Maybe one or two more answers will be added later.

Exercises

1. Show validity of the following PDL-formulas (from lecture 12):

   (a) $\langle \alpha ; \beta \rangle p \leftrightarrow \langle \alpha \rangle \langle \beta \rangle p$

   Answer:

   Let $\mathcal{M}$ be a PDL-model.

   Let x be a state in $\mathcal{M}$. We have $x \models \langle \alpha ; \beta \rangle p$ if and only if there exists a state y such that $(x, y) \in R_{\alpha;\beta} = R_\alpha ; R_\beta$ (the equality holds by the requirements on a PDL-model), which means there is a state u such that $(x, u) \in R_\alpha$ and $(u, y) \in R_\beta$. This means (recall that $y \models p$) that $x \models \langle \alpha \rangle \langle \beta \rangle p$.

   (We can also treat both implications separately.)

   (b) $\langle \alpha \cup \beta \rangle p \leftrightarrow \langle \alpha \rangle p \vee \langle \beta \rangle p$

   Answer:

   Let x be a state in a PDL-model $\mathcal{M}$ and assume $x \models \langle \alpha \cup \beta \rangle p$. This means there exists a state y with $(x, y) \in R_{\alpha \cup \beta} = R_\alpha \cup R_\beta$ and $y \models p$. We have $(x, y) \in R_\alpha$ or $(x, y) \in R_\beta$ (or both). In the first case, we have $x \models \langle \alpha \rangle p$. In the second case, we have $x \models \langle \beta \rangle p$. So we have $x \models \langle \alpha \rangle p$ or $x \models \langle \beta \rangle p$. So $x \models \langle \alpha \rangle p \vee \langle \beta \rangle p$.

   (c) $\langle \alpha^* \rangle p \leftrightarrow p \vee \langle \alpha \rangle \langle \alpha^* \rangle p$

   (d) $\langle p? \rangle q \leftrightarrow p \wedge q$

   Answer:

   Assume that $((W, R), V), x \models \langle p? \rangle q$. Then there exists $y \in W$ with $(x, y) \in R_{?p} = \{(z, z) \mid z \models p\}$ such that $y \models q$. From the definition of $R_{p?}$ follows that $y = x$ and $x \models p$. In addition $x \models q$. Hence $x \models p \wedge q$.

   For proving the other direction of the implication, assume that $((W, R), V), x \models p \wedge q$. Then $x \models p$ and hence $(x, x) \in R_{p?}$. Also $x \models q$, and hence $x \models \langle ?p \rangle q$.

   (e) $[\alpha^*]p \leftrightarrow p \wedge [\alpha^*](p \rightarrow [\alpha]p)$ (induction principle)

2. Show validity of the following PDL-formulas:

   (a) $[\alpha(\beta \cup \gamma)]\varphi \leftrightarrow [\alpha\beta \cup \alpha\gamma]\varphi$

   (b) $[(\alpha \cup \beta)\gamma]\varphi \leftrightarrow [\alpha\gamma \cup \beta\gamma]\varphi$

3. Let $\mathcal{M} = (W, \{R_\alpha \mid \alpha \in \textsc{Prog}\}, V)$ be a PDL-model. Determine the transition relations corresponding to the following programs:

   (a) $\mathsf{if}\ p\ \mathsf{then}\ \alpha\ \mathsf{else}\ \beta$

   Answer:

   Let $\gamma := \mathsf{if}\ p\ \mathsf{then}\ \alpha\ \mathsf{else}\ \beta$.

$$\begin{aligned}
s\,R_\gamma\,t &\iff s\,R_{(p?;\alpha)\cup(\neg p?;\beta)}\,t \\
&\iff s\,(R_{p?;\alpha} \cup R_{\neg p?;\beta})\,t \\
&\iff (s\,R_{p?;\alpha}\,t) \lor (s\,R_{\neg p?;\beta}\,t) \\
&\iff (\mathcal{M}, s \vDash p \land s\,R_\alpha\,t) \lor (\mathcal{M}, s \nvDash p \land s\,R_\beta\,t)
\end{aligned}$$

   (b) $\mathsf{while}\ p\ \mathsf{do}\ \alpha$

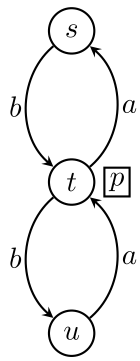
   Answer:

   Let $\delta := \mathsf{while}\ p\ \mathsf{do}\ \alpha$.

