advanced logic
2018 02 25
lecture 7

the book so far

basic modal logic
slides and MLOM chapter 2

bisimulations
slides and MLOM chapter 3

decidability
slides and MLOM chapter 4 and 7.1

expressivity, extension global box
slides and MLOM chapter 7.4

proof systems
slides and MLOM chapter 5

overview

- decidability: standard translation
- proof systems

overview

- decidability: standard translation
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standard translation: idea

translate being true in world x as predicate P holds for x

translate the accessibility relation R as a binary predicate R

examples

$$\mathrm{ST}_x(\Diamond p) = \exists y.\, Rxy \wedge Py$$

$$\mathrm{ST}_x(\Box p) = \forall y.\, (Rxy \to Py)$$

$$\mathrm{ST}_x(p \to \Diamond p) = Px \to \exists y(Rxy \wedge Py)$$

$$\mathrm{ST}_x(\Box\Diamond(p \vee q)) = \forall y.\, (Rxy \to \exists z.\, (Ryz \wedge (Pz \vee Qz)))$$

standard translation: more precisely

from basic modal logic to first-order predicate logic

$\mathsf{ST}_x(p) = Px$

$\mathsf{ST}_x(\bot) = \bot$

$\mathsf{ST}_x(\neg\phi) = \neg\mathsf{ST}_x(\phi)$

$\mathsf{ST}_x(\phi \wedge \psi) = \mathsf{ST}_x(\phi) \wedge \mathsf{ST}_x(\psi)$

$\mathsf{ST}_x(\Diamond\phi) = \exists y(Rxy \wedge \mathsf{ST}_y(\phi))$

$\mathsf{ST}_x(\Box\phi) = \forall y(Rxy \rightarrow \mathsf{ST}_y(\phi))$

standard translation: using only two variables

$$\mathsf{ST}_a(\Diamond\phi) = \exists b(Rab \wedge \mathsf{ST}_b(\phi))$$

$$\mathsf{ST}_a(\Box\phi) = \forall b(Rab \rightarrow \mathsf{ST}_b(\phi))$$

$$\mathsf{ST}_b(\Diamond\phi) = \exists a(Rba \wedge \mathsf{ST}_a(\phi))$$

$$\mathsf{ST}_b(\Box\phi) = \forall a(Rba \rightarrow \mathsf{ST}_a(\phi))$$

decidability via translation

the translation preserves satisfiability

$M, w \vDash \phi$ if and only if $\mathsf{ST}_x(\phi)[x := w]$ is satisfiable

first-order predicate logic using only two variables is decidable

decidability: important to know

effective finite model property:

tree unravelling, nearsightedness, via satisfiability, effective

sequents

tableaux

translation to decidable fragment of pred1:

translation on easy examples

overview

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recall: the modal tautologies are exactly defined by

extension:

a tautology for first-order propositional logic is a modal tautology

modal distribution:

$\models \Box(p \to q) \to \Box p \to \Box q$

modus ponens:

if $\models \phi \to \psi$ and $\models \phi$ then $\models \psi$

necessitation:

if $\models \phi$ then $\models \Box\phi$

substitution:

if $\models \phi$ then $\models \phi^\sigma$

proof systems

we may consider different proof systems:

natural deduction, sequent calculus, Hilbert systems

here we will use Hilbert systems

a proof is a sequence of numbered formulas,

where every formula is either an axiom

or the result of applying a derivation rule

soundness and completeness: intuitive definition

we have a model-theoretic notion of truth with notation $\models$

$\models \phi$ if ϕ is true in all models

we make a proof system with notation $\vdash$

$\vdash \phi$ if there is a derivation ending with $\vdash \phi$

a proof system is sound with respect to the semantics if

what we can derive is true: if $\vdash$ then $\models$

the proof system is complete with respect to the semantics if

we can derive what is true: if $\models$ then $\vdash$

soundness and completeness: $\vdash$ if and only if $\models$

provability and derivability

soundness and completeness for prop1:

$\models \phi$ if and only if $\vdash \phi$

soundness and completeness for pred1:

$\models \phi$ if and only if $\vdash \phi$

difficult proof rules

in prop1: implication introduction (because of the scope)

in prop1: disjunction elimination

in pred1: for all introduction (because of the side condition)

in pred1: exists elimination (because of the side condition)

proof system for basic modal logic

extension:
$\vdash \phi$ if ϕ is a tautology from prop1

modal distribution (!):
$\vdash \Box(\phi \to \psi) \to \Box\phi \to \Box\psi$

modus ponens:
if $\vdash \phi \to \psi$ and $\vdash \phi$ then $\vdash \psi$

necessitation:
if $\vdash \phi$ then $\vdash \Box\phi$

substitution:
if $\vdash \phi$ then $\vdash \phi^\sigma$ for every substitution σ

example derivation

1. $\phi \to \psi$ assumption
2. $\Box(\phi \to \psi)$ necessitation, 1
3. $\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$ modal distribution
4. $\Box\phi \to \Box\psi$ modus ponens, 3, 2

admissible rule: definition

if provability of $\phi_1, \ldots, \phi_n$ implies provability of ψ,

then we have an admissible rule :

$$\frac{\phi_1 \ldots \phi_n}{\psi}$$

admissible rule DISTR

we have shown the following rule (scheme) DISTR to be admissible:

$$\frac{\phi \to \psi}{\Box\phi \to \Box\psi}$$

we can use DISTR in derivations