( \mathcal{T} = (T, <) \) is not dense. We prove that \( \mathcal{T} \not\models \lozenge p \to \lozenge \lozenge p \).

By the assumption, there are points \( t, v \) such that \( t < v \) with no point in between, i.e., there is no point \( u \) such that \( t < u < v \). We falsify the formula in \( t \) by defining a valuation \( V \) such that \( t \models \lozenge p \) and \( t \not\models \lozenge \lozenge p \). We take \( V(p) = \{v\} \). Then indeed \( t \models \lozenge p \) (as \( t < v \) and \( v \models p \)). Moreover, \( t \not\models \lozenge \lozenge p \) since there is no point \( u \) with \( t < u \) and \( u \models \lozenge p \), because that would require \( u < v \) (since by the definition of \( V \), \( p \) only holds in \( v \)), and the existence of a point \( u \) with \( t < u < v \) was excluded.