advanced logic
2019 03 04
lecture 9
overview

- temporal logic using temporal frames
- basic temporal logic
- towards multi-modal logic
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definition: a frame $\mathcal{F} = (T, <)$ is a **temporal frame** if

$<$ is irreflexive: not $t < t$ for all $t \in T$, and

$<$ is transitive: if $t < u$ and $u < v$ then $t < v$ for all $t, u, v \in T$

definition: a **temporal model** is

a model based on a temporal frame,

so a temporal frame $(T, <)$ with a valuation $V : \text{Var} \rightarrow \mathcal{P}(T)$
temporal frame: example

\( \mathcal{N} = (\mathbb{N}, <) \) is a temporal frame

\[ \mathcal{N} \models \Diamond \Box p \rightarrow \Box \Diamond p \]

\[ \mathcal{N} \not\models \Box \Diamond p \rightarrow \Diamond \Box p \]
a temporal frame \((T, <)\) is

**right-linear** if \((x < y) \land (x < z) \rightarrow (y < z) \lor (y = z) \lor (z < y)\)

**right-branching** if there exist \(x, y, z\) such that \(x < y\) and \(x < z\) but \(\neg(y < z) \land y \neq z \land \neg(z < y)\)

**discrete** if \((x < y) \rightarrow \exists z : x < z \land \neg \exists u : (x < u) \land (u < z)\)

**dense** if \(x < z \rightarrow \exists y (x < y \land y < z)\)
are those frame properties modally definable?

right-linearity is modally definable by

\[(\Diamond p \land \Diamond q) \rightarrow \Diamond(p \land \Diamond q) \lor \Diamond(p \land q) \lor \Diamond(\Diamond p \land q)\]

right-branchingness is not modally definable

discreteness is definable in basic temporal logic (later)

density is modally definable by \(\Diamond p \rightarrow \Diamond\Diamond p\)
new operators

we will consider new(?) operators with a time-intuition

next: $\otimes$

$\mathcal{M}, t \models \otimes \phi \iff \exists v \; t < v \land (\neg \exists u : t < u < v) \land \mathcal{M}, v \models \phi$

next is not definable in basic modal logic

until: $U$

$\mathcal{M}, t \models \phi U \psi \iff \exists v : t < v \land \mathcal{M}, v \models \psi \land \forall u : t < u < v \rightarrow \mathcal{M}, u \models \phi$

until is not definable in basic modal logic
diamond and next and until

in a temporal frame

\( \diamond p \) is equivalent to \( \top Up \)

\( \otimes p \) is equivalent to \( \perp Up \)
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second approach to temporal logic

formulas inductively defined by:

\[ p | \bot | \neg \phi | \phi \land \psi | \langle F \rangle \phi | \langle P \rangle \phi \]

truth and validity for basic temporal formulas in temporal frames

\[ M, t \models p \text{ iff } t \in V(p) \]

\[ M, t \models \bot \text{ never} \]

\[ M, t \models \neg \phi \text{ iff } M, t \not\models \phi \]

\[ M, t \models \phi \land \psi \text{ iff } M, t \models \phi \text{ and } M, t \models \psi \]

\[ M, t \models \langle F \rangle \phi \text{ iff for some } u \text{ with } t < u \text{ we have } M, u \models \phi \]

\[ M, t \models \langle P \rangle \phi \text{ iff for some } s \text{ with } s < t \text{ we have } M, s \models \phi \]
we can define $[F]\phi$ as $\neg\langle F \rangle \neg \phi$ and $[P]\phi$ as $\neg\langle P \rangle \neg \phi$

alternatively we can define

$\mathcal{M}, t \models [F]\phi$ iff for all $u$ with $t < u$ we have $\mathcal{M}, u \models \phi$

$\mathcal{M}, t \models [P]\phi$ iff for all $s$ with $s < t$ we have $\mathcal{M}, s \models \phi$
is basic temporal logic an extension?

yes: ‘past’ cannot be defined in basic modal logic

why?
defining properties in basic temporal logic

we can define discreteness for linear temporal frames by

\[ q \rightarrow \langle F \rangle[P](q \lor \langle F \rangle q) \]

using next we can define discreteness by

\[ \langle F \rangle T \rightarrow \otimes T \]

we can define right-linearity by

\[ \langle P \rangle \langle F \rangle q \rightarrow \langle P \rangle q \lor q \lor \langle F \rangle q \]
what to do with bisimulation?

the notion of bisimulation for basic modal logic

is too weak for basic temporal logic

intuitively, also the past should be taken into account
adapt bisimulation to basic temporal logic

∅ ≠ Z ⊆ T × T' is a temporal bisimulation between \( M = (T, R, V) \) and \( M' = (T', R', V') \) if for all \((t, t') \in Z\) we have:

- \( t \in V(p) \) iff \( t' \in V'(p) \) for all \( p \in \text{Var} \)
- if \( Rtu \) then there is \( u' \) such that \( R't'u' \) and \((u, u') \in Z\)
- if \( R't'u' \) then there is \( u \) such that \( Rtu \) and \((u, u') \in Z\)
- if \( Rst \) then there is \( s' \) such that \( R's't' \) and \((s, s') \in Z\)
- if \( R's't' \) then there is \( s \) such that \( Rst \) and \((s, s') \in Z\)
$w$ and $w'$ are bisimilar using a temporal bisimulation

iff

in $w$ and in $w'$ the same basic temporal formulas are true

for finitely branching models

this is a special case of a more general result
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multi-modal logic: formulas

we assume a set of labels $\mathcal{I}$

for every label $i$ there is a modality $\langle i \rangle$

we consider for every label $i$ a diamond-formula $\langle i \rangle \phi$

so the formulas of multi-modal logic are, given $\mathcal{I}$, inductively defined by

$p \mid \bot \mid \neg \phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \langle i \rangle \phi \mid [i] \phi$ for $i \in \mathcal{I}$
instances of multi-modal logic

we already know various instances of multi-modal logic:

$I = \emptyset$ gives propositional logic

$I = \{0\}$ gives basic modal logic

$I = \{F, P\}$ gives temporal logic with temporal model $R_F = R_P^{-1}$

$I = \text{Prog}(A)$ gives propositional dynamic logic with PDL-model
exercises

until not definable in basic modal logic in temporal frames

until not definable in temporal modal logic in temporal frames

next not definable in basic modal logic in temporal frames

define discrete in temporal logic

use next to define discrete in temporal logic

define right-linearity in temporal logic
multi-modal logic: book

MLOM chapter 10.1

especially interesting if there is some connection between the $R_i$

such as: past is the inverse of future: $R_F = R_P^{-1}$