

Automorphic Lie Algebras with Dihedral Symmetry

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Abstract

Automorphic Lie Algebras are interesting because of their fundamental nature and their role in our understanding of symmetry. Particularly crucial is their description and classification as it allows us to understand and apply them in different contexts, from mathematics to physical sciences. While the problem of classification of Automorphic Lie Algebras with dihedral symmetry was already considered in the past (see for instance [5], [6], and [7]), it was never addressed in full generality, where one takes any two representations of the dihedral group. Indeed, all results so far have been obtained under the simplifying assumption that one can use the same dihedral representation to define an action on either the space of vectors or matrices, and over the polynomials. In this paper we present a complete classification of Automorphic Lie Algebras with dihedral symmetry, starting from any two representations. In the light of this result we show that Automorphic Lie Algebras with dihedral symmetry are isomorphic, thereby extending the uniformity result of Lombardo and Sanders [7] and Bury [1]. Furthermore, we consider also the case of invariant vectors and compute the corresponding Molien functions; their knowledge allows us to compute symmetric rational maps related to the energy of Skyrmions with dihedral symmetry.

1 Introduction

Automorphic Lie Algebras were introduced in order to classify integrable partial differential equations [4], [5], [6]. In fact, the inverse scattering - spectral transform method, used to integrate these equations, requires a pair of matrices with functions as entries (Lax Pair). Since a general pair of such matrices gives rise to an under determined system of differential equations, one requires additional constraints. By a well established scheme introduced by Mikhailov [9],

and further developed in [6], this can be achieved by imposing a group symmetry on the matrices. Algebras consisting of all such symmetric matrices are called *Automorphic Lie Algebras*, by analogy with automorphic functions. Since their introduction they have been extensively studied (see, for instance, [1] and [2], to mention two recent PhD theses on the subject).

Automorphic Lie Algebras with dihedral symmetry were already considered in the past (see for instance [5], [6], and [7]); they are interesting because of their fundamental nature. They behave almost like $\mathfrak{sl}_2(\mathbb{C})$ algebras, that is, like semi-simple, finite dimensional algebras, but possess at the same time symmetry properties with respect to the dihedral group, which enrich their structure and allow to interpret them also like a symmetric version of Kac-Moody type algebras, that is, infinite dimensional algebras. From the point of view of applications, dihedral symmetry has been the most extensively studied in the context of integrable systems (e.g. [9], [5], [14], [10]) and it seems one of the most relevant in physical systems (although there are also many physical systems with icosahedral symmetry). The classification of Automorphic Lie Algebras with dihedral symmetry is therefore timely.

In the present paper we investigate dihedral symmetry in spaces of 2-dimensional vectors and 2×2 matrices over a polynomial ring in 2 variables, X and Y . For any two representations of the dihedral group one can define an action on the aforementioned spaces by simultaneously acting on either the vectors or matrices with one of the representations and acting on the polynomials with the other. With the aid of classical invariant theory and representation theory for finite groups, we construct the spaces of dihedral-invariant vectors and matrices. In particular, in the matrix case we find an Automorphic Lie Algebra, and we show that the Lie algebra structure does not depend on the initially chosen representations of the dihedral group, thereby extending the uniformity result of Lombardo and Sanders [7], and Bury [1]. Indeed, the main result can be formulated as follows (see Theorem 4 in Section 6.2.3 for details):

Let \mathbb{D}_N be the dihedral group and let its action on the space of 2×2 matrices over a polynomial ring in 2 variables, X and Y , be defined by simultaneously acting on the matrices and on the polynomials with any two irreducible representations of \mathbb{D}_N . Then, the resulting Automorphic Lie Algebras are isomorphic as Lie algebras and the Lie algebra structure does not depend on the initially chosen representations.

This result allows us to provide a complete classification of $\mathfrak{sl}_2(\mathbb{C})$ -based Automorphic Lie Algebras with dihedral symmetry. This is an important step towards the complete classification of Automorphic Lie Algebras, as the simplifying assumption that the representations are the same can no longer be made when considering higher dimensional Lie algebras.

2 Set Up

Let G be a finite group. We define an action of this group on various vector spaces, making them G -modules.

- A linear representation

$$\tau : G \rightarrow \text{GL}(\mathbb{C}^k)$$

yields an action of G on \mathbb{C}^k , given by

$$g : v \mapsto \tau(g)v \quad v \in \mathbb{C}^k, g \in G.$$

We adopt the notation convention $\tau_g := \tau(g)$.

