so far and alternatives

formulas: quite a few alternative approaches possible and used

for example: negation only on atoms and further defined inductively

valuation: alternative approach could be (not used)

\[ \sigma : W \rightarrow \mathcal{P}(\text{Var}) \]

semantics: Kripke models

alternative approaches possible and used

semantics in terms of sets of worlds, game semantics
overview

- preservation of truth and validity
- characterizations of frame properties
- bisimulations
overview

- preservation of truth and validity
- characterizations of frame properties
- bisimulations
prop1: schemes preservation under substitution

sometimes we consider a formula as a scheme:

\[ p \rightarrow q \rightarrow p \]

\[ (r \land s) \rightarrow \bot \rightarrow (r \land s) \]

let \( \sigma : \text{Var} \rightarrow \text{Form} \) be a substitution and \( v : \text{Var} \rightarrow \{0, 1\} \) a valuation

if \( \llbracket \phi \rrbracket_v = 1 \) then not necessarily \( \llbracket \phi^\sigma \rrbracket_v = 1 \)

if \( \llbracket \phi \rrbracket = 1 \) (or: \( \models \phi \)) then \( \models \phi^\sigma \)
definition: substitution for propositional variables

a mapping \( \sigma : \text{Var} \rightarrow \text{Form} \) induces a mapping \((\cdot)^\sigma : \text{Form} \rightarrow \text{Form}\):

\[
\begin{align*}
p^\sigma &= \sigma(p) \\
\bot^\sigma &= \bot \\
\top^\sigma &= \top \\
(X\phi)^\sigma &= X(\phi^\sigma) \quad \text{for } X \in \{\neg, \Box, \Diamond\} \\
(\phi \star \psi)^\sigma &= (\phi^\sigma) \star (\psi^\sigma) \quad \text{for } \star \in \{\lor, \land, \rightarrow\} 
\end{align*}
\]
modal prop1: what validity is preserved under substitution?

let $\sigma : \text{Var} \rightarrow \text{Form}$ be a substitution

if $((W, R), V), w \models \phi$ then not necessarily $((W, R), V), w \models \phi^\sigma$

if $((W, R), V) \models \phi$ then not necessarily $((W, R), V) \models \phi^\sigma$

if $(W, R) \models \phi$ then $(W, R) \models \phi^\sigma$?

if so, then also: if $\models \phi$ then $\models \phi^\sigma$
validity in a frame is preserved under substitution

we will see:

if $F \models \phi$ then $F \models \phi^\sigma$ for any substitution $\sigma$

how do we prove this?
validity in a frame is preserved under substitution

we will see:

if $\mathcal{F} \models \phi$ then $\mathcal{F} \models \phi^\sigma$ for any substitution $\sigma$

how do we prove this?

assume $\mathcal{F} \models \phi$ with $\mathcal{F} = (W, R)$ and let $\sigma$ be a substitution
validity in a frame is preserved under substitution

we will see:

if $\mathcal{F} \models \phi$ then $\mathcal{F} \models \phi^\sigma$ for any substitution $\sigma$

how do we prove this?

assume $\mathcal{F} \models \phi$ with $\mathcal{F} = (W, R)$ and let $\sigma$ be a substitution

let $V$ be a valuation and let $w \in W$

we aim to show $\mathcal{F}, V, w \models \phi^\sigma$
validity in a frame is preserved under substitution

we will see:

if $\mathcal{F} \models \phi$ then $\mathcal{F} \models \phi^\sigma$ for any substitution $\sigma$

how do we prove this?

assume $\mathcal{F} \models \phi$ with $\mathcal{F} = (\mathcal{W}, R)$ and let $\sigma$ be a substitution

let $V$ be a valuation and let $w \in \mathcal{W}$

we aim to show $\mathcal{F}, V, w \models \phi^\sigma$

we would like to use $\mathcal{F}, V^\sigma, w \models \phi$

with $V^\sigma$ mapping $p$ to the set of worlds in which $\sigma(p)$ is true

use ‘alternative semantics’
recall: alternative semantics

Let $\mathcal{M} = (W, R, V)$ be a model.