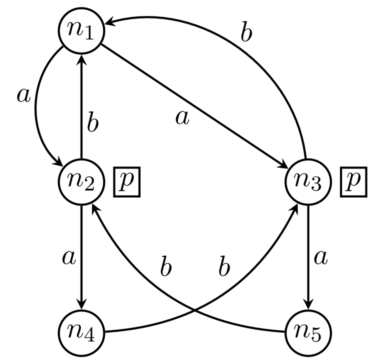
$$\begin{aligned}
s\,R_\delta\,t &\iff s\,R_{(p?;\alpha)^*;\neg p?}\,t \\
&\iff s\,(R_{(p?;\alpha)^*}; R_{\neg p?})\,t \\
&\iff \exists u (s\,R_{(p?;\alpha)^*}\,u \land u\,R_{\neg p?}\,t) \\
&\iff s\,R_{(p?;\alpha)^*}\,t \land \mathcal{M}, t \vDash \neg p \\
&\iff \exists r_0, \ldots, r_n\,(s = r_0 \land \forall i \in \mathbb{N}_{<n}\,(r_i\,R_{p?;\alpha}\,r_{i+1}) \land r_n = t) \land \mathcal{M}, t \nvDash p \\
&\iff \exists r_0, \ldots, r_n\,(s = r_0 \land \forall i \in \mathbb{N}_{<n}\,(\mathcal{M}, r_i \vDash p \land r_i\,R_\alpha\,r_{i+1}) \land r_n = t) \land \mathcal{M}, t \nvDash p
\end{aligned}$$

4. (From exam 2017 06 08.)

   Consider the models $\mathcal{M}$ and $\mathcal{N}$ defined by:



$\mathcal{M}$



$\mathcal{N}$

(a) Show that there is no modal formula distinguishing state n_3 in model $\mathcal{N}$ from state t in model $\mathcal{M}$.

Answer:

We show that t and n_3 are bisimilar, and we know that bisimilar states have the same modal theory: if pointed models $\mathcal{X}, x$ and $\mathcal{X}', x'$ are bisimilar then, for all modal formulas φ, it holds that $\mathcal{X}, x \vDash \varphi$ if and only if $\mathcal{X}', x' \vDash \varphi$.

Define the relation $G \subseteq W^{\mathcal{M}} \times W^{\mathcal{N}}$ by

$$G := \{(s, n_4), (s, n_5), (t, n_2), (t, n_3), (u, n_1)\}.$$

We show that G is a bisimulation:

- First of all, we notice that G satisfies the requirement of atomic harmony: for all $(x, x') \in G$ and all propositional variables q we have $\mathcal{M}, x \vDash q$ iff $\mathcal{N}, x' \vDash q$.
- To verify the zig-condition of G, for every pair $(x, x') \in G$, for every $i \in \{a, b\}$, and for every $y \in W^{\mathcal{M}}$ with $R_i^{\mathcal{M}} xy$, we have to find a point $y' \in W^{\mathcal{N}}$ such that $R_i^{\mathcal{N}} x'y'$ and $(y, y') \in G$. This we indicate by $\begin{array}{c|c} x & x' \\ \hline y & y' \end{array} i$.

