overview

- derivations in modal logic
- soundness and completeness results
- modelling knowledge

proof system for basic modal logic

extension:
\[ \vdash \phi \text{ if } \phi \text{ is a tautology from prop1} \]

modal distribution:
\[ \vdash \Box(p \rightarrow q) \rightarrow \Box p \rightarrow \Box q \]

modus ponens:
if \( \vdash \phi \rightarrow \psi \) and \( \vdash \phi \) then \( \vdash \psi \)

necessitation:
if \( \vdash \phi \) then \( \vdash \Box \phi \)

substitution:
if \( \vdash \phi \) then \( \vdash \phi^\sigma \) for every substitution \( \sigma \)
We consider the following proposition:

\[ \phi \rightarrow \psi \]

We derive the following steps:

1. \[ \phi \rightarrow \psi \] (assumption)
2. \[ \Box(\phi \rightarrow \psi) \] (necessitation, 1)
3. \[ \Box(p \rightarrow q) \rightarrow \Box p \rightarrow \Box q \] (modal distribution)
4. \[ \Box(\phi \rightarrow \psi) \rightarrow \Box \phi \rightarrow \Box \psi \] (substitution, 3)
5. \[ \Box \phi \rightarrow \Box \psi \] (modus ponens, 4, 2)

We use the substitution rule explicitly, MLOM uses it implicitly.

**Admissible rule**

In the previous example, we used a tautology of prop1 of the form \( l \rightarrow r \)
we followed the following pattern:

- \( l^\sigma \) derivable
- \( l^\sigma \rightarrow r^\sigma \) instance of a tautology from prop1 so derivable
- \( r^\sigma \) derivable by modus ponens

We can generalize this pattern and formulate an admissible rule.

**Example MLOM p52**

If \( \vdash \phi \rightarrow \psi \) then \( \lozenge \phi \rightarrow \lozenge \psi \)

1. \( \phi \rightarrow \psi \) (assumption)
2. \( (a \rightarrow b) \rightarrow (\neg b \rightarrow \neg a) \) (tauto)
3. \( (\phi \rightarrow \psi) \rightarrow (\neg \psi \rightarrow \neg \phi) \) (substitution, 2)
4. \( \neg \psi \rightarrow \neg \phi \) (modus ponens, 1, 3)
5. \( \Box \neg \psi \rightarrow \Box \neg \phi \) (we use the previous example!)
6. \( (\Box \neg \psi \rightarrow \Box \neg \phi) \rightarrow (\neg \Box \neg \phi \rightarrow \neg \Box \neg \psi) \) (substitution, 2)
7. \( \neg \Box \neg \phi \rightarrow \neg \Box \neg \psi \) (modus ponens, 5, 6)

We conclude: if \( \vdash \phi \rightarrow \psi \) then \( \vdash \neg \Box \neg \phi \rightarrow \neg \Box \neg \psi \)

From the duality of \( \Box \) and \( \lozenge \) we get: if \( \vdash \phi \rightarrow \psi \) then \( \vdash \lozenge \phi \rightarrow \lozenge \psi \)

**Admissible rule: PROP**

If provability of \( \phi_1, \ldots, \phi_n \) implies provability of \( \psi \),
then the following is an admissible rule:

\[
\begin{align*}
\phi_1 & \ldots \phi_n \\
\psi
\end{align*}
\]

The proof rule PROP is defined as follows:

If \( (\phi_1 \land \ldots \land \phi_n) \rightarrow \psi \) is a tautology
then for every substitution \( \sigma \) the following is an admissible proof rule:

\[
\begin{align*}
\phi_1^\sigma & \ldots \phi_n^\sigma \\
\psi^\sigma
\end{align*}
\]
PROP is admissible (for $n = 3$)

assume $\phi_1 \rightarrow \phi_2 \rightarrow \phi_3 \rightarrow \psi$ is a tautology from prop1; derivation:

1. $\phi_1 \rightarrow \phi_2 \rightarrow \phi_3 \rightarrow \psi$ (tauto)
2. $\phi_1^\sigma \rightarrow \phi_2^\sigma \rightarrow \phi_3^\sigma \rightarrow \psi^\sigma$ (subst, 1)
3. $\phi_1^\sigma$ (assumption)
4. $\phi_2^\sigma$ (assumption)
5. $\phi_3^\sigma$ (assumption)
6. $\phi_2^\sigma \rightarrow \phi_3^\sigma \rightarrow \psi^\sigma$ (modus ponens, 2, 3)
7. $\phi_3^\sigma \rightarrow \psi^\sigma$ (modus ponens, 4, 6)
8. $\psi^\sigma$ (modus ponens, 5, 7)

so PROP is an admissible rule and we can use it in our derivations

example MLOM p52

we show $\vdash \lozenge \phi \land \Box (\phi \rightarrow \psi) \rightarrow \lozenge \phi$

first we rewrite the formula:

$\lozenge \phi \land \Box (\phi \rightarrow \psi) \rightarrow \lozenge \psi$

$\Box (\phi \rightarrow \psi) \land \lozenge \phi \rightarrow \lozenge \psi$

$\Box (\phi \rightarrow \psi) \rightarrow \lozenge \phi \rightarrow \lozenge \psi$

$\Box (\phi \rightarrow \psi) \rightarrow (\neg \Box \neg \phi \rightarrow \neg \Box \neg \psi)$

$\Box (\phi \rightarrow \psi) \rightarrow (\Box \neg \psi \rightarrow \Box \neg \phi)$

we almost see the pattern of a tautology from prop1:

$(a \rightarrow b) \rightarrow (\neg b \rightarrow \neg a)$

admissible rule: DISTR

we have shown: if $\vdash \phi \rightarrow \psi$ then $\vdash \Box \phi \rightarrow \Box \psi$

hence the following rule, called DISTR, is admissible:

$\phi \rightarrow \psi$

$\Box \phi \rightarrow \Box \psi$

so DISTR is an admissible rule and we can use it in our derivations

example MLPM p52, continued

we show $\vdash \Box (\phi \rightarrow \psi) \rightarrow (\Box \neg \psi \rightarrow \Box \neg \phi)$
overview

• derivations in modal logic
• soundness and completeness results
• modelling knowledge

proof system \( K \) for basic modal logic

use notation \( K \) instead of \( \Box \), and multi-modal setting

extension:
\[ \vdash \phi \text{ if } \phi \text{ is a tautology from prop1} \]

modal distribution:
\[ \vdash K_i(p \rightarrow q) \rightarrow K_i p \rightarrow K_i q \]

modus ponens:
if \( \vdash \phi \rightarrow \psi \) and \( \vdash \phi \) then \( \vdash \psi \)

necessitation:
if \( \vdash \phi \) then \( \vdash K_i\phi \)

substitution:
if \( \vdash \phi \) then \( \vdash \phi^\sigma \) for every substitution \( \sigma \)

we extend system \( K \) step by step

four times soundness and completeness

system \( T \): truth axiom added (for every \( i \))
\[ A_{1i} : K_i p \rightarrow p \]

system \( S4 \): positive introspection added (for every \( i \))
\[ A_{2i} : K_i p \rightarrow K_i K_i p \]

system \( S5 \): negative introspection added (for every \( i \))
\[ A_{3i} : \neg K_i p \rightarrow K_i \neg K_i p \]

system \( T \): truth axiom added (for every \( i \))
\[ A_{1i} : K_i p \rightarrow p \]

system \( S4 \): positive introspection added (for every \( i \))
\[ A_{2i} : K_i p \rightarrow K_i K_i p \]

system \( S5 \): negative introspection added (for every \( i \))
\[ A_{3i} : \neg K_i p \rightarrow K_i \neg K_i p \]

\( K \) is sound and complete for all epistemic frames

\( T \) is sound and complete for all reflexive frames

\( S4 \) is sound and complete for all reflexive-transitive frames

\( S5 \) is sound and complete for all reflexive-transitive-symmetric frames
we consider system $K$ with 2 agents and show that

$K_1K_2p \rightarrow K_2K_1p$ is not derivable

we give a countermodel:

states $a, b, c$

accessibility relations $R_1 = \{(a, b)\}$ and $R_2 = \{(a, a), (b, c)\}$

valuation: $V(p) = \{c\}$

then $a \models K_1K_2p$ but $a \not\models K_2K_1p$

by soundness we conclude $\not\models K_1K_2p \rightarrow K_2K_1p$

derivations in modal logic

soundness and completeness results

modelling knowledge

example A fully informed

agent A is fully informed about $p$

in world where $p$ holds we have:

$p, K_Ap, \neg K_Qp \land \neg K_Q\neg p, K_Q(K_Ap \lor K_A\neg p), K_A(K_Ap \lor K_A\neg p)$

update of the model after question $p?$ and answer yes
muddy children: after public announcement

muddy children: after all silent to question 1

muddy children: after all silent to question 2

card game: initial model
card game: update after 1 announces not to have white