We define $[\phi]_\mathcal{M} \subseteq W$, the interpretation of a formula $\phi$ in the model $\mathcal{M}$, inductively by

$$
\begin{align*}
[p]_\mathcal{M} &= V(p) & (p \in \text{Var}) \\
[⊥]_\mathcal{M} &= \emptyset \\
[\top]_\mathcal{M} &= W \\
[\neg \phi]_\mathcal{M} &= W \setminus [\phi]_\mathcal{M} \\
[\phi \lor \psi]_\mathcal{M} &= [\phi]_\mathcal{M} \cup [\psi]_\mathcal{M} \\
[\phi \land \psi]_\mathcal{M} &= [\phi]_\mathcal{M} \cap [\psi]_\mathcal{M} \\
[\phi \rightarrow \psi]_\mathcal{M} &= \neg [\phi]_\mathcal{M} \cup [\psi]_\mathcal{M} \\
[\Diamond \phi]_\mathcal{M} &= \{ w \in W | \exists v R w v \land v \in [\phi]_\mathcal{M} \} \\
[\Box \phi]_\mathcal{M} &= \{ w \in W | \forall v R w v \Rightarrow v \in [\phi]_\mathcal{M} \}
\end{align*}
$$

(for $X \subseteq W$ we write $\neg X$ to denote the complement of $X$, i.e., $\neg X = W \setminus X$)
lemma: intuition about interpretation is correct

the interpretation $[\phi]_M$ of formula $\phi$ in model $M = (W, R, V)$ is the set of worlds in which $\phi$ is true:

**Lemma:** $M, w \models \phi$ if and only if $w \in [\phi]_M$

consequence of the lemma:

$M \models \phi$ if and only if $[\phi]_M = W$
evaluating substitution instances

let $\mathcal{M} = (W, R, V)$ be a model, and let $\sigma : \text{Var} \rightarrow \text{Form}$ be a substitution

definition: a substitution $\sigma$ applied to a valuation:

$V^\sigma(p) = \llbracket \sigma(p) \rrbracket_\mathcal{M}$

Lemma:

$\llbracket \phi^\sigma \rrbracket(W, R, V) = \llbracket \phi \rrbracket(W, R, V^\sigma)$
Theorem: validity is closed under substitution

If $F \models \phi$ then $F \models \phi^\sigma$ for any substitution $\sigma$.

Assume $F \models \phi$ with $F = (W, R)$ and let $\sigma$ be a substitution.

We have to show $F \models \phi^\sigma$.

Let $V$ be a valuation and let $w \in W$.

We have $F, V^\sigma, w \models \phi$.

Hence $w \in [\phi]_{F, V^\sigma}$.

Hence $w \in [\phi^\sigma]_{F, V}$.

Hence $F, V, w \models \phi^\sigma$.

Hence $F \models \phi^\sigma$. 
consequences of this theorem

take a propositional tautology, apply a substitution, then we have a modal tautology

how can we use this?

for example: consider the substitution \( \delta \) with \( \delta(p) = \neg p \)

\[ \mathcal{F} \models p \rightarrow \Diamond p \text{ if and only if } \mathcal{F} \models \Box p \rightarrow p \]

\[ \mathcal{F} \models \Diamond p \rightarrow \Box \Diamond p \text{ if and only if } \mathcal{F} \models \Diamond \Box p \rightarrow \Box p \]
preservation of truth and validity

local truth is preserved by modus ponens:

\[
\text{if } \mathcal{M}, w \models \phi \rightarrow \psi \text{ and } \mathcal{M}, w \models \phi \text{ then } \mathcal{M}, w \models \psi
\]

global truth is preserved by modus ponens and by necessitation:

\[
\text{if } \mathcal{M} \models \phi \text{ then } \mathcal{M} \models \Box \phi
\]

frame validity is preserved by modus ponens, necessitation, and substitution:

\[
\text{if } \mathcal{F} \models \phi \text{ then } \mathcal{F} \models \phi^\sigma
\]
the modal tautologies are exactly defined by

**extension:**
a tautology for first-order propositional logic is a modal tautology

**modal distribution:**
\[ \models \Box(p \rightarrow q) \rightarrow \Box p \rightarrow \Box q \]

**modus ponens:**
if \( \models \phi \rightarrow \psi \) and \( \models \phi \) then \( \models \psi \)

**necessitation:**
if \( \models \phi \) then \( \models \Box \phi \)

**substitution:**
if \( \models \phi \) then \( \models \phi^\sigma \)
example frame characterization

if $\mathcal{F} = (W, R)$ is symmetric then $\mathcal{F} \models \lozenge \Box p \rightarrow p$
**example frame characterization**

If $\mathcal{F} = (W, R)$ is symmetric then $\mathcal{F} \models \lozenge \Box p \rightarrow p$

Let $\mathcal{F} = (W, R)$ be a symmetric frame
if $\mathcal{F} = (W, R)$ is symmetric then $\mathcal{F} \models \Diamond \Box p \rightarrow p$

let $\mathcal{F} = (W, R)$ be a symmetric frame

let $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$, and let $w \in W$
example frame characterization

if $\mathcal{F} = (W, R)$ is symmetric then $\mathcal{F} \models \Diamond \Box p \rightarrow p$

let $\mathcal{F} = (W, R)$ be a symmetric frame

let $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$, and let $w \in W$

assume $\mathcal{M}, w \models \Diamond \Box p$; we have to show $\mathcal{M}, w \models p$
example frame characterization

If $\mathcal{F} = (W, R)$ is symmetric then $\mathcal{F} \models \Diamond \Box p \to p$

Let $\mathcal{F} = (W, R)$ be a symmetric frame.