- For any Lie group $\mathcal{G} \subset \text{GL}(\mathbb{C}^k)$ of linear transformations, we have the adjoint representation on its Lie algebra, denoted $\text{Ad} : \mathcal{G} \rightarrow \text{Aut}(\mathfrak{g})$, and defined as conjugation: $\text{Ad}(A)M = AMA^{-1}$, $A \in \mathcal{G}$, $M \in \mathfrak{g}$. If the finite group G is represented within the Lie group, $\tau : G \rightarrow \mathcal{G} \subset \text{GL}(\mathbb{C}^k)$, then we can compose the two maps to get an action of the finite group on the Lie algebra:

$$\text{Ad} \circ \tau : G \rightarrow \text{Aut}(\mathfrak{g}).$$

It is important to note that, in this construction, G acts by Lie algebra isomorphisms. That is, the Lie bracket is respected by G . Notice also that in case $\mathfrak{g} = \mathfrak{sl}_k(\mathbb{C})$ one can neglect the requirement that $\tau(G) \subset \text{SL}_k(\mathbb{C})$. Indeed, the trace is preserved by conjugation with any invertible matrix.

- Let $\mathbb{C}[X, Y]$ be the ring of polynomials in two variables. A two-dimensional representation

$$\sigma : G \rightarrow \text{GL}(\mathbb{C}^2)$$

can be used to define an action on the polynomial ring:

$$g : p(X, Y) \mapsto p(\sigma_{g^{-1}}(X, Y)), \quad p(X, Y) \in \mathbb{C}[X, Y], g \in G.$$

- The tensor product of two G -modules V and W is a G -module itself, with the action $g(v \otimes w) = gv \otimes gw$, $v \in V$, $w \in W$ and $g \in G$. In particular, we are interested in the action of G on $\mathfrak{g} \otimes \mathbb{C}[X, Y]$, which, by the above, becomes

$$g : M \otimes p(X, Y) \mapsto \tau_g M \tau_{g^{-1}} \otimes p(\sigma_{g^{-1}}(X, Y)).$$

The space $\mathfrak{g} \otimes \mathbb{C}[X, Y]$ is a Lie algebra with the bracket $[M \otimes p, M' \otimes p'] := [M, M'] \otimes pp'$. Because the action on both the factors \mathfrak{g} and $\mathbb{C}[X, Y]$ behave so well, regarding the structure, the action on the tensor product also respects the bracket:

$$g([M \otimes p, M' \otimes p']) = [g(M \otimes p), g(M' \otimes p')].$$

The objects of interest in this paper are the spaces of invariants and covariants:

Definition 1. Let χ be a one-dimensional representation of G , and V a G -module. An element $v \in V$ is called χ -**covariant** if $gv = \chi(g)v$. If χ is the trivial representation then v is called **invariant**.

The space of χ -covariants in V will be denoted by V_G^χ , the space of all covariants by V_G and the subspace of invariants by V^G . Both V^G and V_G are Lie algebras. For the invariants this follows directly from the fact that G preserves the bracket: if $v, w \in V^G$ and $g \in G$ then $g[v, w] = [gv, gw] = [v, w]$. To see that the space of all covariants is closed under the bracket as well, one notices that the product of two one-dimensional representations, say χ and ψ , is again a one-dimensional representation (since it is again a homomorphism). One finds $[V_G^\chi, V_G^\psi] \subset V_G^{\chi\psi} \subset V_G$.

Definition 2. The special cases

$$(\mathfrak{g} \otimes \mathbb{C}[X, Y])^G$$

are called **Automorphic Lie Algebras** c.f. [4], [6], [7].

An important classical result in invariant theory is Molien's Theorem (see for example [13]). We state it in a form tailored to our particular needs.

Theorem 1 (Molien). If V is a G -module with character ψ , and G also acts on $\mathbb{C}[X, Y]$ using a two-dimensional representation σ in the way shown above, then the Poincaré series for the space of χ -covariants of $V \otimes \mathbb{C}[X, Y]$ is given by

$$P((V \otimes \mathbb{C}[X, Y])_G^\chi, t) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi^*(g)\psi(g)}{\det(1 - \sigma_g^{-1}t)},$$

where the $*$ stands for complex conjugation.

The best known special case of this theorem is $P(\mathbb{C}[X, Y]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - \sigma_g^{-1}t)}$, which gives the Poincaré series for invariant forms.

3 The Dihedral Group \mathbb{D}_N

The dihedral group of order $2N$ can be defined abstractly by

$$\mathbb{D}_N = \langle r, s \mid r^N = s^2 = (rs)^2 = 1 \rangle.$$

If one thinks of the symmetries of a regular N -gon, then r stands for rotation and s for reflection.

