Exercises

1. We consider basic modal logic (BML) and aribitrary frames.

   Show that the property of right-branching (slides of lecture 8) is not definable using a formula from basic modal logic.

   Answer:

   A frame (W, R) is said to be right-branching if there exist states $x, y, z \in W$ such that Rxy and Rzu but ($y \neq z$ and not Ryz and not Rzy).

   Suppose that there is a formula ϕ that characterizes the property of being right-branching.

   We consider the two frames $\mathcal{F} = (\{x, y, z\}, \{(x, y), (x, z)\})$ and $\mathcal{F}' = (\{x', y'\}, \{(x', y')\})$. We observe that $\mathcal{F}$ is right-branching and $\mathcal{F}'$ is not right-branching.

   We aim to show $\mathcal{F}' \models \phi$. We take an arbitrary valuation V' for $\mathcal{F}'$, and we take an arbitrary point $a \in \{x', y'\}$. (It is not a correct approach to take a *particular* valuation on $\mathcal{F}'$.)

   We carry over the valuation V' to the frame $\mathcal{F}$ as follows. Let P be exactly the set of propositional variables true in state x' according to V', and let Q be exactly the set of proposition variables true in state y' according to V'. We define V to be the valuation for frame $\mathcal{F}$ such that P is exactly the set of propositional variables true in state x'. Q is exactly the set of propositional variables true in state y', and Q is exactly the set of propositional variables true in state z'.

   We define $Z = \{(x, x'), (y, y'), (y, z')\}$. $Z is a bisimulation.$

   Now we consider two cases: $a = x'$ or $a = y'$.

   If $a = x'$, we reason as follows. Because ϕ characterizes being right-branching, and $\mathcal{F}$ is right-branching, we have $\mathcal{F} \models \phi$ and hence $\mathcal{F}, V, x \models \phi$. Because using Z we have $\mathcal{F}, V, x \leftrightarrows \mathcal{F}', V'x'$, and being bisimilar implies being modally equivalent (theorem due to Hennesy and Milner), we have $\mathcal{F}', V', x' \models \phi$.

Similarly, using that we have $\mathcal{F}, V, y \leftrightarrows \mathcal{F}', V', y'$, we find that $\mathcal{F}', V', y' \models \phi$.

We conclude that ϕ holds in $\mathcal{F}$ for any valuation and in any state, that is: $\mathcal{F}' \models \phi$.

Because $\mathcal{F}'$ is not right-branching, this is in contradition with the assumption that ϕ characterizes being right-branching.

We conclude that such a ϕ does not exist.

2. We consider basic modal logic (BML) and temporal fames.

   Show that the operator until is not definable.

   Show that the operator next is not definable.

3. We consider basic temporal logic.

   Prove or disprove the validity of the following formulas in the temporal frame $\mathcal{N} = (\mathbb{N}, <)$ of the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ with the usual ordering $<$:

   (a) $\langle F \rangle [F] \bot$

   Answer:

   We show $\mathcal{N} \not\models \langle F \rangle [F] \bot$. We aim to show that there is a valuation V, and there is a state x such that $\mathcal{N}, V, x \not\models \langle F \rangle [F] \bot$. We can take V to be the empty valuation. (The valuation does not play a role in the formula.) We take the state $x = 1$. For every $y > x$ there is a $z > y$ (namely: take $z = y + 1$) for which (as for any state) $z \not\models \bot$. So $x \not\models \langle F \rangle [F] \bot$. Hence $\mathcal{N} \not\models \langle F \rangle [F] \bot$. Note that it is not important that we take $x = 1$; we could have taken any other element of $\mathbb{N}$.

   (b) $\langle P \rangle \top \rightarrow \langle P \rangle [P] \top$

   (c) $\langle P \rangle \langle F \rangle q \rightarrow (\langle P \rangle q \vee q \vee \langle F \rangle q)$

   Answer:

   We show that $\mathcal{N} \vDash \langle P \rangle \langle F \rangle q \rightarrow (\langle P \rangle q \vee q \vee \langle F \rangle q)$. Let V be an arbitrary valuation on $\mathcal{N}$, let n be an arbitrary point of the model $(\mathcal{N}, V)$, and assume $n \vDash \langle P \rangle \langle F \rangle q$. Goal is $n \vDash \langle P \rangle q \vee q \vee \langle F \rangle q$. By the assumption, there is $m < n$ such that $m \vDash \langle F \rangle q$. So there are m, k such that $m < n$ and $m < k$ and $k \vDash q$. Now, by the construction of $\mathbb{N}$, there are three possible cases:

   if $n > k$, then $n \vDash \langle P \rangle q$;

if $n = k$, then $n \vDash q$;
if $n < k$, then $n \vDash \langle F \rangle q$;
and so $n \vDash \langle P \rangle q \lor q \lor \langle F \rangle q$.