Let $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$, and let $w \in W$.

Assume $\mathcal{M}, w \models \Diamond \Box p$; we have to show $\mathcal{M}, w \models p$.

Because $w \models \Diamond \Box p$ there is a $v \in W$ with $Rwv$ and $v \models \Box p$. 
example frame characterization

if $\mathcal{F} = (W, R)$ is symmetric then $\mathcal{F} \models \Diamond \Box p \rightarrow p$

let $\mathcal{F} = (W, R)$ be a symmetric frame

let $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$, and let $w \in W$

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because $\mathcal{F}$ is symmetric we have $Rvw$
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because $v \models \Box p$ we have $w \models p$

so $\mathcal{M}, w \models \Diamond \Box p \rightarrow p$
example frame characterization

if $\mathcal{F} = (\mathcal{W}, \mathcal{R})$ is symmetric then $\mathcal{F} \models \Diamond \Box p \rightarrow p$

let $\mathcal{F} = (\mathcal{W}, \mathcal{R})$ be a symmetric frame

let $\mathcal{M} = (\mathcal{F}, \mathcal{V})$ be a model based on $\mathcal{F}$, and let $w \in \mathcal{W}$

assume $\mathcal{M}, w \models \Diamond \Box p$; we have to show $\mathcal{M}, w \models p$

because $w \models \Diamond \Box p$ there is a $v \in \mathcal{W}$ with $Rwv$ and $v \models \Box p$

because $\mathcal{F}$ is symmetric we have $Rvw$

because $v \models \Box p$ we have $w \models p$

so $\mathcal{M}, w \models \Diamond \Box p \rightarrow p$

so $\mathcal{F} \models \Diamond \Box p \rightarrow p$
example frame characterization

if $\mathcal{F} \models \lozenge \Box p \rightarrow p$ then $\mathcal{F}$ is symmetric
example frame characterization

if $\mathcal{F} \models \lozenge \square p \rightarrow p$ then $\mathcal{F}$ is symmetric

assume $\mathcal{F} = (W, R)$ is not symmetric; we will show $\mathcal{F} \not\models \lozenge \square p \rightarrow p$
example frame characterization

if $F \models \Diamond \Box p \rightarrow p$ then $F$ is symmetric

assume $F = (W, R)$ is not symmetric; we will show $F \not\models \Diamond \Box p \rightarrow p$

because $F$ is not symmetric there are $a, b \in W$ such that $Rab$ but not $Rba$

(note: a non-symmetric frame has at least two states)
example frame characterization

if $\mathcal{F} \models \Diamond \Box p \rightarrow p$ then $\mathcal{F}$ is symmetric

assume $\mathcal{F} = (W, R)$ is not symmetric; we will show $\mathcal{F} \not\models \Diamond \Box p \rightarrow p$

because $\mathcal{F}$ is not symmetric there are $a, b \in W$ such that $Rab$ but not $Rba$

(note: a non-symmetric frame has at least two states)

define $V$ such that $V(p) = \{x \in W \mid Rbx\}$ and let $\mathcal{M} = (\mathcal{F}, V)$
example frame characterization

if $\mathcal{F} \models \Diamond \Box p \rightarrow p$ then $\mathcal{F}$ is symmetric

assume $\mathcal{F} = (W, R)$ is not symmetric; we will show $\mathcal{F} \not\models \Diamond \Box p \rightarrow p$

because $\mathcal{F}$ is not symmetric there are $a, b \in W$ such that $Rab$ but not $Rba$

(note: a non-symmetric frame has at least two states)

define $V$ such that $V(p) = \{x \in W \mid Rbx\}$ and let $\mathcal{M} = (\mathcal{F}, V)$

then we have $\mathcal{M}, b \models \Box p$
example frame characterization

if $\mathcal{F} \models \Diamond \Box p \rightarrow p$ then $\mathcal{F}$ is symmetric

assume $\mathcal{F} = (W, R)$ is not symmetric; we will show $\mathcal{F} \not\models \Diamond \Box p \rightarrow p$