All linear representations of \mathbb{D}_N are characterised by the character table, which contains an N^{th} root of unity $\omega := e^{\frac{2\pi i}{N}}$.

Character table of \mathbb{D}_N , N odd

	1	$[r]$	$[r^2]$	\dots	$[r^{\frac{N-1}{2}}]$	$[s]$
χ_1	1	1	1	\dots	1	1
χ_2	1	1	1	\dots	1	-1
ψ_1	2	$\omega + \omega^{N-1}$	$\omega^2 + \omega^{N-2}$	\dots	$\omega^{\frac{N-1}{2}} + \omega^{\frac{N+1}{2}}$	0
ψ_2	2	$\omega^2 + \omega^{N-2}$	$\omega^4 + \omega^{N-4}$	\dots	$\omega^{2\frac{N-1}{2}} + \omega^{2\frac{N+1}{2}}$	0
\vdots	\vdots	\vdots	\vdots	\dots	\vdots	\vdots
$\psi_{\frac{N-1}{2}}$	2	$\omega^{\frac{N-1}{2}} + \omega^{\frac{N+1}{2}}$	$\omega^{2\frac{N-1}{2}} + \omega^{2\frac{N+1}{2}}$	\dots	$\omega^{(\frac{N-1}{2})^2} + \omega^{(\frac{N+1}{2})^2}$	0

Character table of \mathbb{D}_N , N even

	1	$[r]$	$[r^2]$	\dots	$[r^{N/2-1}]$	$[s]$	$[rs]$
χ_1	1	1	1	\dots	1	1	1
χ_2	1	1	1	\dots	1	-1	-1
χ_3	1	-1	1	\dots	1	1	-1
χ_4	1	-1	1	\dots	1	-1	1
ψ_1	2	$\omega + \omega^{N-1}$	$\omega^2 + \omega^{N-2}$	\dots	$\omega^{N/2-1} + \omega^{N/2+1}$	0	0
ψ_2	2	$\omega^2 + \omega^{N-2}$	$\omega^4 + \omega^{N-4}$	\dots	$\omega^{2(N/2-1)} + \omega^{2(N/2+1)}$	0	0
\vdots	\vdots	\vdots	\vdots	\dots	\vdots	\vdots	\vdots
$\psi_{N/2-1}$	2	$\omega^{N/2-1} + \omega^{N/2+1}$	$\omega^{2(N/2-1)} + \omega^{2(N/2+1)}$	\dots	$\omega^{(N/2-1)^2} + \omega^{(N/2+1)^2}$	0	0

One can summarise the two-dimensional characters by

$$\psi_j(r^m) = \omega^{mj} + \omega^{-mj}, \quad \psi_j(sr^m) = 0.$$

The characters of the faithful (injective) representations are precisely those ψ_j for which

$\gcd(j, N) = 1$, where \gcd stands for *greatest common divisor*.

Note that

$$\psi_j = \psi_{j+N} = \psi_{-j}. \quad (1)$$

When we want explicit matrices for a representation $\tau : G \rightarrow \mathrm{GL}(\mathbb{C}^2)$ with character ψ_j , there is an entire equivalence class to choose from. Let

$$\tau_r = \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{N-j} \end{pmatrix} \quad \tau_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2)$$

Another useful choice is the real orthogonal one:

$$\tau_r = \begin{pmatrix} \cos(j\theta) & -\sin(j\theta) \\ \sin(j\theta) & \cos(j\theta) \end{pmatrix} \quad \tau_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\omega = e^{i\theta}$. In what follows, we will consider the first one and we call it *standard*.

If one knows the space of invariants V^G , when G is represented by τ , then one can easily find the space of invariants belonging to an equivalent representation τ' , because the invertible matrix T that relates the two representations, $T\tau_g = \tau'_g T$, also relates the spaces of invariants: $V^{\tau'(G)} = TV^{\tau(G)}$.

Lemma 1 ([4], [2]). *Let $\rho : G \rightarrow \mathrm{GL}(V)$ be any representation of G and $H \subset G$ a normal subgroup. Then restriction gives a representation*

$$\tilde{\rho} : \frac{G}{H} \rightarrow \mathrm{GL}(V^H),$$

where $\tilde{\rho}_{gH} = \rho_g|_{V^H}$.

Corollary 1. *If V is a \mathbb{D}_N module, then by restriction it is a $\mathbb{Z}_N \subset \mathbb{D}_N$ module and*

$$V^{\mathbb{Z}_N} = V^{\mathbb{D}_N} \oplus V_{\mathbb{D}_N}^{\chi_2}.$$

Moreover, the projections on the terms are $\frac{1}{2}(1+s)$ and $\frac{1}{2}(1-s)$ respectively.