4. We consider basic temporal logic.

Let τ and γ abbreviate the following formulas:

$$\tau = (\langle F \rangle [F] q \land \langle F \rangle \neg q) \to \langle F \rangle (\neg q \land [F] q)$$
$$\gamma = (\langle F \rangle [F] q \land \langle F \rangle \neg q) \to \langle F \rangle ([F] q \land [P] \langle F \rangle \neg q)$$

Consider the temporal frames $\mathcal{Z} = (\mathbb{Z}, <)$, $\mathcal{Q} = (\mathbb{Q}, <)$, and $\mathcal{R} = (\mathbb{R}, <)$ of the integers, rational and real numbers, respectively, with their usual orderings.

(a) Show that τ is valid in $\mathcal{Z}$.

Answer:

We show that $(\mathbb{Z}, <) \vDash \tau$. Let V be an arbitrary valuation on the frame $(\mathbb{Z}, <)$, and consider an arbitrary point x in the model $\mathcal{M} = (\mathbb{Z}, <, V)$. Assume $\mathcal{M}, x \vDash \langle F \rangle [F] q \land \langle F \rangle \neg q$. Goal is to show $\mathcal{M}, x \vDash \langle F \rangle (\neg q \land [F] q)$. By the assumption, there are integers $z, y > x$ such that $\mathcal{M}, z \vDash [F] q$ and $\mathcal{M}, y \vDash \neg q$. It follows that $y \leq z$, for if $y > z$ then $\mathcal{M}, y \vDash q$ would be forced by $\mathcal{M}, z \vDash [F] q$, contradicting $\mathcal{M}, y \vDash \neg q$. So $y \leq z$. Now let u be the largest integer in the set $\{y, y+1, \ldots, z-1, z\}$ such that $\mathcal{M}, u \vDash \neg q$ (such an integer exists because y satisfies it and the set is finite). Then $\mathcal{M}, u \vDash \neg q$ by assumption. And, for all $v > u$ we see $\mathcal{M}, v \vDash q$, either by maximality of u (if $v \leq z$), or by $\mathcal{M}, z \vDash \Box q$ (if $v > z$). Hence $\mathcal{M}, u \vDash [F] q$. We found a point $u > x$ such that $\mathcal{M}, u \vDash \neg q \land [F] q$, and we conclude $\mathcal{M}, x \vDash \langle F \rangle (\neg q \land [F] q)$.

(b) Show that τ is not valid in $\mathcal{Q}$.

Answer:

We show that $(\mathbb{Q}, <) \nvDash \tau$. Define the valuation $V(q) = \{x \in \mathbb{Q} \mid x \geq 1\}$, and let $\mathcal{M} = (\mathbb{Q}, <, V)$. We prove that $\mathcal{M}, 0 \nvDash \tau$, that is, $\mathcal{M}, 0 \vDash \langle F \rangle [F] q \land \langle F \rangle \neg q$ and $\mathcal{M}, 0 \nvDash \langle F \rangle (\neg q \land [F] q)$. We have $\mathcal{M}, 0 \vDash \langle F \rangle [F] q$ because $\mathcal{M}, 1 \vDash [F] q$ by the definition of V. Also $\mathcal{M}, 0 \vDash \langle F \rangle \neg q$ because, for example, $\mathcal{M}, \frac{1}{2} \vDash \neg q$.

Now suppose $\mathcal{M}, 0 \vDash \langle F \rangle (\neg q \land [F] q)$ (we show this leads to a contradiction). Then exists $x > 0$ such that $\mathcal{M}, x \vDash \neg q$ (and so, by definition of V, $x < 1$) and $\mathcal{M}, x \vDash [F] q$. Let $z = \frac{x+1}{2}$. Then

$\mathcal{M}, z \models q$ by $x \models [F]q$ and $z > x$. However, $z < 1$ (since $x < 1$) and so $\mathcal{M}, z \models \neg q$ (by definition of V). Contradiction, and we conclude $\mathcal{M}, 0 \not\models \langle F\rangle(\neg q \wedge [F]q)$.

