advanced logic

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lecture 5

overview

- transforming and constructing models
- truth
- towards decidability

modal equivalence implies bisimilarity for finitely branching

theorem: for finitely branching models,

if two states are modally equivalent, then they are bisimilar

proof: take modal equivalence as bisimulation

recall: why is the condition needed?

disjoint union of models

assume mutually disjoint models $\mathcal{M}_i = (W_i, R_i, V_i)$

definition: the disjoint sum $\cup_i \mathcal{M}_i$ is the model with

$W = \cup W_i$

$R = \cup R_i$

$V(p) = \cup V_i(p)$

then: w in W_i is bisimilar with w in $\cup_i \mathcal{M}_i$

global box

define the operator A for global box as follows:

$\mathcal{M}, w \vDash A\phi$ if and only if $\mathcal{M}, u \vDash \phi$ for all u

claim: we cannot define A in basic modal logic

that is: there is no formula $\zeta(p)$ depending on p such that

$\mathcal{M}, w \vDash A\phi$ if and only if $\mathcal{M}, w \vDash \zeta(\phi)$ for all $\mathcal{M}, w$

use of disjoint union: global box is not definable

suppose a basic modal logic formula $\zeta(p)$ for global box exists

that is: $\mathcal{M}, w \models Ap$ if and only if $\mathcal{M}, w \models \zeta(p)$

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define $\mathcal{M}_1$ such that p holds in all worlds

define $\mathcal{M}_2$ such that p holds in no world

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let w_1 be a world in $\mathcal{M}_1$, then $\mathcal{M}_1, w_1 \models Ap$ and hence $\mathcal{M}_1, w_1 \models \zeta(p)$

use of disjoint union: global box is not definable

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$\mathcal{M}_1, w_1$ is bisimilar to $\mathcal{M}_1 \cup \mathcal{M}_2, w_1$

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let w_1 be a world in $\mathcal{M}_1$, then $\mathcal{M}_1, w_1 \models Ap$ and hence $\mathcal{M}_1, w_1 \models \zeta(p)$

$\mathcal{M}_1, w_1$ is bisimilar to $\mathcal{M}_1 \cup \mathcal{M}_2, w_1$

so $\mathcal{M}_1 \cup \mathcal{M}_2, w_1 \models \zeta(p)$ and hence $\mathcal{M}_1 \cup \mathcal{M}_2, w_1 \models Ap$

that means p holds in any state in $\mathcal{M}_1 \cup \mathcal{M}_2$

and in particular p in every state in $\mathcal{M}_2$

contradiction; hence such $\zeta(p)$ does not exist

example tree unravelling

$W = \{1\}$, $R = \{(1,1)\}$ and $V(p) = \emptyset$ for all p

the tree unravelling of 1 is an 'infinite branch'

$W = \{1,2\}$, $R = \{(1,1),(1,2)\}$ and $V(p) = \emptyset$ for all p

the tree unravelling of 1 is an 'infinite comb'

tree unravelling

definition: unravelling of world s in (W, R, V) is the model (W', R', V'):

W' consists of $(s_1, \ldots, s_n)$ with $s_1 = s$ and $Rs_i s_{i+1}$

R' relates $(s_1, \ldots, s_n)$ to $(s_1, \ldots, s_n, s_{n+1})$ if $Rs_n s_{n+1}$

$V'(p) = \{(s_1, \ldots, s_n) \mid s_n \in V(p)\}$

then: a state $(s_1, \ldots, s_n)$ in (W', R', V') is bisimilar to s_n in (W, R, V)

every pointed model $\mathcal{M}, w$ can be given as a rooted tree

nearsighted: finite depth

for every formula ϕ of modal depth k:

if ϕ is satisfiable then it is satisfiable in a tree model $\mathcal{M}$ with root w

and we can restrict attention to the part of $\mathcal{M}$ that is reachable from w in at most k steps
$\mathcal{M}, w \models \phi$

example bisimulation contraction

$W = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 3), (3, 3), (3, 4)\}$, $V(p) = \emptyset$ for all p

do we have a bisimulation relating 2 and 4?

do we have a bisimulation relating 1 and 2?

do we have a bisimulation relating 1 and 3?

the bisimulation contraction has states $|1|$ and $|2|$
and has accessibility relation $\{(|1|, |1|), (|1|, |2|)\}$

bisimulation contraction

consider a model $\mathcal{M} = (W, R, V)$

the relation $s \underline{\leftrightarrow} t$ is an equivalence (refl, sym, trans) relation

definition: bisimulation contraction of (W, R, V) is the model (W', R', V'):