If N is even, there is also the subgroup $\mathbb{D}_{N/2} \subset \mathbb{D}_N$ and, up to isomorphism,

$$V^{\mathbb{D}_{N/2}} = V^{\mathbb{D}_N} \oplus V_{\mathbb{D}_N}^{\chi_3}, \quad V^{\mathbb{D}_{N/2}} = V^{\mathbb{D}_N} \oplus V_{\mathbb{D}_N}^{\chi_4}.$$

Proof. The corollary follows if one applies the lemma to the normal subgroups $H = \mathrm{Ker}\chi_i$, $i = 2, 3, 4$, of \mathbb{D}_N . These groups are \mathbb{Z}_N , $\mathbb{D}_{N/2}$ and $\mathbb{D}_{N/2}$ respectively. Moreover, they all have order N hence the quotients $\frac{G}{H}$ have order 2 and thus equal \mathbb{Z}_2 . The decomposition of the \mathbb{Z}_2

modules V^H into irreducible components proves the corollary. \square

Corollary 1 was inspired by the work of Larry Smith on polynomial invariants of finite groups (see [13]).

4 Covariant forms

We define the binary forms

$$\alpha := XY, \quad \beta := \frac{X^N + Y^N}{2}, \quad \gamma := \frac{X^N - Y^N}{2}.$$

Where we need to consider different powers we use a subscript:

$$\beta_M := \frac{X^M + Y^M}{2}, \quad \gamma_M := \frac{X^M - Y^M}{2}.$$

Lemma 2 (Covariant forms). *Suppose the representation σ has character ψ_j and $\gcd(j, N) = 1$ (that is, σ is faithful). With the standard choice of matrices given in (2), the invariant forms are*

$$\mathbb{C}[X, Y]^{\mathbb{D}_N} = \mathbb{C}[\alpha, \beta], \quad \mathbb{C}[X, Y]_{\mathbb{D}_N}^{\chi_2} = \mathbb{C}[\alpha, \beta]\gamma$$

and, if N is even,

$$\mathbb{C}[X, Y]_{\mathbb{D}_N}^{\chi_3} = \mathbb{C}[\alpha, \beta]\beta_{N/2}, \quad \mathbb{C}[X, Y]_{\mathbb{D}_N}^{\chi_4} = \mathbb{C}[\alpha, \beta]\gamma_{N/2}.$$

In the case that σ is not faithful, one only needs to replace N by $N' = \frac{N}{\gcd(j, N)}$.

Proof. We will use Corollary 1. If \mathbb{Z}_N is generated by our standard choice of matrix, $\sigma_r = \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{N-j} \end{pmatrix}$, then one can find that $\mathbb{C}[X, Y]^{\mathbb{Z}_N} = \bigoplus_{i=0}^{N-1} \mathbb{C}[X^N, Y^N]\alpha^i$. This space has Poincaré series

$$\begin{aligned} P\left(\bigoplus_{i=0}^{N-1} \mathbb{C}[X^N, Y^N]\alpha^i, t\right) &= \frac{1 + t^2 + \dots + t^{2N-4} + t^{2N-2}}{(1 - t^N)^2} \\ &= \frac{1 - t^{2N}}{(1 - t^2)(1 - t^N)^2} = \frac{1 + t^N}{(1 - t^2)(1 - t^N)} = P(\mathbb{C}[\alpha, \beta] \oplus \mathbb{C}[\alpha, \beta]\gamma, t). \end{aligned}$$

We check that $\mathbb{C}[\alpha, \beta] \subset \mathbb{C}[X, Y]^{\mathbb{D}_N}$ and $\mathbb{C}[\alpha, \beta]\gamma \subset \mathbb{C}[X, Y]_{\mathbb{D}_N}^{\chi_1}$ by letting the generators r and s act on an arbitrary element. Equality now follows from Corollary 1.

For the final two spaces of covariants we use the fact that we already found the invariant forms. By Corollary 1

$$\begin{aligned} P(\mathbb{C}[X, Y]_{\mathbb{D}_N}^{\chi_3}, t) &= P(\mathbb{C}[X, Y]_{\mathbb{D}_N}^{\chi_4}, t) = P(\mathbb{C}[X, Y]^{\mathbb{D}_{N/2}}, t) - P(\mathbb{C}[X, Y]^{\mathbb{D}_N}, t) = \\ &= \frac{1}{(1-t^2)(1-t^{N/2})} - \frac{1}{(1-t^2)(1-t^N)} = \frac{t^{N/2}}{(1-t^2)(1-t^N)}. \end{aligned}$$

Checking that the proposed spaces have exactly this Poincaré series, and consist of the correct type of covariants finishes the proof. \square

For $a \in \mathbb{Z}$, define

$$[a] := (a + N\mathbb{Z}) \cap \{0, \dots, N-1\}. \quad (3)$$

In the rest of the paper we will use the notation $X^{[h]}$; note that the exponent does not denote the equivalence class $h + N\mathbb{Z}$ but rather its representative in $\{0, 1, \dots, N-1\}$, according to (3). Note also that the condition $\gcd(j, N) = 1$ implies that ψ_{hj} can be any of the irreducible characters when h varies, and visa versa.

