advanced logic
2019 02 11
lecture 3

so far and alternatives

formulas: quite a few alternative approaches possible and used

for example: negation only on atoms and further defined inductively

valuation: alternative approach could be (not used)

$\sigma : W \to \mathcal{P}(Var)$

semantics: Kripke models

alternative approaches possible and used

semantics in terms of sets of worlds, game semantics

overview

- preservation of truth and validity
- characterizations of frame properties
- bisimulations

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- preservation of truth and validity
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- bisimulations

prop1: schemes preservation under substitution

sometimes we consider a formula as a scheme:

$p \to q \to p$

$(r \wedge s) \to \bot \to (r \wedge s)$

let $\sigma : Var \to Form$ be a substitution and $v : Var \to \{0,1\}$ a valuation

if $[\![\phi]\!]_v = 1$ then not necessarily $[\![\phi^\sigma]\!]_v = 1$

if $[\![\phi]\!] = 1$ (or: $\models \phi$) then $\models \phi^\sigma$

definition: substitution for propositional variables

a mapping $\sigma : Var \to Form$ induces a mapping $(\cdot)^\sigma : Form \to Form$:

$$\begin{array}{rcll} p^\sigma & = & \sigma(p) & \\ \bot^\sigma & = & \bot & \\ \top^\sigma & = & \top & \\ (X\phi)^\sigma & = & X(\phi^\sigma) & \quad \text{for } X \in \{\neg, \Box, \Diamond\} \\ (\phi * \psi)^\sigma & = & (\phi^\sigma) \star (\psi^\sigma) & \quad \text{for } \star \in \{\vee, \wedge, \to\} \end{array}$$

modal prop1: what validity is preserved under substitution?

let $\sigma : Var \to Form$ be a substitution

if $((W,R),V), w \models \phi$ then not necessarily $((W,R),V), w \models \phi^\sigma$

if $((W,R),V) \models \phi$ then not necessarily $((W,R),V) \models \phi^\sigma$

if $(W,R) \models \phi$ then $(W,R) \models \phi^\sigma$?

if so, then also: if $\models \phi$ then $\models \phi^\sigma$

validity in a frame is preserved under substitution

we will see:

if $\mathcal{F} \models \phi$ then $\mathcal{F} \models \phi^\sigma$ for any substitution σ

how do we prove this?

assume $\mathcal{F} \models \phi$ with $\mathcal{F} = (W,R)$ and let σ be a substitution

let V be a valuation and let $w \in W$

we aim to show $\mathcal{F}, V, w \models \phi^\sigma$

we would like to use $\mathcal{F}, V^\sigma, w \models \phi$

with V^σ mapping p to the set of worlds in which $\sigma(p)$ is true

use ‘alternative semantics’

recall: alternative semantics

Let $\mathcal{M} = (W, R, V)$ be a model.
We define $[\![\phi]\!]_\mathcal{M} \subseteq W$, the *interpretation* of a formula ϕ in the model $\mathcal{M}$, inductively by

$$
\begin{aligned}
[\![p]\!]_\mathcal{M} &= V(p) && (p \in \mathit{Var}) \\
[\![\bot]\!]_\mathcal{M} &= \emptyset \\
[\![\top]\!]_\mathcal{M} &= W \\
[\![\neg\phi]\!]_\mathcal{M} &= W \setminus [\![\phi]\!]_\mathcal{M} \\
[\![\phi \vee \psi]\!]_\mathcal{M} &= [\![\phi]\!]_\mathcal{M} \cup [\![\psi]\!]_\mathcal{M} \\
[\![\phi \wedge \psi]\!]_\mathcal{M} &= [\![\phi]\!]_\mathcal{M} \cap [\![\psi]\!]_\mathcal{M} \\
[\![\phi \to \psi]\!]_\mathcal{M} &= \neg[\![\phi]\!]_\mathcal{M} \cup [\![\psi]\!]_\mathcal{M} \\
[\![\Diamond\phi]\!]_\mathcal{M} &= \{w \in W \mid \exists v Rwv \wedge v \in [\![\phi]\!]_\mathcal{M}\} \\
[\![\Box\phi]\!]_\mathcal{M} &= \{w \in W \mid \forall v Rwv \Rightarrow v \in [\![\phi]\!]_\mathcal{M}\}
\end{aligned}
$$

