the book so far

basic modal logic
slides and MLOM chapter 2

bisimulations
slides and MLOM chapter 3

decidability
slides and MLOM chapter 4 and 7.1

expressivity, extension global box
slides and MLOM chapter 7.4

proof systems
slides and MLOM chapter 5

overview

- decidability: standard translation
- proof systems

overview

- decidability: standard translation
- proof systems
**standard translation: idea**

translate being true in world $x$ as predicate $P$ holds for $x$

translate the accessibility relation $R$ as a binary predicate $R$

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**examples**

$ST_x(\diamond p) = \exists y. Rxy \land Py$

$ST_x(\Box p) = \forall y. (Rxy \rightarrow Py)$

$ST_x(p \rightarrow \diamond p) = Px \rightarrow \exists y(Rxy \land Py)$

$ST_x(\Box(\diamond (p \lor q))) = \forall y. (Rxy \rightarrow \exists z. (Ryz \land (Pz \lor Qz)))$

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**standard translation: more precisely**

from basic modal logic to first-order predicate logic

$ST_x(p) = Px$

$ST_x(\bot) = \bot$

$ST_x(\neg \phi) = \neg ST_x(\phi)$

$ST_x(\phi \land \psi) = ST_x(\phi) \land ST_x(\psi)$

$ST_x(\diamond \phi) = \exists y(Rxy \land ST_y(\phi))$

$ST_x(\Box \phi) = \forall y(Rxy \rightarrow ST_y(\phi))$

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**standard translation: using only two variables**

$ST_a(\diamond \phi) = \exists b(Rab \land ST_b(\phi))$

$ST_a(\Box \phi) = \forall b(Rab \rightarrow ST_b(\phi))$

$ST_b(\diamond \phi) = \exists a(Rba \land ST_a(\phi))$

$ST_b(\Box \phi) = \forall a(Rba \rightarrow ST_a(\phi))$
decidability via translation

the translation preserves satisfiability

$$M, w \models \phi$$ if and only if $$\text{ST}_x(\phi)[x := w]$$ is satisfiable

first-order predicate logic using only two variables is decidable

decidability: important to know

effective finite model property:

tree unravelling, nearsightedness, via satisfiability, effective sequents

tableaux

translation to decidable fragment of pred1:

translation on easy examples

overview

• decidability: standard translation

• proof systems

recall: the modal tautologies are exactly defined by

extension:
a tautology for first-order propositional logic is a modal tautology

modal distribution:
$$\models (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$$

modus ponens:
if $$\models \phi \rightarrow \psi$$ and $$\models \phi$$ then $$\models \psi$$

necessitation:
if $$\models \phi$$ then $$\models \Box \phi$$

substitution:
if $$\models \phi$$ then $$\models \phi^\sigma$$
proof systems

we may consider different proof systems:
natural deduction, sequent calculus, Hilbert systems

here we will use Hilbert systems
a proof is a sequence of numbered formulas,
where every formula is either an axiom
or the result of applying a derivation rule

soundness and completeness: intuitive definition

we have a model-theoretic notion of truth with notation $\models$

$\models \phi$ if $\phi$ is true in all models

we make a proof system with notation $\vdash$

$\vdash \phi$ if there is a derivation ending with $\vdash \phi$

a proof system is sound with respect to the semantics if
what we can derive is true: if $\vdash \phi$ then $\models \phi$

the proof system is complete with respect to the semantics if
we can derive what is true: if $\models \phi$ then $\vdash \phi$

soundness and completeness: $\vdash$ if and only if $\models$

provability and derivability

soundness and completeness for prop1:

$\models \phi$ if and only if $\vdash \phi$

soundness and completeness for pred1:

$\models \phi$ if and only if $\vdash \phi$

difficult proof rules

in prop1: implication introduction (because of the scope)
in prop1: disjunction elimination
in pred1: for all introduction (because of the side condition)
in pred1: exists elimination (because of the side condition)
proof system for basic modal logic

extension:
\[ \vdash \phi \text{ if } \phi \text{ is a tautology from prop1} \]

modal distribution (!):
\[ \vdash \Box(\phi \to \psi) \to \Box\phi \to \Box\psi \]

modus ponens:
if \( \vdash \phi \to \psi \) and \( \vdash \phi \) then \( \vdash \psi \)

necessitation:
if \( \vdash \phi \) then \( \vdash \Box\phi \)

substitution:
if \( \vdash \phi \) then \( \vdash \phi^\sigma \) for every substitution \( \sigma \)

example derivation

1. \( \phi \to \psi \) \hspace{1cm} \text{assumption}
2. \( \Box(\phi \to \psi) \) \hspace{1cm} \text{necessitation, 1}
3. \( \Box(\phi \to \psi) \to \Box\phi \to \Box\psi \) \hspace{1cm} \text{modal distribution}
4. \( \Box\phi \to \Box\psi \) \hspace{1cm} \text{modus ponens, 3, 2}

admissible rule: definition

if provability of \( \phi_1, \ldots, \phi_n \) implies provability of \( \psi \),
then we have an admissible rule:
\[
\begin{array}{c}
\phi_1 \ldots \phi_n \\
\hline
\psi
\end{array}
\]

admissible rule DISTR

we have shown the following rule (scheme) DISTR to be admissible:
\[
\begin{array}{c}
\phi \to \psi \\
\hline
\Box\phi \to \Box\psi
\end{array}
\]

we can use DISTR in derivations