5 Covariant vectors

Theorem 2 (Covariant vectors). *Suppose the characters of σ and τ are ψ_j and ψ_{hj} respectively, where $j, h \in \mathbb{Z}$, $\gcd(j, N) = 1$ and $h \notin N/2\mathbb{Z}$. That is, σ is irreducible and faithful and τ is irreducible. With the standard choice of matrices given in (2), the spaces of covariant vectors are given by*

$$(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N} = \left\langle \begin{pmatrix} X^{[h]} \\ Y^{[h]} \end{pmatrix}, \begin{pmatrix} Y^{[N-h]} \\ X^{[N-h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta],$$

$$(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_2} = \left\langle \begin{pmatrix} X^{[h]} \\ -Y^{[h]} \end{pmatrix}, \begin{pmatrix} Y^{[N-h]} \\ -X^{[N-h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta].$$

If N is even then,

$$(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_3} = \left\langle \begin{pmatrix} Y^{[N/2-h]} \\ X^{[N/2-h]} \end{pmatrix}, \begin{pmatrix} X^{[N/2+h]} \\ Y^{[N/2+h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta],$$

$$(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_4} = \left\langle \begin{pmatrix} Y^{[N/2-h]} \\ -X^{[N/2-h]} \end{pmatrix}, \begin{pmatrix} X^{[N/2+h]} \\ -Y^{[N/2+h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta].$$

Moreover, the vectors between the brackets are independent over the ring $\mathbb{C}[\alpha, \beta]$.

Proof. By Corollary 1,

$$(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{Z}_N} = (\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N} \oplus (\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi^2},$$

where \mathbb{Z}_N is represented as the rotations $\langle r \rangle \subset \mathbb{D}_N$. The cyclic group is easy enough to find the invariant vectors directly. Applying the projections $\frac{1}{2}(1 + s)$ and $\frac{1}{2}(1 - s)$ on these vectors gives the desired dihedral invariant and covariant vectors.

The space of invariant vectors is a module over the ring of invariant forms. When searching for \mathbb{Z}_N -invariant vectors, one can therefore look for invariants modulo powers of the \mathbb{Z}_N -invariant forms XY , X^N and Y^N . Moreover, we represent \mathbb{Z}_N by diagonal matrices (which is possible because \mathbb{Z}_N is abelian). Hence, if \mathbb{Z}_N acts on $V = \langle e_1, e_2 \rangle$ then $\mathbb{Z}_N e_i \subset \langle e_i \rangle$. Therefore, one only needs to investigate the vectors $X^d e_i$ and $Y^d e_i$ for $d \in \{0, \dots, N-1\}$ and $i \in \{1, 2\}$.

Recall that $\sigma_r = \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{N-j} \end{pmatrix}$ and $\tau_r = \begin{pmatrix} \omega^{hj} & 0 \\ 0 & \omega^{N-hj} \end{pmatrix}$, so that

$$rX^d e_1 = \omega^{hj} \omega^{-dj} X^d e_1.$$

We want to solve $\omega^{hj} \omega^{-dj} = 1$, i.e. $dj - hj = (d - h)j \in N\mathbb{Z}$. Because $\gcd(j, N) = 1$ this is equivalent to $d - h \in N\mathbb{Z}$. Hence $d \in (h + N\mathbb{Z}) \cap \{0, \dots, N-1\} = [h]$. That is, $X^{[h]} e_1$ is invariant under the action of $\langle r \rangle \cong \mathbb{Z}_N$.

Let us now consider the next one, $rY^d e_1 = \omega^{hj} \omega^{dj} Y^d e_1$. We solve $dj + hj = (d + h)j \in N\mathbb{Z}$ i.e. $d + h \in N\mathbb{Z}$. This implies $d \in (N - h + N\mathbb{Z}) \cap \{0, \dots, N-1\} = [N - h]$, thus $rY^{[N-h]} e_1$ is invariant.