(for $X \subseteq W$ we write $\neg X$ to denote the *complement* of X, i.e., $\neg X = W \setminus X$)

lemma: intuition about interpretation is correct

the interpretation $[\![\phi]\!]_\mathcal{M}$ of formula ϕ in model $\mathcal{M} = (W, R, V)$

is the set of worlds in which ϕ is true:

Lemma: $\mathcal{M}, w \models \phi$ if and only if $w \in [\![\phi]\!]_\mathcal{M}$

consequence of the lemma:

$\mathcal{M} \models \phi$ if and only if $[\![\phi]\!]_\mathcal{M} = W$

evaluating substitution instances

let $\mathcal{M} = (W, R, V)$ be a model, and let $\sigma : \mathit{Var} \to \mathit{Form}$ be a substitution

definition: a substitution σ applied to a valuation:

$V^\sigma(p) = [\![\sigma(p)]\!]_\mathcal{M}$

Lemma:

$[\![\phi^\sigma]\!]_{(W,R,V)} = [\![\phi]\!]_{(W,R,V^\sigma)}$

theorem: validity is closed under substitution

if $\mathcal{F} \models \phi$ then $\mathcal{F} \models \phi^\sigma$ for any substitution σ

assume $\mathcal{F} \models \phi$ with $\mathcal{F} = (W, R)$ and let σ be a substitution

we have to show $\mathcal{F} \models \phi^\sigma$

let V be a valuation and let $w \in W$

we have $\mathcal{F}, V^\sigma, w \models \phi$

hence $w \in [\![\phi]\!]_{\mathcal{F},V^\sigma}$

hence $w \in [\![\phi^\sigma]\!]_{\mathcal{F},V}$

hence $\mathcal{F}, V, w \models \phi^\sigma$

hence $\mathcal{F} \models \phi^\sigma$

consequences of this theorem

take a propositional tautology, apply a substitution, then we have a modal tautology

how can we use this?

for example: consider the substitution δ with $\delta(p) = \neg p$

$\mathcal{F} \models p \to \Diamond p$ if and only if $\mathcal{F} \models \Box p \to p$

$\mathcal{F} \models \Diamond p \to \Box\Diamond p$ if and only if $\mathcal{F} \models \Diamond\Box p \to \Box p$

preservation of truth and validity

local truth is preserved by modus ponens:

if $\mathcal{M}, w \models \phi \to \psi$ and $\mathcal{M}, w \models \phi$ then $\mathcal{M}, w \models \psi$

global truth is preserved by modus ponens and by necessitation:

if $\mathcal{M} \models \phi$ then $\mathcal{M} \models \Box\phi$

frame validity is preserved by modus ponens, necessitation, and substitution:

if $\mathcal{F} \models \phi$ then $\mathcal{F} \models \phi^\sigma$

the modal tautologies are exactly defined by

extension:
a tautology for first-order propositional logic is a modal tautology

modal distribution:
$\models \Box(p \to q) \to \Box p \to \Box q$

modus ponens:
if $\models \phi \to \psi$ and $\models \phi$ then $\models \psi$

necessitation:
if $\models \phi$ then $\models \Box\phi$

substitution:
if $\models \phi$ then $\models \phi^\sigma$

example frame characterization

if $\mathcal{F} = (W, R)$ is symmetric then $\mathcal{F} \models \Diamond\Box p \to p$

let $\mathcal{F} = (W, R)$ be a symmetric frame

let $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$, and let $w \in W$

assume $\mathcal{M}, w \models \Diamond\Box p$; we have to show $\mathcal{M}, w \models p$

because $w \models \Diamond\Box p$ there is a $v \in W$ with Rwv and $v \models \Box p$

because $\mathcal{F}$ is symmetric we have Rvw

because $v \models \Box p$ we have $w \models p$

so $\mathcal{M}, w \models \Diamond\Box p \to p$

so $\mathcal{F} \models \Diamond\Box p \to p$

example frame characterization

if $\mathcal{F} \models \Diamond\Box p \to p$ then $\mathcal{F}$ is symmetric

assume $\mathcal{F} = (W, R)$ is not symmetric; we will show $\mathcal{F} \not\models \Diamond\Box p \to p$

because $\mathcal{F}$ is not symmetric there are $a, b \in W$ such that Rab but not Rba

