Remark
We first prove a fact that is used in the answers below.

Fact. Let $i,j \in I$, with $I$ some index set (for example the set of PDL-programs), and let $\mathcal{F} = \{(W, \{R_k\}_{k \in I})\}$ be an $I$-frame. Then

$$\mathcal{F} \models [i]p \rightarrow [j]p \iff R_j \subseteq R_i$$  \hfill (1)

Below we only need the direction $\Leftarrow$, but let us prove both directions here:

Proof.

($\Rightarrow$) Assume $\mathcal{F} \models [i]p \rightarrow [j]p$ and let $s,t$ be such that $s R_j t$. We have to prove $s R_i t$. Let $V$ be a valuation such that $V(p) = W \setminus \{t\}$. In the model $\mathcal{M} = (\mathcal{F}, V)$ it then holds that $\mathcal{M}, s \not\models [j]p$. Since $[i]p \rightarrow [j]p$ is valid in $\mathcal{F}$ it also holds in $\mathcal{M}, s$. That means that also $\mathcal{M}, s \not\models [i]p$, which is only the case if $s R_i t$.

($\Leftarrow$) Assume $R_j \subseteq R_i$ holds in $\mathcal{F}$. Consider a model $\mathcal{M}$ on $\mathcal{F}$, and a point $s$, and assume $\mathcal{M}, s \models [i]p$. We have to prove $\mathcal{M}, s \models [j]p$. To do so, let $t$ with $s R_j t$ be arbitrary (goal: $\mathcal{M}, t \models p$). By the assumption $R_j \subseteq R_i$ it also holds that $s R_i t$, and hence, by the assumption $\mathcal{M}, s \models [i]p$, we also obtain $\mathcal{M}, t \models p$. $\dashv$

Of course, with (1) we also get:

$$\mathcal{F} \models [i]p \leftrightarrow [j]p \iff R_i = R_j .$$ \hfill (2)

Exercises

1. (a)  
(b) Assume that $((W, R), V), x \models (\exists p)q$. Then there exists $y \in W$ with $(x, y) \in R? p = \{(z, z) \mid z \models p\}$ such that $y \models q$. From the
definition of $R_p$ follows that $y = x$ and $x \models p$. In addition $x \models q$.
Hence $x \models p \land q$.

For the other direction of the implication, assume that $((W, R), V), x \models p \land q$. Then $x \models p$ and hence $(x, x) \in R_p$. Also $x \models q$, and hence $x \models (?p)q$.

(c)

2. (a) We prove that $[\alpha(\beta \cup \gamma)]\varphi \leftrightarrow [\alpha \beta \cup \alpha \gamma]\varphi$ is a PDL-tautology. Using the characterization (2), for this it suffices to show $R_{\alpha(\beta \cup \gamma)} = R_{\alpha \beta \cup \alpha \gamma}$. This boils down to showing a familiar identity in relational algebra, namely that relational composition left-distributes over the union of relations, that is, for binary relations $R, S, T$, we have $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$:

$$x (R \circ (S \cup T)) y \iff \exists v ((x R v \land v (S \cup T) y)$$
$$\iff \exists v ((x R v \land v S y) \lor (x R v \land v T y))$$
$$\iff \exists v ((x R v \land v S y) \lor \exists v (x R v \land v T y))$$
$$\iff \exists v (x (R \circ S) y \lor x (R \circ T) y)$$
$$\iff \exists v (x (R \circ S) \cup (R \circ T) y)$$

(b) The proof that $[(\alpha \cup \beta)\gamma]\varphi \leftrightarrow [\alpha \gamma \cup \beta \gamma]\varphi$ is valid in all PDL-frames is analogous to the previous item.