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example frame characterization

if $\mathcal{F} \models \Diamond \Box p \rightarrow p$ then $\mathcal{F}$ is symmetric

assume $\mathcal{F} = (W, R)$ is not symmetric; we will show $\mathcal{F} \not\models \Diamond \Box p \rightarrow p$

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(note: a non-symmetric frame has at least two states)

define $V$ such that $V(p) = \{x \in W \mid Rbx\}$ and let $\mathcal{M} = (\mathcal{F}, V)$

then we have $\mathcal{M}, b \models \Box p$

because $Rab$ we have $\mathcal{M}, a \models \Diamond \Box p$

because not $Rba$ we have $a \notin V(p)$ so we have $\mathcal{M}, a \not\models p$
example frame characterization

if $\mathcal{F} \models \Diamond \Box p \to p$ then $\mathcal{F}$ is symmetric

assume $\mathcal{F} = (W, R)$ is not symmetric; we will show $\mathcal{F} \not\models \Diamond \Box p \to p$

because $\mathcal{F}$ is not symmetric there are $a, b \in W$ such that $Rab$ but not $Rba$

(note: a non-symmetric frame has at least two states)

define $V$ such that $V(p) = \{x \in W \mid Rbx\}$ and let $\mathcal{M} = (\mathcal{F}, V)$

then we have $\mathcal{M}, b \models \Box p$

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because not $Rba$ we have $a \not\in V(p)$ so we have $\mathcal{M}, a \not\models p$

so $\mathcal{M}, a \not\models \Diamond \Box p \to p$
example frame characterization

if $\mathcal{F} \models \Diamond \Box p \rightarrow p$ then $\mathcal{F}$ is symmetric

assume $\mathcal{F} = (W, R)$ is not symmetric; we will show $\mathcal{F} \not\models \Diamond \Box p \rightarrow p$

because $\mathcal{F}$ is not symmetric there are $a, b \in W$ such that $Rab$ but not $Rba$

(note: a non-symmetric frame has at least two states)

define $V$ such that $V(p) = \{x \in W \mid Rbx\}$ and let $\mathcal{M} = (\mathcal{F}, V)$

then we have $\mathcal{M}, b \models \Box p$

because $Rab$ we have $\mathcal{M}, a \models \Diamond \Box p$

because not $Rba$ we have $a \notin V(p)$ so we have $\mathcal{M}, a \not\models p$

so $\mathcal{M}, a \not\models \Diamond \Box p \rightarrow p$

so $\mathcal{F} \not\models \Diamond \Box p \rightarrow p$
frame characterization

we have shown:

the formula $\diamondsuit \square p \rightarrow p$ characterizes the frame property symmetry

in general:

the formula $\phi$ characterizes the frame property $P$ means

$\mathcal{F}$ has property $P$ if and only if $\mathcal{F} \models \phi$
overview

- preservation of truth and validity
- characterizations of frame properties
- bisimulations
distinguishable and indistinguishable states

1 \rightarrow 2
\downarrow \downarrow
4 \rightarrow 3

we can distinguish state 1 from state 3
we do not succeed in distinguishing state 2 from state 4
can we prove that it is not possible to distinguish state 2 from state 4?
bisimulation: definition

Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be models.

A non-empty relation $Z \subseteq W \times W'$ is a bisimulation, notation $Z : \mathcal{M} \leftrightarrow \mathcal{M}'$, if for all pairs $(w, w') \in Z$ we have the following:

- $w \in V(p)$ if and only if $w' \in V'(p)$
- if $Rwv$ then for some $v' \in W'$ we have $R'w'v'$ and $vZv'$
- if $R'w'v'$ then for some $v \in W$ we have $Rwv$ and $vZv'$
bisimulation: base

\[ M = (W, R, V) \quad M' = (W', R', V') \]

If \( wZw' \) then for all \( p \in \text{Var} \) we have \( w \in V(p) \) if and only if \( w' \in V'(p) \).
bisimulation: base

\[ M = (W, R, V) \quad \text{and} \quad M' = (W', R', V') \]

If \( wZw' \) then for all \( p \in \text{Var} \) we have \( w \in V(p) \) if and only if \( w' \in V'(p) \)
bisimulation: zig

