the book so far

- basic modal logic
  slides and MLOM chapter 2
- bisimulations
  slides and MLOM chapter 3
- decidability
  slides and MLOM chapter 4 and 7.1
- expressivity, extension, global box
  slides and MLOM chapter 7.4
- proof systems
  slides and MLOM chapter 5

overview

- decidability: standard translation
- proof systems

overview

- decidability: standard translation
- proof systems
standard translation: idea

translate being true in world \( x \) as predicate \( P \) holds for \( x \)

translate the accessibility relation \( R \) as a binary predicate \( R \)

standard translation: more precisely

from basic modal logic to first-order predicate logic

\[
\begin{align*}
ST_x(p) &= P_x \\
ST_x(\bot) &= \bot \\
ST_x(\neg \phi) &= \neg ST_x(\phi) \\
ST_x(\phi \land \psi) &= ST_x(\phi) \land ST_x(\psi) \\
ST_x(\diamond \phi) &= \exists y. Rxy \land Py \\
ST_x(\Box \phi) &= \forall y. (Rxy \to Py) \\
ST_x(p \to \diamond \phi) &= Px \to \exists y. (Rxy \land Py) \\
ST_x(\Box \diamond (p \lor q)) &= \forall y. (Rxy \to \exists z. (Ryz \land (Pz \lor Qz)))
\end{align*}
\]

standard translation: using only two variables

\[
\begin{align*}
ST_a(\diamond \phi) &= \exists b. (Rab \land ST_b(\phi)) \\
ST_a(\Box \phi) &= \forall b. (Rab \to ST_b(\phi)) \\
ST_b(\diamond \phi) &= \exists a. (Rba \land ST_a(\phi)) \\
ST_b(\Box \phi) &= \forall a. (Rba \to ST_b(\phi))
\end{align*}
\]
decidability via translation

the translation preserves satisfiability

\( M, w \models \phi \) if and only if \( \text{ST}_x(\phi)[x := w] \) is satisfiable

first-order predicate logic using only two variables is decidable

decidability

effective finite model property:

tree unravelling, nearsightedness, via satisfiability, effective sequents

tableaux

translation to decidable fragment of pred1:

translation on easy examples

overview

decidability: standard translation

proof systems

recall: the modal tautologies are exactly defined by

extension:

a tautology for first-order propositional logic is a modal tautology

modal distribution:

\( \models \Box(p \to q) \to \Box p \to \Box q \)

modus ponens:

if \( \models \phi \to \psi \) and \( \models \phi \) then \( \models \psi \)

necessitation:

if \( \models \phi \) then \( \models \Box \phi \)

substitution:

if \( \models \phi \) then \( \models \phi^\sigma \)
proof systems

we may consider different proof systems:
natural deduction, sequent calculus, Hilbert systems

here we will use Hilbert systems

a proof is a sequence of numbered formulas,
where every formula is either an axiom
or the result of applying a derivation rule

soundness and completeness: intuitive definition

we have a model-theoretic notion of truth with notation ⊨
⊨ φ if φ is true in all models

we make a proof system with notation ⊢
⊢ φ if there is a derivation ending with ⊢ φ

a proof system is sound with respect to the semantics if
what we can derive is true: if ⊢ then ⊨
the proof system is complete with respect to the semantics if
we can derive what is true: if ⊨ then ⊢
soundness and completeness: ⊢ if and only if ⊨

provability and derivability

soundness and completeness for prop1:
⊨ φ if and only if ⊢ φ

soundness and completeness for pred1:
⊨ φ if and only if ⊢ φ

proof system for basic modal logic

extension:
⊢ φ if φ is a tautology from prop1

modal distribution:
⊢ □(p → q) → □p → □q

modus ponens:
if ⊢ φ → ψ and ⊢ φ then ⊢ ψ

necessitation:
if ⊢ φ then ⊢ □φ

substitution:
if ⊢ φ then ⊢ φσ for every substitution σ
examples

if \( \vdash \phi \rightarrow \psi \) then \( \vdash \Box \phi \rightarrow \Box \psi \)

if \( \vdash \Box (\phi \land \psi) \) then \( \vdash \Box \phi \land \Box \psi \)