Similarly one finds the invariant vectors $Y^{[h]} e_2$ and $X^{[N-h]} e_2$, resulting in the space

$$\begin{aligned} (\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{Z}_N} &= \left\langle \begin{pmatrix} X^{[h]} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Y^{[h]} \end{pmatrix}, \begin{pmatrix} Y^{[N-h]} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X^{[N-h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[X, Y]^{\mathbb{Z}_N} = \\ &= \left\langle \begin{pmatrix} X^{[h]} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Y^{[h]} \end{pmatrix}, \begin{pmatrix} Y^{[N-h]} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X^{[N-h]} \end{pmatrix}, \gamma \begin{pmatrix} X^{[h]} \\ 0 \end{pmatrix}, \gamma \begin{pmatrix} 0 \\ Y^{[h]} \end{pmatrix}, \gamma \begin{pmatrix} Y^{[N-h]} \\ 0 \end{pmatrix}, \gamma \begin{pmatrix} 0 \\ X^{[N-h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta] \end{aligned}$$

where we have used the fact $\mathbb{C}[X, Y]^{\mathbb{Z}_N} = \mathbb{C}[\alpha, \beta] \oplus \mathbb{C}[\alpha, \beta]\gamma$, which follows from Lemma 2 and Corollary 1, to obtain the last expression.

It turns out that there are some relations between the vectors. First note that $[h] + [N - h] \in N\mathbb{Z} \cap \{0, \dots, 2N - 2\} = \{0, N\}$. By the assumption $h \notin N\mathbb{Z}$ we have $[h] + [N - h] = N$. Now one finds that

$$\gamma \begin{pmatrix} X^{[h]} \\ 0 \end{pmatrix} = f(\alpha, \beta) \begin{pmatrix} X^{[h]} \\ 0 \end{pmatrix} + g(\alpha, \beta) \begin{pmatrix} Y^{[N-h]} \\ 0 \end{pmatrix}$$

if $f(\alpha, \beta) = \beta$ and $g(\alpha, \beta) = -\alpha^{[h]}$, hence this vector is redundant. Similarly, we find $\gamma Y^{[N-h]} = \alpha^{[N-h]} X^{[h]} - \beta Y^{[N-h]}$ and see that the vector $\gamma \begin{pmatrix} Y^{[N-h]} \\ 0 \end{pmatrix}$ is redundant as well. Investigation of the second component of the vectors gives the same result. We conclude that

$$(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{Z}_N} = \left\langle \begin{pmatrix} X^{[h]} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Y^{[h]} \end{pmatrix}, \begin{pmatrix} Y^{[N-h]} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X^{[N-h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta],$$

and the remaining vectors between the brackets are independent over the ring $\mathbb{C}[\alpha, \beta]$. Indeed, let $f, g \in \mathbb{C}[\alpha, \beta]$ and consider the equation

$$fX^{[h]} + gY^{[N-h]} = 0.$$

If the equation is multiplied by $Y^{[h]}$ we find

$$f\alpha^{[h]} + gY^N = f\alpha^{[h]} + g(\beta - \gamma) = 0.$$

Now one can use the fact that the sum $\mathbb{C}[\alpha, \beta] \oplus \mathbb{C}[\alpha, \beta]\gamma$ is direct to see that $g = 0$, and hence $f = 0$. Similarly $fY^{[h]} + gX^{[N-h]} = 0$ implies $f = g = 0$. Thus the Poincaré series takes the form

$$P((\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{Z}_N}, t) = \frac{2t^{[h]} + 2t^{[N-h]}}{(1-t^2)(1-t^N)}.$$

To obtain \mathbb{D}_N -invariants we apply the projection $\frac{1}{2}(1+s)$ from Corollary 1. Observe that the \mathbb{D}_N -invariant polynomials move through this operator so that one only needs to compute

$$\frac{1}{2}(1+s) \begin{pmatrix} X^{[h]} \\ 0 \end{pmatrix} = \frac{1}{2}(1+s) \begin{pmatrix} 0 \\ Y^{[h]} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} X^{[h]} \\ Y^{[h]} \end{pmatrix}$$

and

$$\frac{1}{2}(1+s) \begin{pmatrix} Y^{[N-h]} \\ 0 \end{pmatrix} = \frac{1}{2}(1+s) \begin{pmatrix} 0 \\ X^{[N-h]} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} Y^{[N-h]} \\ X^{[N-h]} \end{pmatrix},$$

concluding that

$$(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N} = \left\langle \begin{pmatrix} X^{[h]} \\ Y^{[h]} \end{pmatrix}, \begin{pmatrix} Y^{[N-h]} \\ X^{[N-h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta]. \quad (4)$$

If one uses the other projection, $\frac{1}{2}(1-s)$, one finds

$$(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_2} = \left\langle \begin{pmatrix} X^{[h]} \\ -Y^{[h]} \end{pmatrix}, \begin{pmatrix} Y^{[N-h]} \\ -X^{[N-h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta]. \quad (5)$$

Since $(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N} \oplus (\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_2} = (\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{Z}_N}$ by Corollary 1, the Poincaré series of $(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N}$ and $(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_2}$ add up to the one of $(\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{Z}_N}$. Hence, the projected vectors are independent over $\mathbb{C}[\alpha, \beta]$ as well.