(c) Show that γ is not valid in $\mathcal{Q}$.

Answer:

Let γ denote the temporal formula $(\langle F\rangle[F]q \wedge \langle F\rangle\neg q) \rightarrow \langle F\rangle([F]q \wedge [P]\langle F\rangle\neg q)$.

Let $V = \{x \in \mathbb{Q} \mid x > \sqrt{2}\}$, and $\mathcal{M} = (\mathbb{Q}, <, V)$. We disprove validity of γ in $(\mathbb{Q}, <)$ by proving $\mathcal{M}, 0 \not\models \gamma$, that is, $\mathcal{M}, 0 \models \langle F\rangle[F]q \wedge \langle F\rangle\neg q$ and $\mathcal{M}, 0 \not\models \langle F\rangle([F]q \wedge [P]\langle F\rangle\neg q)$. We have $\mathcal{M}, 0 \models \langle F\rangle[F]q$ because $\mathcal{M}, 2 \models [F]q$ (all $x > 2$ are also above $\sqrt{2}$). And $\mathcal{M}, 0 \models \langle F\rangle\neg q$ follows from $\mathcal{M}, 1 \models \neg q$ (by definition of V and $1 < \sqrt{2}$).

Towards a contradiction, assume $\mathcal{M}, 0 \models \langle F\rangle([F]q \wedge [P]\langle F\rangle\neg q)$. Then there exists $x > 0$ such that $\mathcal{M}, x \models [F]q$ and $\mathcal{M}, x \models [P]\langle F\rangle\neg q$. So for all $y > x$, $\mathcal{M}, y \models q$, i.e., $y > \sqrt{2}$ $(*)$, and also $(**)$ for all $z < x$, $\mathcal{M}, z \models \langle F\rangle\neg q$.

Now it is impossible that $x < \sqrt{2}$ because by the nature of the rational numbers ($\mathbb{Q}$ is dense in $\mathbb{R}$), if $x < \sqrt{2}$, then there exists $a \in \mathbb{Q}$ with $x < a < \sqrt{2}$, contradicting $(*)$.

Also $x \neq \sqrt{2}$ since $\sqrt{2}$ is not rational.

So $x > \sqrt{2}$. Let $a \in \mathbb{Q}$ be such that $\sqrt{2} < a < x$. Then, by $(**)$, $\mathcal{M}, a \models \langle F\rangle\neg q$, which implies there is $b \in \mathbb{Q}$ such that $\sqrt{2} < a < b$ and $\mathcal{M}, b \models \neg q$, contradicting the definition of V.

(d) Is γ valid in $\mathcal{R}$?

Answer:

Here we need the property of the real numbers saying that every non-empty set X of real numbers that is bounded from below has an infimum.Note that the rational numbers do not have this property: The set $\{x \in \mathbb{Q} \mid x > \sqrt{2}\}$ has a lower bound (for example, $\frac{41}{29}$), but it does not have a (rational) greatest lower bound.

(This depends on from which set we may take the border-value.)

Let $(S, \leq)$ be a partially ordered set, and let $X \subseteq S$ with $X \neq \varnothing$. A *lower bound* of X is an element $b \in S$ such that $b \leq x$ for all $x \in X$. The *greatest* lower bound (also called *infimum*) of X is a lower bound a of X such that $a \leq b$, for all lower bounds b of X.

We show that $(\mathbb{R}, <) \models \gamma$. Let V be an arbitrary valuation on $(\mathbb{R}, <)$, and $\mathcal{M} = (\mathbb{R}, <, V)$. Let $x \in \mathbb{R}$ be an arbitrary point of $\mathcal{M}$ and assume $\mathcal{M}, x \models \langle F \rangle [F] q \wedge \langle F \rangle \neg q$. We must see: $\mathcal{M}, x \models \langle F \rangle ([F] q \wedge [P] \langle F \rangle \neg q)$. By the assumption, exists $a, b \in \mathbb{R}$ with $x < a$ and $x < b$ such that $\mathcal{M}, a \models [F] q$ and $\mathcal{M}, b \models \neg q$. By the latter, $c \not\models [F] q$, for all $c < b$.

Now let A denote the set $A = \{y \in \mathbb{R} \mid \mathcal{M}, y \models [F] q\}$. Then $A \neq \varnothing$ since $a \in A$. Moreover, A is bounded from below by b, and hence it has an infimum, which we denote by $\inf(A)$.