$\mathcal{M} = (W, R, V)$

$\mathcal{M}' = (W', R', V')$

if $wZw'$ and $Rwv$ then there exists $v' \in W'$ such that $R'w'v'$ and $vZv'$
bisimulation: zig

\[ M = (W, R, V) \]

\[ M' = (W', R', V') \]

if \( wZw' \) and \( Rwv \) then there exists \( v' \in W' \) such that \( R'w'v' \) and \( vZv' \)
bisimulation: zig

\[ M = (W, R, V) \quad \exists v' \in W' \quad M' = (W', R', V') \]

if \( wZw' \) and \( Rwv \) then there exists \( v' \in W' \) such that \( R'w'v' \) and \( vZv' \)
bisimulation: zig

\[ \mathcal{M} = (W, R, V) \quad \exists v' \in W' \quad \mathcal{M}' = (W', R', V') \]

if \( wZw' \) and \( Rwv \) then there exists \( v' \in W' \) such that \( R'w'v' \) and \( vZv' \)
bisimulation: zig

\[ M = (W, R, V) \quad \exists v' \in W' \quad M' = (W', R', V') \]

If \( wZw' \) and \( Rwv \) then there exists \( v' \in W' \) such that \( R'w'v' \) and \( vZv' \)
bisimulation: zag

\[ M = (W, R, V) \quad \text{and} \quad M' = (W', R', V') \]

If \( wZw' \) and \( R'w'v' \) then there exists \( v \in W \) such that \( Rvw \) and \( vZv' \)
bisimulation: $\alpha$-

$$
\mathcal{M} = (W, R, V) \\
\mathcal{M}' = (W', R', V')$

if $wZw'$ and $R'w'v'$ then there exists $v \in W$ such that $Rwv$ and $vZv'$
bisimulation: zag

\[ M = (W, R, V) \quad \exists v \in W \quad M' = (W', R', V') \]

if \( wZw' \) and \( R'w'v' \) then there exists \( v \in W \) such that \( Rwv \) and \( vZv' \)
bisimulation: zag

\[ M = (W, R, V) \]

\[ \exists v \in W \]

\[ M' = (W', R', V') \]

if \( wZw' \) and \( R'w'v' \) then there exists \( v \in W \) such that \( Rwv \) and \( vZv' \)
bisimulation: zag

\[ M = (W, R, V) \]

if \( wZw' \) and \( R'w'v' \) then there exists \( v \in W \) such that \( Rwv \) and \( vZv' \)
bisimilarity: definition

two models $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ are bisimilar, notation $\mathcal{M} \leftrightarrow \mathcal{M}'$, if there exists a bisimulation $Z \subseteq W \times W'$

two pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ are bisimilar, notation $\mathcal{M}, w \leftrightarrow \mathcal{M}', w'$ or $w \leftrightarrow w'$, if there exists a bisimulation $Z : \mathcal{M} \leftrightarrow \mathcal{M}'$ such that $(w, w') \in Z$
example bisimilar states

states 2 and 4 are bisimilar
example bisimilar states

1 → 2
↓   ↓
4 → 3

states 2 and 4 are bisimilar

$B_1 = \{(2, 4), (3, 3)\}$

$B_2 = \{(1, 1), (2, 4), (4, 2), (3, 3)\}$

$B_3 = \{(1, 1), (2, 2), (2, 4), (3, 3), (4, 2), (4, 4)\}$
example bisimilar states

\[ \mathcal{N} = (\mathbb{N}, S) \]

\[ S = \{(n, n+1) \mid n \in \mathbb{N}\} \]

\[ V(p) = \{2n \mid n \in \mathbb{N}\} \]

\[ \mathcal{F} = (\{e, o\}, R) \]

\[ R = \{(e, o), (o, e)\} \]

\[ U(p) = \{e\} \]

the states 0 and e are bisimilar
non-bisimilarity: example
non-bisimilarity: example

```
  a
  ↓  ↓
  b  b
  ↓  ↓
p  c  p  q
  ↓  ↓
c  d  q

  1
  ↓  2  3
  2  4  3  5
  ↓  5
  p  q
```
non-bisimilarity: example
non-bisimilarity: example
non-bisimilarity: example

\[ \begin{array}{ccc}
  a & \overrightarrow{\downarrow} & b \\
  & \overrightarrow{\downarrow} & \overrightarrow{\downarrow} \\
  p & c & q \\
\end{array} \]

\[ \begin{array}{ccc}
  b & \overrightarrow{\downarrow} & 1 \\
  & \overrightarrow{\downarrow} & \overrightarrow{\downarrow} \\
  2 & 3 & q \\
\end{array} \]

\[ \begin{array}{ccc}
  a & \overrightarrow{\downarrow} & b \\
  & \overrightarrow{\downarrow} & \overrightarrow{\downarrow} \\
  2 & 4 & p \\
\end{array} \]