overview

- derivations in modal logic
- soundness and completeness results
- modelling knowledge
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proof system for basic modal logic

extension:
\[ \vdash \phi \text{ if } \phi \text{ is a tautology from prop1} \]

modal distribution:
\[ \vdash \Box(p \rightarrow q) \rightarrow \Box p \rightarrow \Box q \]

modus ponens:
\[ \text{if } \vdash \phi \rightarrow \psi \text{ and } \vdash \phi \text{ then } \vdash \psi \]

necessitation:
\[ \text{if } \vdash \phi \text{ then } \vdash \Box \phi \]

substitution:
\[ \text{if } \vdash \phi \text{ then } \vdash \phi^\sigma \text{ for every substitution } \sigma \]
if ⊢ φ → ψ then ⊢ □φ → □ψ

1. φ → ψ (assumption)

2. □(φ → ψ) (necessitation, 1)

3. □(p → q) → □p → □q (modal distribution)

4. □(φ → ψ) → □φ → □ψ (subst, 3)

5. □φ → □ψ (modus ponens, 4, 2)

we use the substitution rule explicitly, MLOM uses it implicitly
example MLOM p52

if ⊢ φ → ψ then ⊢ ◊φ → ◊ψ

1. φ → ψ  (assumption)

2. (a → b) → (¬b → ¬a)  (tauto)

3. (φ → ψ) → (¬ψ → ¬φ)  (substitution, 2)

4. ¬ψ → ¬φ  (modus ponens, 1, 3)

5. □¬ψ → □¬φ  (we use the previous example !)

6. (□¬ψ → □¬φ) → (¬□¬φ → ¬□¬ψ)  (substitution, 2)

7. ¬□¬φ → ¬□¬ψ  (modus ponens, 5, 6)

we conclude: if ⊢ φ → ψ then ⊢ ¬□¬φ → ¬□¬ψ

from the duality of □ and ◊ we get: if ⊢ φ → ψ then ⊢ ◊φ → ◊ψ
admissible rule

in the previous example we used a tautology of prop1 of the form \( l \rightarrow r \)

we followed the following pattern:

\( l^{\sigma} \) derivable

\( l^{\sigma} \rightarrow r^{\sigma} \) instance of a tautology from prop1 so derivable

\( r^{\sigma} \) derivable by modus ponens

we can generalize this pattern and formulate an admissible rule
admissible rule: PROP

if provability of $\phi_1, \ldots, \phi_n$ implies provability of $\psi$, then the following is an admissible rule:

$$\frac{\phi_1 \cdots \phi_n}{\psi}$$

the proof rule PROP is defined as follows:

if $(\phi_1 \land \ldots \land \phi_n) \rightarrow \psi$ is a tautology

then for every substitution $\sigma$ the following is an admissible proof rule:

$$\frac{\phi_1^\sigma \cdots \phi_n^\sigma}{\psi^\sigma}$$
PROP is admissible (for \( n = 3 \))

assume \( \phi_1 \rightarrow \phi_2 \rightarrow \phi_3 \rightarrow \psi \) is a tautology from prop1; derivation:

1. \( \phi_1 \rightarrow \phi_2 \rightarrow \phi_3 \rightarrow \psi \) (tauto)

2. \( \phi_1^\sigma \rightarrow \phi_2^\sigma \rightarrow \phi_3^\sigma \rightarrow \psi^\sigma \) (subst, 1)

3. \( \phi_1^\sigma \) (assumption)

4. \( \phi_2^\sigma \) (assumption)

5. \( \phi_3^\sigma \) (assumption)

6. \( \phi_2^\sigma \rightarrow \phi_3^\sigma \rightarrow \psi^\sigma \) (modus ponens, 2, 3)

7. \( \phi_3^\sigma \rightarrow \psi^\sigma \) (modus ponens, 4, 6)

8. \( \psi^\sigma \) (modus ponens, 5, 7)

so PROP is an admissible rule and we can use it in our derivations
we have shown: if $\vdash \phi \rightarrow \psi$ then $\vdash \Box \phi \rightarrow \Box \psi$

hence the following rule, called DISTR, is admissible:

$$
\frac{\phi \rightarrow \psi}{
\Box \phi \rightarrow \Box \psi
}$$

so DISTR is an admissible rule and we can use it in our derivations
we show $\vdash \diamond \phi \land \Box (\phi \rightarrow \psi) \rightarrow \diamond \phi$

first we rewrite the formula:

$\diamond \phi \land \Box (\phi \rightarrow \psi) \rightarrow \diamond \psi$

$\Box (\phi \rightarrow \psi) \land \diamond \phi \rightarrow \diamond \psi$

$\Box (\phi \rightarrow \psi) \rightarrow \diamond \phi \rightarrow \diamond \psi$

$\Box (\phi \rightarrow \psi) \rightarrow (\neg \Box \neg \phi \rightarrow \neg \Box \neg \psi)$

$\Box (\phi \rightarrow \psi) \rightarrow (\Box \neg \psi \rightarrow \Box \neg \phi)$

we almost see the pattern of a tautology from prop1:

$(a \rightarrow b) \rightarrow (\neg b \rightarrow \neg a)$
we show ⊢ □(φ → ψ) → (□¬ψ → □¬φ)
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proof system $K$ for basic modal logic
use notation $K$ instead of $\Box$, and multi-modal setting

extension:
$\vdash \phi$ if $\phi$ is a tautology from prop1

modal distribution:
$\vdash K_i(p \to q) \to K_ip \to K_iq$

modus ponens:
if $\vdash \phi \to \psi$ and $\vdash \phi$ then $\vdash \psi$

necessitation:
if $\vdash \phi$ then $\vdash K_i\phi$

substitution:
if $\vdash \phi$ then $\vdash \phi^\sigma$ for every substitution $\sigma$
we extend system K step by step

system $T$: truth axiom added (for every $i$)
A1$_i$: $K_i p \rightarrow p$

system $S4$: positive introspection added (for every $i$)
A2$_i$: $K_i p \rightarrow K_i K_i p$

system $S5$: negative introspection added (for every $i$)
A3$_i$: $\neg K_i p \rightarrow K_i \neg K_i p$
four times soundness and completeness

$K$ is sound and complete for all epistemic frames

$T$ is sound and complete for all reflexive frames

$S4$ is sound and complete for all reflexive-transitive frames

$S5$ is sound and complete for all reflexive-transitive-symmetric frames
we consider system $K$ with 2 agents and show that $K_1 K_2 p \rightarrow K_2 K_1 p$ is not derivable

we give a countermodel:

states $a, b, c$

accessibility relations $R_1 = \{(a, b)\}$ and $R_2 = \{(a, a), (b, c)\}$

valuation: $V(p) = \{c\}$

then $a \models K_1 K_2 p$ but $a \not\models K_2 K_1 p$

by soundness we conclude $\not\models K_1 K_2 p \rightarrow K_2 K_1 p$
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example A fully informed

agent A is fully informed about $p$

in world where $p$ holds we have:

$$p, \, K_A p, \, \neg K_Q p \land \neg K_Q \neg p, \, K_Q (K_A p \lor K_A \neg p), \, K_A (K_A p \lor K_A \neg p)$$

update of the model after question $p$? and answer yes
muddy children: initial model
muddy children: after public announcement
muddy children: after all silent to question 1
muddy children: after all silent to question 2
card game: initial model
card game: update after 1 announces not to have white