\( \vdash (\Diamond \phi \land \Box (\phi \rightarrow \psi)) \rightarrow \Diamond \psi \)

admissible rule: definition

if provability of \( \phi_1, \ldots, \phi_n \) implies provability of \( \psi \),
then we have an admissible rule:

\[
\frac{\phi_1 \ldots \phi_n}{\psi}
\]

soundness and completeness

the proof system \( K \) is sound and complete with respect to all frames

\( \vdash_k \phi \iff \vdash \phi \)

two useful admissible rules

PROP is a useful admissible proof rule (scheme):

if \( (\phi_1 \land \ldots \land \phi_n) \rightarrow \psi \) is a tautology
then for every substitution \( \sigma \) the following is an admissible proof rule:

\[
\frac{\phi_1^\sigma \ldots \phi_n^\sigma}{\psi^\sigma}
\]

(DISTR) is a useful admissible proof rule (scheme):

\[
\frac{\phi \rightarrow \psi}{\Box \phi \rightarrow \Box \psi}
\]
consider the following axioms

distribution
**DB:** \( \square(p \rightarrow q) \rightarrow (\square p \rightarrow \square q) \)

truth axiom or veridicality
**A1:** \( \square p \rightarrow p \)

positive introspection
**A2:** \( \square p \rightarrow \square \square p \)

negative introspection
**A3:** \( \neg \square p \rightarrow \square \neg \square p \)

---

frame correspondences

for all frames \( \mathcal{F} = (W, R) \) we have:

\( \mathcal{F} \models A1 \) iff \( R \) is reflexive

\( \mathcal{F} \models A2 \) iff \( R \) is transitive

\( \mathcal{F} \models A3 \) iff \( R \) is euclidean

\( R \) is euclidean if: \( \forall xyz (Rxy \land Rxz \rightarrow Ryz) \)

reflexive frames are characterized by A1

reflexive-transitive frames are characterized by A1 and A2

frames with equivalence relations are characterized by A1 and A2 and A3

---

we extend system K step by step

**system \( T \):** truth axiom added

**A1:** \( \square p \rightarrow p \)

**system \( S4 \):** positive introspection added

**A2:** \( \square p \rightarrow \square \square p \)

**system \( S5 \):** negative introspection added

**A3:** \( \neg \square p \rightarrow \square \neg \square p \)

---

proof system: remark

we use Hilbert systems \( K, T, S4, S5 \)

we can easily imagine other similar Hilbert proof systems

there are usually various options in the design of a proof system
four times soundness and completeness

\( K \) is sound and complete for all frames
\( T \) is sound and complete for all reflexive frames
\( S4 \) is sound and complete for all reflexive-transitive frames
\( S5 \) is sound and complete for all frames with \( R \) an equivalence relation

**example**

show that \( A3 = \neg \Box p \rightarrow \Box \neg \Box p \) is not derivable in \( S4 \)
we give a reflexive-transitive frame in which \( A3 \) is not true
then we use soundness and completeness: if not true then not derivable
the frame we use: \( \{ (w, v), ((w, w), (w, v), (v, v)) \} \)
with the valuation \( V(p) = \{ v \} \) we have \( A3 \) not true in \( w \)
\( w \models \neg \Box p \) because \( (w, w) \in R \) and \( w \not\models p \)
\( v \models \Box p \) hence \( v \not\models \neg \Box p \) and because also \( (w, v) \in R \) we have \( w \not\models \Box p \)

**derivation in proof system T: example**

1. \( \Box r \rightarrow r \)
2. \( \Box(p \rightarrow q) \rightarrow (p \rightarrow q) \)
3. \( (a \rightarrow (b \rightarrow c)) \rightarrow ((a \land b) \rightarrow c) \)
4. \( (\Box(p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow ((\Box(p \rightarrow q) \land p) \rightarrow q) \)
5. \( (\Box(p \rightarrow q) \land p) \rightarrow q \)

how should we annotate the steps?

**derivation in \( S5 \): example**

we prove \( \vdash_{S5} \neg \Box \neg \Box p \rightarrow p \)
1. \( \neg \Box p \rightarrow \Box \neg \Box p \) (A3)
2. \( \neg \Box \neg \Box p \rightarrow \Box p \) (PROP, 1)
3. \( \Box p \rightarrow p \) (A1)
4. \( \neg \Box \neg \Box p \rightarrow p \) (PROP, 2, 3)

Which tautologies from prop1 do we use?
\( (\neg a \rightarrow b) \rightarrow (\neg b \rightarrow a) \), and
\( ((a \rightarrow b) \land (b \rightarrow c)) \rightarrow (a \rightarrow c) \)