the book so far

basic modal logic
slides and MLOM chapter 2

bisimulations
slides and MLOM chapter 3

decidability
slides and MLOM chapter 4 and 7.1

expressivity, extension global box
slides and MLOM chapter 7.4

proof systems
slides and MLOM chapter 5
overview

- decidability: standard translation
- proof systems
overview

- decidability: standard translation
- proof systems
standard translation: idea

translate being true in world $x$ as predicate $P$ holds for $x$

translate the accessibility relation $R$ as a binary predicate $R$
examples

\[ ST_x(\Diamond p) = \exists y. Rxy \land Py \]

\[ ST_x(\Box p) = \forall y. (Rxy \rightarrow Py) \]

\[ ST_x(p \rightarrow \Diamond p) = Px \rightarrow \exists y(Rxy \land Py) \]

\[ ST_x(\Box \Diamond(p \lor q)) = \forall y. (Rxy \rightarrow \exists z. (Ryz \land (Pz \lor Qz))) \]
standard translation: more precisely

from basic modal logic to first-order predicate logic

\( \text{ST}_x(p) = Px \)

\( \text{ST}_x(\bot) = \bot \)

\( \text{ST}_x(\neg \phi) = \neg \text{ST}_x(\phi) \)

\( \text{ST}_x(\phi \land \psi) = \text{ST}_x(\phi) \land \text{ST}_x(\psi) \)

\( \text{ST}_x(\Diamond \phi) = \exists y (Rxy \land \text{ST}_y(\phi)) \)

\( \text{ST}_x(\Box \phi) = \forall y (Rxy \rightarrow \text{ST}_y(\phi)) \)
standard translation: using only two variables

\[ \text{ST}_a(\diamond \phi) = \exists b (Rab \land \text{ST}_b(\phi)) \]

\[ \text{ST}_a(\square \phi) = \forall b (Rab \rightarrow \text{ST}_b(\phi)) \]

\[ \text{ST}_b(\diamond \phi) = \exists a (Rba \land \text{ST}_a(\phi)) \]

\[ \text{ST}_b(\square \phi) = \forall a (Rba \rightarrow \text{ST}_b(\phi)) \]
decidability via translation

the translation preserves satisfiability

\( M, w \models \phi \) if and only if \( ST_x(\phi)[x := w] \) is satisfiable

first-order predicate logic using only two variables is decidable
decidability

effective finite model property:
tree unravelling, nearsightedness, via satisfiability, effective sequents
tableaux

translation to decidable fragment of pred1:
translation on easy examples
overview

- decidability: standard translation
- proof systems
recall: the modal tautologies are exactly defined by

extension:
a tautology for first-order propositional logic is a modal tautology

modal distribution:
\[ \models \Box(p \to q) \to \Box p \to \Box q \]

modus ponens:
if \( \models \phi \to \psi \) and \( \models \phi \) then \( \models \psi \)

necessitation:
if \( \models \phi \) then \( \models \Box \phi \)

substitution:
if \( \models \phi \) then \( \models \phi^\sigma \)
proof systems

we may consider different proof systems:

natural deduction, sequent calculus, Hilbert systems

here we will use Hilbert systems

a proof is a sequence of numbered formulas,

where every formula is either an axiom

or the result of applying a derivation rule
soundness and completeness: intuitive definition

we have a model-theoretic notion of truth with notation \( \models \)
\( \models \phi \) if \( \phi \) is true in all models

we make a proof system with notation \( \vdash \)
\( \vdash \phi \) if there is a derivation ending with \( \vdash \phi \)

a proof system is sound with respect to the semantics if
what we can derive is true: if \( \vdash \) then \( \models \)

the proof system is complete with respect to the semantics if
we can derive what is true: if \( \models \) then \( \vdash \)

soundness and completeness: \( \vdash \) if and only if \( \models \)
provability and derivability

soundness and completeness for prop1:
\[ \models \phi \text{ if and only if } \vdash \phi \]

soundness and completeness for pred1:
\[ \models \phi \text{ if and only if } \vdash \phi \]
proof system for basic modal logic

definitions:

extension:
\[ \vdash \phi \text{ if } \phi \text{ is a tautology from prop1} \]

modal distribution:
\[ \vdash \Box(p \rightarrow q) \rightarrow \Box p \rightarrow \Box q \]

modus ponens:
if \[ \vdash \phi \rightarrow \psi \text{ and } \vdash \phi \text{ then } \vdash \psi \]

necessitation:
if \[ \vdash \phi \text{ then } \vdash \Box \phi \]

substitution:
if \[ \vdash \phi \text{ then } \vdash \phi^\sigma \text{ for every substitution } \sigma \]
examples