  $\begin{array}{c|c} s & n_4 \\ \hline t & n_3 \end{array} b$ $\begin{array}{c|c} s & n_5 \\ \hline t & n_2 \end{array} b$ $\begin{array}{c|c} t & n_2 \\ \hline s & n_4 \end{array} a$ $\begin{array}{c|c} t & n_2 \\ \hline u & n_1 \end{array} b$ $\begin{array}{c|c} t & n_3 \\ \hline s & n_5 \end{array} a$ $\begin{array}{c|c} t & n_3 \\ \hline u & n_1 \end{array} b$ $\begin{array}{c|c} u & n_1 \\ \hline t & n_3 \end{array} a$

- Similarly for diagrams showing the zag condition (when a step $R_i^{\mathcal{N}} x'y'$ has to be matched by a step $R_i^{\mathcal{M}} xy$) we write

$\begin{array}{c|c} x & x' \\ \hline y & y' \end{array} i.$

$\begin{array}{c|c} s & n_4 \\ \hline t & n_3 \end{array} b \quad \begin{array}{c|c} s & n_5 \\ \hline t & n_2 \end{array} b \quad \begin{array}{c|c} s & n_5 \\ \hline t & n_2 \end{array} b \quad \begin{array}{c|c} t & n_2 \\ \hline s & n_4 \end{array} a \quad \begin{array}{c|c} t & n_2 \\ \hline u & n_1 \end{array} b \quad \begin{array}{c|c} t & n_3 \\ \hline s & n_5 \end{array} a \quad \begin{array}{c|c} t & n_3 \\ \hline u & n_1 \end{array} b \quad \begin{array}{c|c} u & n_1 \\ \hline t & n_2 \end{array} a \quad \begin{array}{c|c} u & n_1 \\ \hline t & n_3 \end{array} a$

(One diagram more than for zig due to two outgoing a-steps from n_1.)

(b) Let $\widehat{\mathcal{N}}$ be the PDL-extension of model $\mathcal{N}$. Compute the transition relation $\widehat{R}_\beta$ corresponding to the PDL-program $\beta = \mathsf{while}\ p\ \mathsf{do}\ abba$.

Answer:

We use the following:

$\hat{R}_{?p} = \{(n_1, n_1), (n_3, n_3)\}$

$\hat{R}_a = R_a = \{(n_1, n_2), (n_1, n_3), (n_2, n_4), (n_3, n_5)\}$

$\hat{R}_b = R_b = \{(n_2, n_1), (n_3, n_1), (n_4, n_3), (n_5, n_2)\}$

$\hat{R}_{ab} = \hat{R}_a; \hat{R}_b = \{(n_1, n_2), (n_1, n_3), (n_2, n_4), (n_3, n_5)\}; \{(n_2, n_1), (n_3, n_1), (n_4, n_3), (n_5, n_2)\} = \{(n_1, n_1), (n_2, n_3), (n_3, n_2)\}$

$\hat{R}_{abb} = \hat{R}_{ab}; \hat{R}_b = \{(n_1, n_1), (n_2, n_3), (n_3, n_2)\}; \{(n_2, n_1), (n_3, n_1), (n_4, n_3), (n_5, n_2)\} = \{(n_2, n_1), (n_3, n_1)\}$

$\hat{R}_{abba} = \hat{R}_{abb}; \hat{R}_a = \{(n_2, n_1), (n_3, n_1)\}; \{(n_1, n_2), (n_1, n_3), (n_2, n_4), (n_3, n_5)\} = \{(n_2, n_2), (n_2, n_3), (n_3, n_2), (n_3, n_3)\}$.

$\hat{R}_{?p} = \{(n_2, n_2), (n_3, n_3)\}$

$\hat{R}_{?\neg p} = \{(n_1, n_1), (n_4, n_4), (n_5, n_5)\}$

$\hat{R}_\beta = \hat{R}_{(p?;abba)^*;\neg p?} = (Id \cup \{(n_2, n_3), (n_3, n_2)\}); \{(n_1, n_1), (n_4, n_4), (n_5, n_5)\} = \{(n_1, n_1), (n_4, n_4), (n_5, n_5)\}$

(c) Determine whether the PDL-formula $[\beta]p \leftrightarrow p$ globally holds in $\widehat{\mathcal{N}}$. Prove your answer.