overview

- bisimulations

what can be expressed by the modal language?

when can two pointed models \((M, w)\) and \((M', w')\) be distinguished?

when should two pointed models be considered modally identical?

what is the right semantic equivalence for the basic modal language?
recall: definition bisimulation between models

local harmony

zig

zag

call: definition bisimilarity

two models \( \mathcal{M} = (W, R, V) \) and \( \mathcal{M}' = (W', R', V') \) are bisimilar, notation \( \mathcal{M} \equiv \mathcal{M}' \),

if there exists a bisimulation \( Z \subseteq W \times W' \)

two pointed models \( \mathcal{M}, w \) and \( \mathcal{M}', w' \) are bisimilar

notation \( \mathcal{M}, w \equiv \mathcal{M}', w' \) or \( w \equiv w' \),

if there exists a bisimulation \( Z : \mathcal{M} \equiv \mathcal{M}' \) such that \( (w, w') \in Z \)

property: bisimilarity between models is an equivalence relation

(reflexive, symmetric, and transitive)

remark: bisimulation versus two simulations

definition of a simulation: local harmony and only zig, no zag

question:

is \( \mathcal{M}, w \) bisimilar with \( \mathcal{N}, v \) the same as

\( \mathcal{M}, w \) similar with \( \mathcal{N}, v \) and also \( \mathcal{N}, v \) similar with \( \mathcal{M}, w \)?

answer: no!

consider \( a \) in \( \{a, a_1, a'_1, a_2\} \), \( \{(a, a_1), (a, a'_1), (a_1, a_2)\} \)

and \( b \) in \( \{b, b_1, b_2\} \), \( \{(b, b_1), (b_1, b_2)\} \)

\( a \) simulates \( b \) and \( b \) simulates \( a \) but \( a \) and \( b \) are not bisimilar

think also of the formula ♠☐⊥

diagram: example non-bisimilar states

the states \( a \) and \( 1 \) are not bisimilar
definition bisimulation games for two players

Spoiler S claims \( M, s \) and \( N, t \) to be different

Duplicator D claims they are similar

play consists of a sequence of links, starting with link \( s \sim t \)

at current link \( m \sim n \) (with \( m \) in \( M \) and \( n \) in \( N \))

if \( m \) and \( n \) are different in their atoms then S wins

if not, then S picks a successor \( x \) either of \( m \) or of \( n \)

then D has to find a matching transition to \( y \) in the other model

play continues with next link \( x \sim y \) (or \( y \sim x \))

if a player cannot make a move, he loses; D wins the infinite games

definition modal equivalence

two states \( M, w \) and \( M', w' \) are modally equivalent

if they satisfy exactly the same formulas

theorem: bisimilarity implies modal equivalence

if \( M, w \) and \( M', w' \) are bisimilar then they are modally equivalent

so also: if two states are not modally equivalent, then they are not bisimilar

the proof is by induction on the definition of formulas

frame properties

for some properties \( P \) and formulas \( \phi \) we can show

\( \mathcal{F} \models \phi \) if and only if \( P(\mathcal{F}) \) for every frame \( \mathcal{F} \)

using bisimilarity implies modal equivalence

we can for some properties \( P \) show:

\( P \) is not modally definable
asymmetry (if $R_{xy}$ then $\neg R_{yx}$) is not modally definable

suppose there is a formula $\phi$ characterizing asymmetry

we use the natural numbers frame $\mathcal{N}$ and the even–odd frame $\mathcal{F}$

we have $\mathcal{N} \models \phi$ but not $\mathcal{F} \models \phi$

let $V$ be a valuation for $\mathcal{F}$ and let $x$ be a state in $\mathcal{F}$

we aim to show $\mathcal{F}, V, x \models \phi$ (to derive a contradiction)

we take $x = e$ (other case similar)

we construct $W$ such that $\mathcal{F}, V, e$ and $\mathcal{N}, W, 0$ are bisimilar

hence $\mathcal{F}, V, e$ and $\mathcal{N}, W, 0$ are modally equivalent

because $\mathcal{N} \models \phi$ we have $\mathcal{N}, W, 0 \models \phi$ and hence $\mathcal{F}, V, e \models \phi$

hence $\mathcal{F} \models \phi$ which yields a contradiction

hence there is no formula $\phi$ characterizing asymmetry