advanced logic
2019 02 07
lecture 2

overview

- semantics
- game semantics
- alternative semantics

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formulas of modal logic

$$p \quad | \quad \bot \quad | \quad \phi \to \psi \quad | \quad \Box\phi \quad | \quad \Diamond\phi$$

we could define $\Box$ using $\Diamond$ or vice versa:

$$\Box\phi := \neg\Diamond\neg\phi$$

$$\Diamond\phi := \neg\Box\neg\phi$$

we will consider different (equivalent) sets of connectives

necessity : true in all accessible states or in all possible worlds

possibility : true in at least one accessible state

truth and validity

$$
\begin{array}{lll}
((W,R),V),w & \models & \phi \\
((W,R),V) & \models & \phi \\
(W,R) & \models & \phi \\
 & \models & \phi
\end{array}
$$

what is omitted is implicitly universally quantified

local and global truth: example

$W = \{u, v, w, s\}$

$R = \{(u, v), (v, w), (w, u), (s, s)\}$

$V(p) = \{w, s\}$

$V(q) = \{u, v, w\}$

for which worlds x do we have $\mathcal{M}, x \models p \to \Box p$?

for which worlds x do we have $\mathcal{M}, x \models \Box p \to \Diamond q$?

if possible give another valuation such that $\Diamond p \to p$ is not globally true

if possible give another valuation such that $\Diamond p \to \Box p$ is not globally true

examples of (in)valid axioms and rules

$\models \phi$ if ϕ a tautology of propositional logic

$\models \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

modus ponens: if $\models \phi \to \psi$ and $\models \phi$ then $\models \psi$

necessitation: if $\models \phi$ then $\models \Box\phi$

$\not\models \phi \to \Box\phi$

$\not\models \Box\phi \to \phi$

more examples

$\models \Box(\phi \land \psi) \leftrightarrow \Box\phi \land \Box\psi$

$\models \Diamond(\phi \lor \psi) \leftrightarrow \Diamond\phi \lor \Diamond\psi$

$\models \Box\phi \lor \Box\psi \to \Box(\phi \lor \psi)$

$\not\models \Box(\phi \lor \psi) \to \Box\phi \lor \Box\psi$

$\models \Diamond(\phi \land \psi) \to \Diamond\phi \land \Diamond\psi$

$\not\models \Diamond\phi \land \Diamond\psi \to \Diamond(\phi \land \psi)$

local truth

from world to formula

definition of local truth: $\mathcal{M}, w \models \phi$

from formula to world

characterize a world via a modal formula without variables

modal formulas distinguishing states

consider a frame with four states $\{1, 2, 3, 4\}$ and accessibility relation $1 \to 2 \to 3$ and $4 \to 1$, $4 \to 3$

find for every point i a distinguishing formula ϕ_i
that is, a formula such that $i \models \phi_j$ if and only if $i = j$ for all $i, j \in \{1, 2, 3, 4\}$

semantics and syntax

we will also see various proof systems for modal logic

and soundness and completeness theorems

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game semantics: setting

another approach to local truth

a model $\mathcal{M} = ((W, R), V)$, and a world $w \in W$, and a formula ϕ

there are two players:

Verifier (V) claims that ϕ is true in w

Falsifier (F) claims that ϕ is false in w

a position is a pair (w, ϕ) with $w \in W$ a world and ϕ a formula

a move from a position (w, ϕ) is determined by the main operator of ϕ

game semantics: position determines move

$(t, \phi_1 \vee \phi_2)$: V chooses a disjunct ϕ_i; play continues with (t, ϕ_i)

$(t, \phi_1 \wedge \phi_2)$: F chooses a conjunct ϕ_i; play continues with (t, ϕ_i)

$(t, \Diamond\phi)$: V chooses a successor u of t; play continues with (u, ϕ)

$(t, \Box\phi)$: F chooses a successor u of t; play continues with (u, ϕ)

$(t, \neg\phi)$: players switch roles; play continues with (t, ϕ)

(t, p): if p is true in t then V wins; otherwise F wins

who should but cannot choose a successor loses

game tree

a complete game tree for ϕ and $(\mathcal{M}, w)$

starts with (w, ϕ) and contains all possible moves

game semantics: strategy

a strategy for player P is a method to select moves

player P has a winning strategy

if there is a strategy ensuring that P wins the game

Theorem explaining the connection with the truth definition:

ϕ is true in $\mathcal{M}$ in s iff V has a winning strategy for $\mathcal{M}, s, \phi$

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alternative semantics

intuition:

given a model $\mathcal{M} = (W, R, V)$,

the interpretation $[\![\phi]\!]_\mathcal{M}$ of a formula ϕ

is the set of worlds in which ϕ is true

alternative semantics

Let $\mathcal{M} = (W, R, V)$ be a model.
We define $[\![\phi]\!]_\mathcal{M} \subseteq W$, the *interpretation* of a formula ϕ in the model $\mathcal{M}$, inductively by

$$\begin{aligned}
[\![p]\!]_\mathcal{M} &= V(p) && (p \in Var)\\
[\![\bot]\!]_\mathcal{M} &= \emptyset\\
[\![\top]\!]_\mathcal{M} &= W\\
[\![\neg\phi]\!]_\mathcal{M} &= W \setminus [\![\phi]\!]_\mathcal{M}\\
[\![\phi \vee \psi]\!]_\mathcal{M} &= [\![\phi]\!]_\mathcal{M} \cup [\![\psi]\!]_\mathcal{M}\\
[\![\phi \wedge \psi]\!]_\mathcal{M} &= [\![\phi]\!]_\mathcal{M} \cap [\![\psi]\!]_\mathcal{M}\\
[\![\phi \to \psi]\!]_\mathcal{M} &= \neg[\![\phi]\!]_\mathcal{M} \cup [\![\psi]\!]_\mathcal{M}\\
[\![\Diamond\phi]\!]_\mathcal{M} &= \{w \in W \mid \exists v Rwv \wedge v \in [\![\phi]\!]_\mathcal{M}\}\\
[\![\Box\phi]\!]_\mathcal{M} &= \{w \in W \mid \forall v Rwv \Rightarrow v \in [\![\phi]\!]_\mathcal{M}\}
\end{aligned}$$

(for $X \subseteq W$ we write $\neg X$ to denote the *complement* of X, i.e., $\neg X = W \setminus X$)