For all $y < \inf(A)$: $y \notin A$, so $\mathcal{M}, y \not\models [F] q$, and so $\mathcal{M}, y \models \langle F \rangle \neg q$. Therefore, $\mathcal{M}, \inf(A) \models [P] \langle F \rangle \neg q$.

For all $y > \inf(A)$ we have $\inf(A) < \frac{\inf(A)+y}{2} < y$, and $\mathcal{M}, \frac{\inf(A)+y}{2} \models [F] q$, so $\mathcal{M}, y \models q$. Therefore, $\mathcal{M}, \inf(A) \models [F] q$.

Since $x < b \leq \inf(A)$, we conclude $x \models \langle F \rangle ([P] \langle F \rangle \neg q \wedge [F] q)$ as required.

5. A temporal order $<$ is *dense* if between any two distinct points x and y we can find a third point z. It is *discrete* if to each non-final point x we can associate an *immediate successor* z.

$$\begin{aligned} \textit{density}: &\quad \forall xy\,(x < y \to \exists z\,(x < z \wedge z < y)) \\ \textit{discreteness}: &\quad \forall xy\,(x < y \to \exists z\,(x < z \wedge \neg \exists u\,(x < u \wedge u < z))) \end{aligned}$$

(a) Define an temporal order which is neither dense nor discrete.

Answer:

The idea is to find orders that are dense on some part but discrete on another. For example, the frame $(T, <)$ with $T = \{0, 1\} \cup [2, 3]$ (where $[2, 3] = \{x \in \mathbb{R} \mid 2 \leq x \leq 3\}$) and $<$ the usual order on the real numbers, is neither dense (there is no x such that $0 < x < 1$), nor discrete (since 2 has no immediate successor).

Another example of a temporal frame that is neither dense nor discrete is $(\{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\}, <)$ with $<$ the usual order on the reals. This frame is not discrete since 0 has a successor (for example 1) but no immediate successor. On the other hand, it is not dense since there are no points between 2^{-1} and 1.

(b) Define a temporal order which is both dense and discrete.

Answer:

Any temporal frame $\mathcal{T} = (T, <)$ with $< = \varnothing$ is both dense and discrete, for example $(\{0\}, \varnothing)$.

6. We consider basic temporal logic and temporal frames.

   Show that the operator until is not definable.

7. We consider multi-modal logic.

   Let I be an arbitrary index set, and let $i, j \in I$.

   (a) Prove that the formula $p \to [i]\langle j\rangle p$ characterizes the class of I-frames $\mathcal{F} = (W, \{R_k \mid k \in I\})$ that satisfy the property $R_i \subseteq R_j^{-1}$.
   Answer:
   The multi-modal formula $p \to [i]\langle j\rangle p$ characterizes the property $R_i \subseteq R_j^{-1}$ if

$$\mathcal{F} \vDash p \to [i]\langle j\rangle p \iff R_i \subseteq R_j^{-1}$$

   for all I-frames $\mathcal{F} = (W, \{R_k \mid k \in I\})$ and $i, j \in I$. So let $\mathcal{F}$ be an arbitrary I-frame, and let $i, j \in I$. We prove the two directions:

   ($\Rightarrow$) By contraposition. Assume that $R_i \not\subseteq R_j^{-1}$. We prove $\mathcal{F} \nvDash p \to [i]\langle j\rangle p$. By the assumption there are (not necessarily distinct) points a and b such that $R_i ab$ and $\neg R_j ba$. In order to show that $p \to [i]\langle j\rangle p$ is not valid in $\mathcal{F}$ we have to find a valuation V on $\mathcal{F}$ and a point x such that $\mathcal{M}, x \vDash p$ and $\mathcal{M}, x \nvDash [i]\langle j\rangle p$.
   We choose V to be such that p holds in a only, $V(p) = \{a\}$. Then in the model $\mathcal{M} = (\mathcal{F}, V)$ we have $\mathcal{M}, b \nvDash \langle j\rangle p$ since $\neg R_j ba$. Hence, due to $R_i ab$, also $\mathcal{M}, a \nvDash [i]\langle j\rangle p$. We conclude that $\mathcal{M}, a \nvDash p \to [i]\langle j\rangle p$. Hence $\mathcal{F} \nvDash p \to [i]\langle j\rangle p$.