3. A description of the model by means of set notation:

$$\mathcal{M} = (W^\mathcal{M}, R^\mathcal{M}_a, R^\mathcal{M}_b, V^\mathcal{M})$$
$$W^\mathcal{M} = \{s, t, u\}$$
$$R^\mathcal{M}_a = \{(t, s), (u, t)\}$$
$$R^\mathcal{M}_b = \{(s, t), (t, u)\}$$
$$V^\mathcal{M}(p) = \{t\}$$

(Likewise we will use $\mathcal{N} = (W^\mathcal{N}, R^\mathcal{N}_a, R^\mathcal{N}_b, V^\mathcal{N}).$)

(a) We show that $t$ and $n_3$ are bisimilar, and we know that bisimilar states have the same modal theory: if pointed models $\mathcal{X}, x$ and $\mathcal{X}', x'$ are bisimilar then, for all modal formulas $\varphi$, it holds that $\mathcal{X}, x \models \varphi$ if and only if $\mathcal{X}', x' \models \varphi$. 

2
Define the relation \( G \subseteq W^M \times W^N \) by
\[
G := \{ (s, n_4), (s, n_5), (t, n_2), (t, n_3), (u, n_1) \}.
\]

We show that \( G \) is a bisimulation:
- First of all, we notice that \( G \) satisfies the requirement of atomic harmony: for all \( (x, x') \in G \) and all propositional variables \( q \) we have \( M, x \vDash q \iff N, x' \vDash q \).
- To verify the zig-condition of \( G \), for every pair \( (x, x') \in G \), for every \( i \in \{a, b\} \), and for every \( y \in W^M \) with \( R_i \) \( x,y \), we have to find a point \( y' \in W^N \) such that \( R_i y'y' \) and \( (y, y') \in G \).

This we indicate by \( \xrightarrow{xy} y'. \)

- Similarly for diagrams showing the zag condition (when a step \( R_i y'y' \) has to be matched by a step \( R_i x,y \)) we write \( \xrightarrow{xy} y'. \)

(One diagram more than for zig due to two outgoing a-steps from 1.)

(b) We use the following:
\[
\hat{R}_p = \{(n_1, n_1), (n_3, n_3)\}
\]
\[
\hat{R}_a = R_a = \{(n_1, n_2), (n_1, n_3), (n_2, n_4), (n_3, n_5)\}
\]
\[
\hat{R}_b = R_b = \{(n_2, n_1), (n_3, n_1), (n_4, n_3), (n_5, n_2)\}
\]
\[
\hat{R}_{ab} = \hat{R}_a; \hat{R}_b = \{(n_1, n_2), (n_1, n_3), (n_2, n_4), (n_3, n_5)\} \cap \{(n_2, n_1), (n_3, n_1), (n_4, n_3), (n_5, n_2)\} = \{(n_2, n_1), (n_2, n_3), (n_3, n_2)\}
\]
\[
\hat{R}_{abb} = \hat{R}_{ab}; \hat{R}_b = \{(n_1, n_1), (n_2, n_3), (n_3, n_2)\} \cap \{(n_2, n_1), (n_3, n_1), (n_4, n_3), (n_5, n_2)\} = \{(n_2, n_1), (n_3, n_1)\}
\]
\[ \hat{R}_{abba} = \hat{R}_{abbi}; \hat{R}_a = \{(n_2, n_1), (n_3, n_1)\}; \{(n_1, n_2), (n_1, n_3), (n_2, n_4), (n_3, n_5)\} = \\
\{(n_2, n_2), (n_2, n_3), (n_3, n_2), (n_3, n_3)\}. \]

\[ \hat{R}_{?p} = \{(n_2, n_2), (n_3, n_3)\} \]

\[ \hat{R}_{?-p} = \{(n_1, n_1), (n_4, n_4), (n_5, n_5)\} \]

\[ \hat{R}_\beta = \hat{R}_{(p?;abba)^*;?-p?} = (Id \cup \{(n_2, n_3), (n_3, n_2)\}); \{(n_1, n_1), (n_4, n_4), (n_5, n_5)\} = \\
\{(n_1, n_1), (n_4, n_4), (n_5, n_5)\}. \]