Now that the \mathbb{D}_N -invariant vectors are found, one can compute the Poincaré series of the χ_3 - and χ_4 -covariants, using Corollary 1. First we extend the notation in (3) to

$$[a]_M := (a + M\mathbb{Z}) \cap \{0, 1, \dots, M-1\}$$

so that

$$\begin{aligned} & P\left((\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_3}, t\right) = P\left((\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_4}, t\right) = \\ &= P\left((\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{D}_{N/2}}, t\right) - P\left((\mathbb{C}^2 \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N}, t\right) = \\ &= \frac{t^{[h]_{N/2}} + t^{[N/2-h]_{N/2}}}{(1-t^2)(1-t^{N/2})} - \frac{t^{[h]_N} + t^{[N-h]_N}}{(1-t^2)(1-t^N)} = \frac{(t^{[h]_{N/2}} + t^{[N/2-h]_{N/2}})(1+t^{N/2}) - t^{[h]_N} - t^{[N-h]_N}}{(1-t^2)(1-t^N)} = \\ &= \frac{t^{[h]_{N/2}} + t^{[N/2-h]_{N/2}} + t^{N/2+[h]_{N/2}} + t^{N/2+[N/2-h]_{N/2}} - t^{[h]_N} - t^{[N-h]_N}}{(1-t^2)(1-t^N)}. \end{aligned}$$

The above numerator is simplified by the equalities

$$t^{[h]_{N/2}} + t^{N/2+[h]_{N/2}} - t^{[h]_N} = t^{[N/2+h]_N} \quad (6)$$

$$t^{[N/2-h]_{N/2}} + t^{N/2+[N/2-h]_{N/2}} - t^{[N-h]_N} = t^{[N/2-h]_N} \quad (7)$$

We prove equation 6 by showing that

$$\{[h]_{N/2}, N/2 + [h]_{N/2}\} = \{[h]_N, [N/2 + h]_N\}.$$

Indeed, for some integer l

$$[h]_{N/2} = h + lN/2 \in \begin{cases} (h + N\mathbb{Z}) \cap \{0, \dots, N-1\} = [h]_N & \text{if } l \text{ is even} \\ (N/2 + h + N\mathbb{Z}) \cap \{0, \dots, N-1\} = [N/2 + h]_N & \text{if } l \text{ is odd} \end{cases}$$

and, with the same l ,

$$N/2 + [h]_{N/2} = N/2 + h + lN/2 \in \begin{cases} (N/2 + h + N\mathbb{Z}) \cap \{0, \dots, N-1\} = [N/2 + h]_N & \text{if } l \text{ is even} \\ (h + N\mathbb{Z}) \cap \{0, \dots, N-1\} = [h]_N & \text{if } l \text{ is odd} \end{cases}.$$

Equation 7 follows in the same fashion. The Poincaré series then reads

$$P\left((\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_3}, t\right) = P\left((\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_4}, t\right) = \frac{t^{[N/2-h]_N} + t^{[N/2+h]_N}}{(1-t^2)(1-t^N)}.$$

One can show that

$$\left\langle \begin{pmatrix} Y^{[N/2-h]} \\ X^{[N/2-h]} \end{pmatrix}, \begin{pmatrix} X^{[N/2+h]} \\ Y^{[N/2+h]} \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta] \subset (\mathbb{C}^2 \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_3}$$

by applying r and s to a general element in the left hand side and observing that r acts as multiplication by $\chi_3(r) = -1$ and s as multiplication by $\chi_3(s) = 1$. Likewise for χ_4 .