W' consists of equivalence classes $|s| = \{ t$ such that $s \underline{\leftrightarrow} t \}$

R' relates $|s|$ to $|t|$ if Ruv for some $u \in |s|$ and some $v \in |t|$

$V'(p) = \{ |s| \, | s \in V(p)\}$

then: x in (W, R, V) is bisimilar with $|x|$ in (W', R', V')

truth and validity

$((W, R), V), w \models \phi$

Verifier has winning strategy in w if and only if ϕ is true in w

description of all modal tautologies

truth and frames

if $\mathcal{F} \vDash \phi$ then $\mathcal{F} \vDash \phi^\sigma$ for every substitution σ

for some properties P there is a characterizing formula ϕ:

$\mathcal{F}$ satisfies P if and only if $\mathcal{F} \vDash \phi$

for some properties P we can show:

P is not modally definable

truth and bisimulation

theorem: for finitely branching models:

w and w' are modally equivalent if and only if w and w' are bisimilar

proof $\Leftarrow$: induction on the definition of formulas

proof $\Rightarrow$: take modal equivalence as bisimulation

recall: definition bisimulation games for two players

Spoiler S claims $\mathcal{M}, s$ and $\mathcal{N}, t$ to be different

Duplicator D claims they are similar

play consists of a sequence of links, starting with link $s \sim t$

at current link $m \sim n$ (with m in $\mathcal{M}$ and n in $\mathcal{N}$)

- if m and n are different in their atoms then S wins
- if not, then S picks a successor x either of m or of n
- then D has to find a matching transition to y in the other model
- play continues with next link $x \sim y$ (or $y \sim x$)

if a player cannot make a move, he loses; D wins the infinite games

modal depth of formulas

modal formulas have limited view

definition modal depth $\mathsf{md}(\phi)$ of a formula ϕ:

$\mathsf{md}(p) = \mathsf{md}(\bot) = \mathsf{md}(\top) = 0$

$\mathsf{md}(\neg\phi) = \mathsf{md}(\phi)$

$\mathsf{md}(\phi \vee \psi) = \mathsf{md}(\phi \wedge \psi) = \max\{\mathsf{md}(\phi), \mathsf{md}(\psi)\}$

$\mathsf{md}(\Box\phi) = \mathsf{md}(\Diamond\phi) = \mathsf{md}(\phi) + 1$

intuition modal depth

we need a formula of modal depth k to distinguish states x and y

Spoiler can win the bisimulation game in k rounds

every winning strategy for Spoilers corresponds to a distinguishing formula

games of less than k rounds can be won by Duplicator

formulas of modal depth less than k cannot distinguish between x and y

see for example conference on coalgebraic methods

adequacy for bisimulation games

$\mathcal{M}, s$ and $\mathcal{N}, t$ satisfy the same formules up to modal depth k

if and only if

Duplicator has winning strategy in the k-round game starting in $s \sim t$

(number of rounds is number of links minus one)

if Spoiler can win in k rounds,

then there is a distinguishing formula of modal depth k

bisimulation: further reading

by Wan Fokkink: book

by Rob van Glabbeek: handbook of process algebra chapter 1

by Davide Sangiorgi: book

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validity and satisfiability

validity problem:
given a formula ϕ, is ϕ valid?

satisfiability problem:
given a formula ϕ, is there a model and a world in which ϕ is true?

validity and satisfiability are dual problems:
ϕ is valid if and only if $\neg\phi$ is not satisfiable

prop1

valuation $V : \mathrm{Var} \to \{0, 1\}$

for a given formula with n propositional variables:

$V : \{p_1, \ldots, p_n\} \to \{0, 1\} = \{0, 1\}^{\{p_1, \ldots, p_n\}}$

there are 2^n such valuations

validity of prop1 is decidable with complexity coNP

satisfiability of prop1 is decidable with complexity NP-complete

pred1

validity and satisfiability are both undecidable for pred1

proof for example via Post Correspondence Problem

size of the models

first-order predicate logic needs infinite models

$(\forall x \neg Rxx) \wedge \forall xyz(Rxy \wedge Ryz \to Rxz) \wedge (\forall x \exists y Rxy)$

modal logic needs only finite models

if ϕ is satisfiable,

then it is satisfiable using a model with at most $2^{s(\phi)}$ elements

with $s(\phi)$ the number of subformulas of ϕ

decidability

prop1 is decidable with an exponential algorithm

pred1 is undecidable for example via Post Correspondence Problem

modal logic is decidable

via finite model property, sequents, tableaux, translation

finite model property

(to be discussed later)

suppose ϕ is satisfiable

then ϕ is satisfiable in a model with at most $f(\phi)$ worlds

so we have a decision procedure for satisfiability of ϕ:

compute $f(\phi)$

consider all (finitely many) models with at most $f(\phi)$ worlds, up to isomorphism

see whether in one of them *phi* holds

$f(\phi)$ is $2^{s(\phi)}$ with $s(\phi)$ the number of subformulas of ϕ

sequent

for propositional logic:

$\phi_1, \ldots, \phi_n \Rightarrow \psi_1, \ldots, \psi_m$

valid if in every model

the conjunction of the ϕ_i implies the disjunction of the ψ_i

for modal logic:

$\phi_1, \ldots, \phi_n \Rightarrow \psi_1, \ldots, \psi_m$

valid if in every model, in every world in that model

the conjunction of the ϕ_i implies the disjunction of the ψ_i

sequents: remark

A and B in $A \Rightarrow B$ are multisets of formulas

we work up to associativity, commutativity, and idempotence of both sides

empty conjunction is true

indeed: adding true left does not help

empty disjunction is false

indeed: adding false right does not help

reducing modal sequents: propositional part

$p, q_1, \ldots, q_n \Rightarrow p, r_1, \ldots, r_m$ is valid

$A, \neg\phi \Rightarrow B$ if and only if $A \Rightarrow \phi, B$

$A \Rightarrow \neg\phi, B$ if and only if $A, \phi \Rightarrow B$

$A, \phi \wedge \psi \Rightarrow B$ if and only if $A, \phi, \psi \Rightarrow B$

$A \Rightarrow \phi \wedge \psi, B$ if and only if both $A \Rightarrow \phi, B$ and $A \Rightarrow \psi, B$

$A, \phi \vee \psi \Rightarrow B$ if and only if both $A, \phi \Rightarrow B$ and $A, \psi \Rightarrow B$

$A \Rightarrow \phi \vee \psi, B$ if and only if $A \Rightarrow \phi, \psi, B$

gives a decision procedure for propositional logic: ϕ valid iff $\Rightarrow \phi$ valid

example

$\phi = (p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$

$\Rightarrow \neg(p \wedge q) \vee (p \vee q)$

$\Rightarrow \neg(p \wedge q), (p \vee q)$

$p \wedge q \Rightarrow p \vee q$

$p, q \Rightarrow p \vee q$

$p, q \Rightarrow p, q$

the sequent is valid so ϕ is valid

example

$\phi = (p \vee q) \to (p \wedge q) \equiv \neg(p \vee q) \vee (p \wedge q)$

$\Rightarrow \neg(p \vee q) \vee (p \wedge q)$

$\Rightarrow \neg(p \vee q), (p \wedge q)$

$p \vee q \Rightarrow p \wedge q$

$p \Rightarrow p \wedge q$ and $q \Rightarrow p \wedge q$

$p \Rightarrow p$ and $p \Rightarrow q$ and $q \Rightarrow p$ and $q \Rightarrow q$

the sequent is not valid so ϕ is not valid

transformation of sequents: intuition

$p_1, \ldots, p_n, \Diamond\phi_1, \ldots, \Diamond\phi_m \Rightarrow q_1, \ldots, q_k, \Diamond\psi_1, \ldots, \Diamond\psi_k$

either we can decide validity because of the propositional part

if not, we go to a next world, leaving behind the propositional part

we then see if $\phi_i \Rightarrow \psi_1, \ldots, \psi_k$ is valid for some $i \in \{1, \ldots, m\}$

extension to modal logic

we get a sequent of the form

$p_1, \ldots, p_n, \Diamond\phi_1, \ldots, \Diamond\phi_m \Rightarrow q_1, \ldots, q_k, \Diamond\psi_1, \ldots, \Diamond\psi_k$

such a sequent is valid if and only if

either $p_i = q_j$ for some i and j

or $\phi_i \Rightarrow \psi_1, \ldots, \psi_k$ is valid for some $i \in \{1, \ldots, m\}$