advanced logic
2019 03 04
lecture 9

overview

- temporal logic using temporal frames
- basic temporal logic
- towards multi-modal logic

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recall: definition temporal frame and temporal model

definition: a frame $\mathcal{F} = (T, <)$ is a temporal frame if

$<$ is irreflexive: not $t < t$ for all $t \in T$, and

$<$ is transitive: if $t < u$ and $u < v$ then $t < v$ for all $t, u, v \in T$

definition: a temporal model is

a model based on a temporal frame,

so a temporal frame $(T, <)$ with a valuation $V : \mathsf{Var} \to \mathcal{P}(T)$

temporal frame: example

$\mathcal{N} = (\mathbb{N}, <)$ is a temporal frame

$\mathcal{N} \models \Diamond\Box p \to \Box\Diamond p$

$\mathcal{N} \not\models \Box\Diamond p \to \Diamond\Box p$

properties of temporal frames

a temporal frame $(T, <)$ is

right-linear if $(x < y) \wedge (x < z) \to (y < z) \vee (y = z) \vee (z < y)$

right-branching if there exist x, y, z such that $x < y$ and $x < z$ but $\neg(y < z) \wedge y \neq z \wedge \neg(z < y)$

discrete if $(x < y) \to \exists z :\ x < z \wedge \neg\exists u :\ (x < u) \wedge (u < z)$

dense if $x < z \to \exists y\, (x < y \wedge y < z)$

are those frame properties modally definable?

right-linearity is modally definable by
$(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$

right-branchingness is not modally definable

discreteness is definable in basic temporal logic (later)

density is modally definable by $\Diamond p \to \Diamond\Diamond p$

new operators

we will consider new(?) operators with a time-intuition

next: $\otimes$

$\mathcal{M}, t \vDash \otimes\phi$ iff $\exists v\ t < v \wedge (\neg\exists u :\ t < u < v) \wedge \mathcal{M}, v \vDash \phi$

next is not definable in basic modal logic

until: U

$\mathcal{M}, t \vDash \phi U \psi$ iff $\exists v :\ t < v \wedge \mathcal{M}, v \vDash \psi \wedge \forall u :\ t < u < v \to \mathcal{M}, u \vDash \phi$

until is not definable in basic modal logic

diamond and next and until

in a temporal frame

$\Diamond p$ is equivalent to $\top U p$

$\otimes p$ is equivalent to $\bot U p$

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basic temporal logic

second approach to temporal logic

formulas inductively defined by:

$p \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \langle F \rangle \phi \mid \langle P \rangle \phi$

truth and validity for basic temporal formulas in temporal frames

$\mathcal{M}, t \models p$ iff $t \in V(p)$

$\mathcal{M}, t \models \bot$ never

$\mathcal{M}, t \models \neg\phi$ iff $\mathcal{M}, t \not\models \phi$

$\mathcal{M}, t \models \phi \wedge \psi$ iff $\mathcal{M}, t \models \phi$ and $\mathcal{M}, t \models \psi$

$\mathcal{M}, t \models \langle F \rangle \phi$ iff for some u with $t < u$ we have $\mathcal{M}, u \models \phi$

$\mathcal{M}, t \models \langle P \rangle \phi$ iff for some s with $s < t$ we have $\mathcal{M}, s \models \phi$

basic temporal logic: remark

we can define $[F]\phi$ as $\neg\langle F \rangle \neg\phi$ and $[P]\phi$ as $\neg\langle P \rangle \neg\phi$

alternatively we can define

$\mathcal{M}, t \models [F]\phi$ iff for all u with $t < u$ we have $\mathcal{M}, u \models \phi$

$\mathcal{M}, t \models [P]\phi$ iff for all s with $s < t$ we have $\mathcal{M}, s \models \phi$

is basic temporal logic an extension?

yes: ‘past’ cannot be defined in basic modal logic

why?

defining properties in basic temporal logic

we can define discreteness for linear temporal frames by

$q \to \langle F \rangle [P](q \vee \langle F \rangle q)$

using next we can define discreteness by

$\langle F \rangle \top \to \otimes \top$

we can define right-linearity by

$\langle P \rangle \langle F \rangle q \to \langle P \rangle q \vee q \vee \langle F \rangle q$

what to do with bisimulation?

the notion of bisimulation for basic modal logic

is too weak for basic temporal logic

inituitively, also the past should be taken into account

adapt bisimulation to basic temporal logic

$\emptyset \neq Z \subseteq T \times T'$ is a temporal bisimulation between $\mathcal{M} = (T, R, V)$ and $\mathcal{M}' = (T', R', V')$ if for all $(t, t') \in Z$ we have:

$t \in V(p)$ iff $t' \in V'(p)$ for all $p \in Var$

if Rtu then there is u' such that $R't'u'$ and $(u, u') \in Z$

if $R't'u'$ then there is u such that Rtu and $(u, u') \in Z$

if Rst then there is s' such that $R's't'$ and $(s, s') \in Z$

if $R's't'$ then there is s such that Rst and $(s, s') \in Z$

bisimulation and modal equivalence for basic temporal logic

w and w' are bisimilar using a temporal bisimulation

iff

in w and in w' the same basic temporal formulas are true

for finitely branching models

this is a special case of a more general result

until not definable in basic temporal logic

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multi-modal logic: formulas

we assume a set of labels $\mathcal{I}$

for every label i there is a modality $\langle i \rangle$

we consider for every label i a diamond-formula $\langle i \rangle \phi$

so the formulas of multi-modal logic are, given $\mathcal{I}$, inductively defined by

$p \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \langle i \rangle \phi \mid [i]\phi$ for $i \in \mathcal{I}$

instances of multi-modal logic

we already know various instances of multi-modal logic:

$\mathcal{I} = \emptyset$ gives propositional logic

$\mathcal{I} = \{0\}$ gives basic modal logic

$\mathcal{I} = \{F, P\}$ gives temporal logic with temporal model $R_F = R_P^{-1}$

$\mathcal{I} = Prog(A)$ gives propositional dynamic logic with PDL-model

multi-modal logic: book

MLOM chapter 10.1

especially interesting if there is some connection between the R_i

such as: past is the inverse of future: $R_F = R_P^{-1}$

exercises

until not definable in basic modal logic in temporal frames

until not definable in temporal modal logic in temporal frames

next not definable in basic modal logic in temporal frames

define discrete in temporal logic

use next to define discrete in temporal logic

define right-linearity in temporal logic