(note: a non-symmetric frame has at least two states)

define V such that $V(p) = \{x \in W \mid Rbx\}$ and let $\mathcal{M} = (\mathcal{F}, V)$

then we have $\mathcal{M}, b \models \Box p$

because Rab we have $\mathcal{M}, a \models \Diamond\Box p$

because not Rba we have $a \notin V(p)$ so we have $\mathcal{M}, a \not\models p$

so $\mathcal{M}, a \not\models \Diamond\Box p \to p$

so $\mathcal{F} \not\models \Diamond\Box p \to p$

frame characterization

we have shown:

the formula $\Diamond\Box p \to p$ characterizes the frame property symmetry

in general:

the formula ϕ characterizes the frame property P means

$\mathcal{F}$ has property P if and only if $\mathcal{F} \models \phi$

overview

- preservation of truth and validity
- characterizations of frame properties
- bisimulations

distinguishable and indistinguishable states

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 4 \arrow[r] & 3 \end{tikzcd}$$

we can distinguish state 1 from state 3

we do not succeed in distinguishing state 2 from state 4

can we prove that it is not possible to distinguish state 2 from state 4?

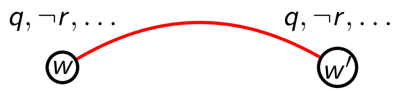
bisimulation: definition

Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be models.

A non-empty relation $Z \subseteq W \times W'$ is a bisimulation,
notation $Z : \mathcal{M} \leftrightarroweq \mathcal{M}'$,
if for all pairs $(w, w') \in Z$ we have the following:

- $w \in V(p)$ if and only if $w' \in V'(p)$
- if Rwv then for some $v' \in W'$ we have $R'w'v'$ and vZv'
- if $R'w'v'$ then for some $v \in W$ we have Rwv and vZv'

bisimulation: base

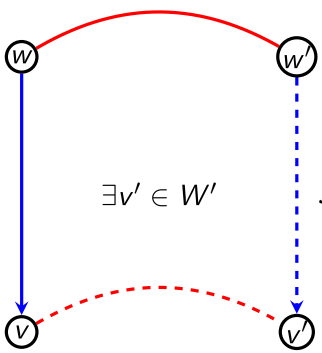


$\mathcal{M} = (W, R, V)$ $\mathcal{M}' = (W', R', V')$

if wZw' then for all $p \in \mathrm{Var}$ we have $w \in V(p)$ if and only if $w' \in V'(p)$

bisimulation: zig

$\mathcal{M} = (W, R, V)$

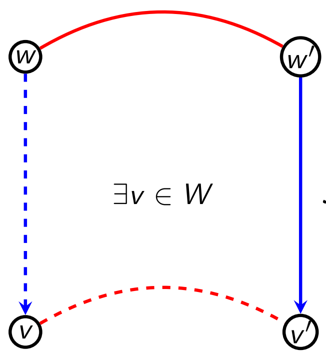


$\mathcal{M}' = (W', R', V')$

if wZw' and Rwv then there exists $v' \in W'$ such that $R'w'v'$ and vZv'

bisimulation: zag

$\mathcal{M} = (W, R, V)$



$\mathcal{M}' = (W', R', V')$

if wZw' and $R'w'v'$ then there exists $v \in W$ such that Rwv and vZv'

bisimularity: definition

two models $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ are bisimilar,

notation $\mathcal{M} \leftrightarroweq \mathcal{M}'$,

if there exists a bisimulation $Z \subseteq W \times W'$

two pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ are bisimilar,

notation $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ or $w \leftrightarroweq w'$,

if there exists a bisimulation $Z : \mathcal{M} \leftrightarroweq \mathcal{M}'$ such that $(w, w') \in Z$

example bisimilar states

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 4 \arrow[r] & 3 \end{tikzcd}$$

states 2 and 4 are bisimilar

$B_1 = \{(2,4), (3,3)\}$

$B_2 = \{(1,1), (2,4), (4,2), (3,3)\}$

$B_3 = \{(1,1), (2,2), (2,4), (3,3), (4,2), (4,4)\}$

example bisimilar states

$\mathcal{N} = (\mathbb{N}, S)$

$S = \{(n, n+1) \mid n \in \mathbb{N}\}$

$V(p) = \{2n \mid n \in \mathbb{N}\}$

$\mathcal{F} = (\{e, o\}, R)$

$R = \{(e, o), (o, e)\}$

$U(p) = \{e\}$

the states 0 and e are bisimilar

non-bisimilarity: example