\[\text{if } \vdash \phi \to \psi \text{ then } \vdash \Box \phi \to \Box \psi\]

\[\text{if } \vdash \Box (\phi \land \psi) \text{ then } \vdash \Box \phi \land \Box \psi\]

\[\vdash (\Diamond \phi \land \Box (\phi \to \psi)) \to \Diamond \psi\]
admissible rule: definition

if provability of $\phi_1, \ldots, \phi_n$ implies provability of $\psi$,
then we have an admissible rule:

$$
\frac{\phi_1 \ldots \phi_n}{\psi}
$$
two useful admissible rules

PROP is a useful admissible proof rule (scheme):

if \((\phi_1 \land \ldots \land \phi_n) \rightarrow \psi\) is a tautology
then for every substitution \(\sigma\) the following is an admissible proof rule:

\[
\begin{array}{c}
\phi_1^\sigma \cdots \phi_n^\sigma \\
\hline \\
\psi^\sigma
\end{array}
\]

(DISTR) is a useful admissible proof rule (scheme):

\[
\begin{array}{c}
\phi \rightarrow \psi \\
\hline \\
\Box \phi \rightarrow \Box \psi
\end{array}
\]
soundness and completeness

the proof system $K$ is sound and complete with respect to all frames

\[ \vdash_k \phi \iff \models \phi \]
consider the following axioms

distribution
DB: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

truth axiom or veridicality
A1: $\Box p \rightarrow p$

positive introspection
A2: $\Box p \rightarrow \Box\Box p$

negative introspection
A3: $\neg\Box p \rightarrow \Box\neg\Box p$
frame correspondences

for all frames $\mathcal{F} = (W, R)$ we have:

$\mathcal{F} \models A_1$ iff $R$ is reflexive

$\mathcal{F} \models A_2$ iff $R$ is transitive

$\mathcal{F} \models A_3$ iff $R$ is euclidean

$R$ is euclidean if: $\forall xyz (Rxy \land Rxz \rightarrow Ryz)$

reflexive frames are characterized by $A_1$

reflexive-transitive frames are characterized by $A_1$ and $A_2$

frames with equivalence relations are characterized by $A_1$ and $A_2$ and $A_3$
we extend system K step by step

system $T$: truth axiom added
A1: $\Box p \rightarrow p$

system $S4$: positive introspection added
A2: $\Box p \rightarrow \Box \Box p$

system $S5$: negative introspection added
A3: $\neg \Box p \rightarrow \Box \neg \Box p$
we use Hilbert systems $K$, $T$, $S4$, $S5$

we can easily imagine other similar Hilbert proof systems

there are usually various options in the design of a proof system
four times soundness and completeness

$K$ is sound and complete for all frames

$T$ is sound and complete for all reflexive frames

$S4$ is sound and complete for all reflexive-transitive frames

$S5$ is sound and complete for all frames with $R$ an equivalence relation
example

show that $A3 = \neg \Box p \rightarrow \Box \neg \Box p$ is not derivable in $S4$

we give a reflexive-transitive frame in which $A3$ is not true

then we use soundness and completeness: if not true then not derivable

the frame we use: $\langle \{w,v\}, \{(w,w),(w,v),(v,v)\} \rangle$

with the valuation $V(p) = \{v\}$ we have $A3$ not true in $w$

$w \models \neg \Box p$ because $(w,w) \in R$ and $w \not\models p$

$v \models \Box p$ hence $v \not\models \neg \Box p$ and because also $(w,v) \in R$ we have $w \not\models \Box p$
derivation in proof system T: example

1. $\Box r \rightarrow r$

2. $\Box (p \rightarrow q) \rightarrow (p \rightarrow q)$

3. $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \land b) \rightarrow c)$

4. $(\Box (p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow ((\Box (p \rightarrow q) \land p) \rightarrow q)$

5. $(\Box (p \rightarrow q) \land p) \rightarrow q$

how should we annotate the steps?
derivation in S5: example

we prove $\vdash_{S5} \neg \Box \neg \Box p \rightarrow p$

1. $\neg \Box p \rightarrow \Box \neg \Box p$ (A3)
2. $\neg \Box \neg \Box p \rightarrow \Box p$ (PROP, 1)
3. $\Box p \rightarrow p$ (A1)
4. $\neg \Box \neg \Box p \rightarrow p$ (PROP, 2, 3)

Which tautologies from prop1 do we use?

$(\neg a \rightarrow b) \rightarrow (\neg b \rightarrow a)$, and

$((a \rightarrow b) \land (b \rightarrow c)) \rightarrow (a \rightarrow c)$