where are we?

basic modal logic
models, bisimulation, invariance, frame properties

basic temporal logic
past and future, temporal bisimulation

multi-modal logic with parameter set $I$ of indices

various instances: $I = \emptyset$, $I = \{0\}$, $I = \{F, P\}$ with temporal models, $I = \{\text{all regular programs over } A\}$ with PDL-models

overview

- propositional dynamic logic
- epistemic logic

overview

- propositional dynamic logic
  - propositional dynamic logic: alternative semantics
  - propositional dynamic logic: bisimulation

- epistemic logic
some formulas that are valid in all PDL-models

\( (\alpha; \beta)p \leftrightarrow (\alpha)(\beta)p \)

\( (\alpha \cup \beta)p \leftrightarrow (\alpha)p \lor (\beta)p \)

\( (\alpha^*)p \leftrightarrow p \lor (\alpha)(\alpha^*)p \)

\( (p?)q \leftrightarrow p \land q \)

\[ \alpha^* \]p \leftrightarrow p \land [\alpha^*](p \rightarrow [\alpha]p) \ (\text{induction principle})

example

how can we express the following property:

\( p \) is alternatively true and false

along all execution paths of a starting in current state with \( p \) true

\[ \neg p \] \land [a^*]( (p \rightarrow [a] \neg p) \land (\neg p \rightarrow [a]p) )

or equivalently:

\[ [(aa)^*]p \land [a(aa)^*] \neg p \]

without the condition on the start state:

\( (p \rightarrow [a] \neg p) \land (\neg p \rightarrow [a]p) \)

example

which property is expressed by the following formula:

\( \langle \text{while } p \text{ do } \alpha \rangle \top \)

while \( p \) do \( \alpha \) terminates if and only if

it is possible by repeated execution of \( \alpha \) to reach state with \( \neg p \)

the formula is equivalent to \( (\alpha^*) \neg p \)

alternative semantics

\( M, w \models (\alpha)\phi \) if and only if

there exists \( w' \) such that \( M, w, w' \models \alpha \) and \( M, w' \models \phi \)

\( M, w, w' \models a \) if \( (w, w') \in R_a \) with \( a \) an atomic program

extend this to sequential composition, choice, iteration, and test

\( M, w, w' \models ?\phi \) if \( w = w' \) and \( M, w \models \phi \)
obvious question

we start with $W$, for every atomic program $a$ a relation $R_a$, and $V$
we get both a semantics according to the PDL-model,
and one following the book

are the two semantics equivalent?

bisimulation for PDL-models

do we need to consider all infinitely many relations?

no, it is sufficient to consider all $R_a$ with $a$ atomic
because PDL-constructions are safe for bisimulation
intersection is not safe for bisimulation
inverse is not safe for bisimulation

overview

- propositional dynamic logic
- epistemic logic

logic of knowledge

Plato (400 BC): true opinion is in general unsufficient for knowledge

see Stanford Encyclopedia of Philosophy
example: muddy children

$k$ out of $n$ children have mud on their head

a child can see all other children, but cannot see himself

the father announces 'at least one of you has mud on his head'

the question 'who has mud on his head?'

is repeatedly asked, and answered the $k$th time

epistemic logic: intuition

$\Box \phi$ means 'I know that $\phi$ is true' and is written $K \phi$

more generally

$[i] \phi$ means 'agent $i$ knows that $\phi$ is true' and is written $K_i \phi$

epistemic logic as an instance of multi-modal logic

we use index set $I = \{1, \ldots, n\}$

the set of epistemic formulas is the set of $I$-multi-modal-formulas:

\[
\phi ::= p \mid \neg \phi \mid \phi \land \psi \mid K_i \phi
\]

the diamond $\langle i \rangle \phi$ is defined as $\neg K_i \neg \phi$

we write $K$ and $\Diamond$ if the index is irrelevant

epistemic frame

an epistemic $n$-frame is a $\{1, \ldots, n\}$-frame

$\mathcal{F} = (W, \{R_1, \ldots, R_n\})$

also written as $\mathcal{F} = (W, R_1, \ldots, R_n)$

intuition of $sR_i t$:

given his information in situation $s$, agent $i$ considers $t$ possible

or: for agent $i$ situation $t$ is an epistemic alternative for $s$

or: $t$ is consistent with the knowledge of $i$ in $s$
an epistemic \( n \)-model is an epistemic \( n \)-frame with a valuation:
\[ M = (W, R_1, \ldots, R_n, V) \]

the interpretation of formulas in a pointed model is as before!
in particular \( M, s \models K_i \phi \) if and only if \( M, t \models \phi \) for all \( t \) with \( s R_i t \)

reconsider the examples

distribution
\( DB_i : K_i(p \rightarrow q) \rightarrow (K_i p \rightarrow K_i q) \)

truth axiom or veridicality
\( A1_i : K_i p \rightarrow p \)

positive introspection
\( A2_i : K p \rightarrow K_i K_i p \)

negative introspection
\( A3_i : \neg K_i p \rightarrow K_i \neg K_i p \)

examples

\( K(p \rightarrow q) \rightarrow (K p \rightarrow K q) \) valid in all frames

\( K p \rightarrow p \) valid in reflexive frames

\( K p \rightarrow KK p \) valid in transitive frames

\( \neg K p \rightarrow K \neg p \) valid in euclidean frames

\( R \) is euclidean if: \( \forall xyz \ (R xy \land Rxz \rightarrow R yz) \)

frame correspondences

for all epistemic \( n \)-frames \( F = (W, R_1, \ldots, R_n) \) we have:

\( F \models A1_i \) for all \( i \) iff every \( R_i \) is reflexive

\( F \models A2_i \) for all \( i \) iff every \( R_i \) is transitive

\( F \models A3_i \) for all \( i \) iff every \( R_i \) is euclidean
reflexive frames are characterized by A1

reflexive-transitive frames are characterized by A1 and A2

frames with equivalence relations are characterized by A1 and A2 and A3