   ($\Leftarrow$) Assume $R_i \subseteq R_j^{-1}$, that is, $R_i uv$ implies $R_j vu$, for all points u and v.
   We have to show that $p \to [i]\langle j\rangle p$ is valid in $\mathcal{F}$. Let V be an arbitrary valuation on $\mathcal{F}$, x an arbitrary point in the model $\mathcal{M} = (\mathcal{F}, V)$, and assume $\mathcal{M}, x \vDash p$. In order to show $\mathcal{M}, x \vDash [i]\langle j\rangle p$, we consider an arbitrary R_i-successor y of x, $R_i xy$, and prove $\mathcal{M}, y \vDash \langle j\rangle p$. By the assumption $R_i \subseteq R_j^{-1}$ we know that $R_j yx$. Hence, since we have $\mathcal{M}, x \vDash p$, it follows that $\mathcal{M}, y \vDash \langle j\rangle p$.

   (b) Use the result of the previous question to show that the formula $\langle i\rangle[j]p \to p$ also characterizes the frame property $R_i \subseteq R_j^{-1}$.
   Answer:

We reason as follows

$$
\begin{aligned}
\mathcal{F} \models \langle i\rangle[j]p \to p &\iff \mathcal{F} \models \langle i\rangle[j]\neg p \to \neg p && (1)\\
&\iff \mathcal{F} \models p \to \neg\langle i\rangle[j]\neg p && (2)\\
&\iff \mathcal{F} \models p \to [i]\neg[j]\neg p && (3)\\
&\iff \mathcal{F} \models p \to [i]\langle j\rangle p && (4)\\
&\iff R_i \subseteq R_j^{-1} && (5)
\end{aligned}
$$

where the steps are justified as follows:

(1) The direction $\Rightarrow$ follows from the fact that validity is closed under substitution; here we substitute $\neg p$ for p. The direction $\Leftarrow$ uses additionally that we may replace subformulas by equivalent subformulas; so from $\mathcal{F} \models \langle i\rangle[j]\neg p \to \neg p$ we infer $\mathcal{F} \models \langle i\rangle[j]\neg\neg p \to \neg\neg p$ and then replace $\neg\neg p$ by p.

(2) These are equivalent since one formula is the contraposition of the other.

(3) $\neg\langle i\rangle[j]\neg p$ is equivalent to $[i]\neg[j]\neg p$.

(4) $\neg[j]\neg p$ is equivalent to $\langle j\rangle p$.

(5) By the result proven in **2.(a)**.

(c) Are the formulas $p \to [i]\langle j\rangle p$ and $\langle i\rangle[j]p \to p$ equivalent? Prove your answer.

Answer:

No, they are not. Clearly, *inside* the class of I-frames with the property $R_i \subseteq R_j^{-1}$ they are equivalent, as we just have shown that they are both valid in that class. However, *outside* this class they need not be equivalent, as we show by the following counterexample.

First we recall the definition of equivalence of modal formulas: Two formulas φ and ψ are *equivalent*, which we denote by $\varphi \equiv \psi$, if $\varphi \leftrightarrow \psi$ is universally valid. In other words, $\varphi \equiv \psi$ when $\mathcal{M}, x \models \varphi$ iff $\mathcal{M}, x \models \psi$ for all models $\mathcal{M}$ and all points x of $\mathcal{M}$.

Now consider the following model $\mathcal{M} = (\{a\}, \{R_1, R_2\}, V)$ with $R_1 = \{(a, a)\}$, $R_2 = \varnothing$, and $V(p) = \varnothing$. Then we have $\mathcal{M}, a \models p \to [1]\langle 2\rangle p$ because of $\mathcal{M}, a \not\models p$. On the other hand $\mathcal{M}, a \models [2]p$ by $R_2 = \varnothing$, and so, by R_1aa, we have $\mathcal{M}, a \models \langle 1\rangle[2]p$. In combination with $\mathcal{M}, a \not\models p$ this gives $\mathcal{M}, a \not\models \langle 1\rangle[2]p \to p$. We conclude that $(p \to [1]\langle 2\rangle p) \not\equiv (\langle 1\rangle[2]p \to p)$.

8. We consider Hoare logic for proving correctness of the gcd-program.

9. (Done in lecture 10.)

