advanced logic
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lecture 7
the book so far

basic modal logic
slides and MLOM chapter 2

bisimulations
slides and MLOM chapter 3

decidability
slides and MLOM chapter 4 and 7.1

expressivity, extension global box
slides and MLOM chapter 7.4

proof systems
slides and MLOM chapter 5
overview

- decidability: standard translation
- proof systems
overview

- decidability: standard translation
- proof systems
standard translation: idea

translate being true in world $x$ as predicate $P$ holds for $x$

translate the accessibility relation $R$ as a binary predicate $R$
examples

\[ \text{ST}_x(\Diamond p) = \exists y. Rxy \land Py \]

\[ \text{ST}_x(\Box p) = \forall y. (Rxy \rightarrow Py) \]

\[ \text{ST}_x(p \rightarrow \Diamond p) = Px \rightarrow \exists y (Rxy \land Py) \]

\[ \text{ST}_x(\Box \Diamond (p \lor q)) = \forall y. (Rxy \rightarrow \exists z. (Ryz \land (Pz \lor Qz))) \]
standard translation: more precisely

from basic modal logic to first-order predicate logic

\[ \text{ST}_x(p) = P_x \]

\[ \text{ST}_x(\bot) = \bot \]

\[ \text{ST}_x(\neg \phi) = \neg \text{ST}_x(\phi) \]

\[ \text{ST}_x(\phi \land \psi) = \text{ST}_x(\phi) \land \text{ST}_x(\psi) \]

\[ \text{ST}_x(\Diamond \phi) = \exists y (Rxy \land \text{ST}_y(\phi)) \]

\[ \text{ST}_x(\Box \phi) = \forall y (Rxy \rightarrow \text{ST}_y(\phi)) \]
standard translation: using only two variables

\[ \text{ST}_a(\Diamond \phi) = \exists b (Rab \land \text{ST}_b(\phi)) \]

\[ \text{ST}_a(\Box \phi) = \forall b (Rab \rightarrow \text{ST}_b(\phi)) \]

\[ \text{ST}_b(\Diamond \phi) = \exists a (Rba \land \text{ST}_a(\phi)) \]

\[ \text{ST}_b(\Box \phi) = \forall a (Rba \rightarrow \text{ST}_a(\phi)) \]
decidability via translation

the translation preserves satisfiability

\[ M, w \models \phi \text{ if and only if } \text{ST}_x(\phi)[x := w] \text{ is satisfiable} \]

first-order predicate logic using only two variables is decidable
decidability: important to know

effective finite model property:

tree unravelling, nearsightedness, via satisfiability, effective sequents

tableaux

translation to decidable fragment of pred1:

translation on easy examples
overview

- decidability: standard translation
- proof systems
recall: the modal tautologies are exactly defined by

extension:
a tautology for first-order propositional logic is a modal tautology

modal distribution:
\[ \vdash \Box(p \rightarrow q) \rightarrow \Box p \rightarrow \Box q \]

modus ponens:
if \( \vdash \phi \rightarrow \psi \) and \( \vdash \phi \) then \( \vdash \psi \)

necessitation:
if \( \vdash \phi \) then \( \vdash \Box \phi \)

substitution:
if \( \vdash \phi \) then \( \vdash \phi^\sigma \)
proof systems

we may consider different proof systems:
natural deduction, sequent calculus, Hilbert systems

here we will use Hilbert systems

a proof is a sequence of numbered formulas,
where every formula is either an axiom
or the result of applying a derivation rule
soundness and completeness: intuitive definition

we have a model-theoretic notion of truth with notation $\models$

$\models \phi$ if $\phi$ is true in all models

we make a proof system with notation $\vdash$

$\vdash \phi$ if there is a derivation ending with $\vdash \phi$

a proof system is sound with respect to the semantics if

what we can derive is true: if $\vdash$ then $\models$

the proof system is complete with respect to the semantics if

we can derive what is true: if $\models$ then $\vdash$

soundness and completeness: $\vdash$ if and only if $\models$
provability and derivability

soundness and completeness for prop1:
\[ \models \phi \text{ if and only if } \vdash \phi \]

soundness and completeness for pred1:
\[ \models \phi \text{ if and only if } \vdash \phi \]
difficult proof rules

in prop1: implication introduction (because of the scope)

in prop1: disjunction elimination

in pred1: for all introduction (because of the side condition)

in pred1: exists elimination (because of the side condition)
proof system for basic modal logic

extension:
⊢ φ if φ is a tautology from prop1

modal distribution (!):
⊢ □(φ → ψ) → □φ → □ψ

modus ponens:
if ⊢ φ → ψ and ⊢ φ then ⊢ ψ

necessitation:
if ⊢ φ then ⊢ □φ

substitution:
if ⊢ φ then ⊢ φ^σ for every substitution σ
example derivation

1. \( \phi \to \psi \)  
   assumption

2. \( \Box(\phi \to \psi) \)  
   necessitation, 1

3. \( \Box(\phi \to \psi) \to \Box\phi \to \Box\psi \)  
   modal distribution

4. \( \Box\phi \to \Box\psi \)  
   modus ponens, 3, 2
admissible rule: definition

if provability of $\phi_1, \ldots, \phi_n$ implies provability of $\psi$, then we have an admissible rule:

\[
\frac{\phi_1 \cdots \phi_n}{\psi}
\]
admissible rule DISTR

we have shown the following rule (scheme) DISTR to be admissible:

\[ \phi \rightarrow \psi \]

\[ \Box \phi \rightarrow \Box \psi \]

we can use DISTR in derivations