   We consider the set of labels $I = \{a, b, c\}$ and the I-frame $\mathcal{F} = (W, \{R_a, R_b, R_c\})$ where:

$$W = \{w_1, w_2, w_3, w_4\} \qquad R_b = \{(w_2, w_3)\}$$
$$R_a = \{(w_1, w_2), (w_3, w_3)\} \qquad R_c = \{(w_2, w_4), (w_4, w_1)\}$$

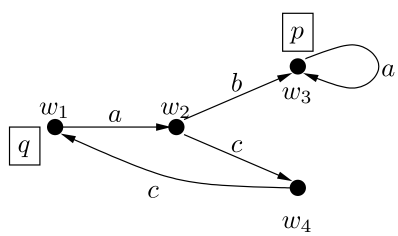
   and let $\mathcal{M} = (\mathcal{F}, V)$ with V the valuation on $\mathcal{F}$ defined by:

$$V(p) = \{w_3\} \quad \text{and} \quad V(q) = \{w_1\}$$

   (a) Draw the model $\mathcal{M}$ as a graph.

   Answer:

   The model $\mathcal{M}$ can be graphically represented by:



   (b) Prove or disprove:

   i. $\mathcal{M} \models [a]p$

   Answer:

   $\mathcal{M} \not\models [a]p$ due to $\mathcal{M}, w_1 \not\models [a]p$: state w_2 is an R_a-successor of w_1 but $w_2 \notin V(p)$.

   ii. $\mathcal{M} \models [a](p \vee (\langle b \rangle \top \wedge \langle c \rangle \top))$

   Answer:

   Let $\chi = [a](p \vee (\langle b \rangle \top \wedge \langle c \rangle \top))$. It holds that $\mathcal{M} \models \chi$ because, as we will show, χ is true in all points of $\mathcal{M}$. First of all, it is clear that $\mathcal{M}, w_2 \models \chi$ and $\mathcal{M}, w_4 \models \chi$, since both w_2 and w_4 are blind with respect to R_a. Moreover $\mathcal{M}, w_1 \models \chi$ since $R_a[w_1] = w_2$ and $\mathcal{M}, w_2 \models p \vee (\langle b \rangle \top \wedge \langle c \rangle \top)$; the latter holds because $\mathcal{M}, w_2 \models \langle b \rangle \top$ (by $R_b w_2 w_3$) and $\mathcal{M}, w_2 \models \langle c \rangle \top$ (by $R_c w_2 w_4$). Finally, we have $\mathcal{M}, w_3 \models \chi$ because $R_a[w_3] = \{w_3\}$ and $\mathcal{M}, w_3 \models p$.

iii. $\mathcal{F} \models [a](\langle a\rangle\top \vee (\langle b\rangle\top \wedge \langle c\rangle\top))$

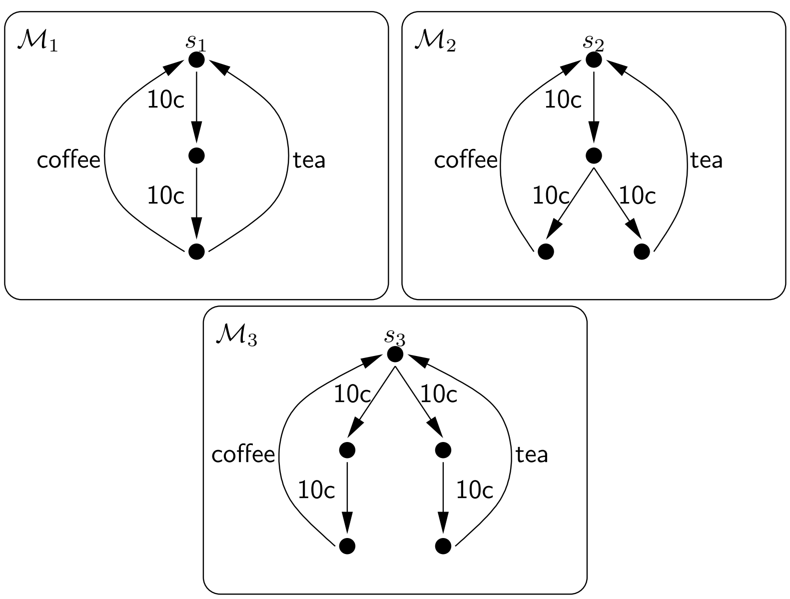
Answer:

Let $\zeta = [a](\langle a\rangle\top \vee (\langle b\rangle\top \wedge \langle c\rangle\top))$. We prove $\mathcal{F} \models \zeta$. Clearly, valuations play no role here; we prove that ζ is valid in all points: $\mathcal{F}, x \models \zeta$ for all $x \in W$.[1] As argued before, in points w_2 and w_4 all formulas of the form $[a]\varphi$ (with φ any formula) are true, the reason being that they have no R_a-successors; so in particular $\mathcal{F}, w_2 \models \zeta$ and $\mathcal{F}, w_4 \models \zeta$. Furthermore, note that $\mathcal{F}, w_2 \models \langle b\rangle\top$ and $\mathcal{F}, w_2 \models \langle c\rangle\top$ because w_2 has an R_b-successor (namely w_3) and an R_c-successor (namely w_4). Hence $\mathcal{F}, w_2 \models \langle b\rangle\top\wedge\langle c\rangle\top$, so $\mathcal{F}, w_2 \models \langle a\rangle\top\vee(\langle b\rangle\top\wedge\langle c\rangle\top)$ and so, as w_3 is the only R_a-successor of w_1, we have $\mathcal{F}, w_1 \models \zeta$. Finally, we see that $\mathcal{F}, w_3 \models \langle a\rangle\top$ as $R_a[w_3] = \{w_3\}$ and so $\mathcal{F}, w_3 \models \langle a\rangle\top \vee (\langle b\rangle\top \wedge \langle c\rangle\top)$, and, by the truth definition for $[a]$ and $R_a[w_3] = \{w_3\}$, we obtain $\mathcal{F}, w_3 \models \zeta$.

10. (Done in lecture 10.)

We consider multi-modal logic.

Let $I = \{\mathsf{10c}, \mathsf{coffee}, \mathsf{tea}\}$ and consider the I-models $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$:



[1] A formula ψ is valid in a point x of a frame $\mathcal{G}$, denoted by $\mathcal{G}, x \models \psi$, iff $\mathcal{G}, V, x \models \psi$ for all valuations V on $\mathcal{G}$.

Give distinguishing formulas (over I) for the processes s_1, s_2 and s_3.

Answer:

For every $i \in \{1,2,3\}$ we give a multimodal formula φ_i over $I = \{\mathsf{10c}, \mathsf{coffee}, \mathsf{tea}\}$ that distinguishes state s_i from s_j for all $j \neq i$ ($j \in \{1,2,3\}$).

$$\begin{aligned}
\varphi_1 &= [\mathsf{10c}][\mathsf{10c}]\langle \mathsf{coffee}\rangle \top \\
\varphi_2 &= [\mathsf{10c}]\neg[\mathsf{10c}]\langle \mathsf{coffee}\rangle \top \\
\varphi_3 &= \neg\varphi_1 \wedge \neg\varphi_2
\end{aligned}$$

Let us verify that $s_i \models \varphi_j$ iff $i = j$.

The bottom point in $\mathcal{M}_1$ satisfies $\langle \mathsf{coffee}\rangle \top$ (note that it also satisfies $[\mathsf{coffee}]\top$). Hence, since all 10c-successors of all 10c-successors of s_1 satisfy $\langle \mathsf{coffee}\rangle \top$, we obtain $s_1 \models \varphi_1$.

To see that $s_2 \not\models \varphi_1$, look at the middle point in $\mathcal{M}_2$; not all of its 10c-successors satisfy $\langle \mathsf{coffee}\rangle \top$, namely the state at the bottom right does not, for that cannot do a coffee-step.

Also $s_3 \not\models \varphi_1$, because the right middle point of $\mathcal{M}_3$ does not satisfy $[\mathsf{10c}]\langle \mathsf{coffee}\rangle \top$, because its right 10c-successor can't do a coffee-step.

We have $s_2 \models \varphi_2$ since the middle point in $\mathcal{M}_2$ does not satisfy $[\mathsf{10c}]\langle \mathsf{coffee}\rangle \top$ (because the state at the bottom right has no coffee-successor).

$s_1 \not\models \varphi_2$ because, as we have seen already, the middle point in $\mathcal{M}_1$ does satisfy $[\mathsf{10c}]\langle \mathsf{coffee}\rangle \top$.

$s_3 \not\models \varphi_2$ because the left 10c-successor of s_3 satisfies $[\mathsf{10c}]\langle \mathsf{coffee}\rangle \top$.

Finally, after all we have shown above, it is clear that φ_3 distinguishes s_3 from s_1 and s_2.