Now let us show that the two vectors between the brackets are independent. Let $f, g \in \mathbb{C}[\alpha, \beta]$ and assume

$$fY^{[N/2-h]} + gX^{[N/2+h]} = 0,$$

$[N/2-h] + [N/2+h] \in N\mathbb{Z} \cap \{0, \dots, 2N-2\} = \{0, N\}$. Since $h \notin N/2\mathbb{Z}$ one finds $[N/2-h] + [N/2+h] = N$. Therefore, multiplication of the equation by $X^{[N/2-h]}$ yields

$$0 = f\alpha^{[N/2-h]} + gX^N = f\alpha^{[N/2-h]} + g(\beta + \gamma) = (f\alpha^{[N/2-h]} + g\beta) \oplus g\gamma \in \mathbb{C}[\alpha, \beta] \oplus \mathbb{C}[\alpha, \beta]\gamma.$$

Since the sum on the right hand side is direct we find $g = 0$, and then also $f = 0$. The independence of the χ_4 -covariant vectors over the ring of invariants is established by the same calculation.

Now that we have shown that the suggested spaces are subsets of the actual covariant spaces and that they moreover have the same Poincaré series, we have shown equality. \square

Remark. If one allows σ to be non-faithful, i.e. $\gcd(j, N) > 1$, several more cases appear. However, they are not more interesting than what we have seen so far, which is why we decided not to include this in the theorem. In words, it is as follows.

If $\tau(\mathbb{D}_N) \subset \sigma(\mathbb{D}_N)$ everything is the same as above except that N will be replaced by $N' = \frac{N}{\gcd(N, j)}$. If on the other hand $\tau(\mathbb{D}_N) \not\subset \sigma(\mathbb{D}_N)$, then the χ_1 - and χ_2 -covariant vector spaces will be zero, while the χ_3 - and χ_4 -covariant vector spaces will be the χ_1 - and χ_2 -covariant vector spaces for the subgroup $\langle r^2, s \rangle \cong \mathbb{D}_{N/2}$. We conclude that *all* covariant vectors are of the form given in the theorem.

6 Automorphic Lie Algebras with dihedral symmetry

6.1 A subalgebra present in any Automorphic Lie Algebra

If the invariant vectors are known, one can use these to construct a subalgebra of the Automorphic Lie Algebra, which has brackets relations that are very similar to the base algebra \mathfrak{g} , in this case $\mathfrak{sl}_2(\mathbb{C})$. The idea is basically to conjugate the basis of $\mathfrak{sl}_2(\mathbb{C})$ with a matrix T consisting of invariant vectors. Indeed, for such T we have $\tau_g T(\sigma_{g^{-1}}(X, Y)) = T(X, Y)$ for all $g \in G$. Hence, for any constant matrix M , we have $g(TMT^{-1}) = TMT^{-1}$.

This conjugation introduces poles at the zeros of $\det(T)$ which is undesirable at this point. However, one can remove these singularities thanks to the following

Proposition 1. *If T is a square matrix whose columns are equivariant vectors then*

$$\det(T) \in \mathbb{C}[X, Y]_G^\chi$$

where $\chi(g) = \frac{1}{\det(\tau_g)}$.

Proof. Let $g \in G$. Then

$$\det(T(X, Y)) = \det(\tau_g T(\sigma_{g^{-1}}(X, Y))) = \det(\tau_g) \det(T(\sigma_{g^{-1}}(X, Y))) = \det(\tau_g) g \det(T(X, Y)).$$

This shows that g acts on $\det(T)$ as multiplication by $\chi(g)$. Moreover, by definition of a group action, $\chi : G \rightarrow \mathbb{C}^*$ must be homomorphic, hence a 1-dimensional representation. \square

Because $\det(T)$ is a χ -covariant form, we may multiply the invariant matrix TMT^{-1} by it and obtain a χ -covariant matrix without any poles. If one prefers an *invariant* matrix that has no poles, one can, in addition, multiply by a χ^* -covariant form.

Let us apply these ideas to the dihedral case. Define

$$T := \begin{pmatrix} X^{[h]} & Y^{[N-h]} \\ Y^{[h]} & X^{[N-h]} \end{pmatrix}$$

and observe that $\det(T) = \gamma$. Now we apply $\gamma^2 \text{Ad}(T)$ to the $\mathfrak{sl}_2(\mathbb{C})$ basis

$$e_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and obtain

$$\begin{aligned}\mathbf{e}_0 &= \gamma^2 \text{Ad}(T)e_0 = \gamma \begin{pmatrix} \beta & -2X^{[h]}Y^{[N-h]} \\ 2X^{[N-h]}Y^{[h]} & -\beta \end{pmatrix}, \\ \mathbf{e}_+ &= \gamma^2 \text{Ad}(T)e_+ = \gamma \begin{pmatrix} -\alpha^{[h]} & X^{2[h]} \\ -Y^{2[h]} & \alpha^{[h]} \end{pmatrix}, \\ \mathbf{e}_- &= \gamma^2 \text{Ad}(T)e_- = \gamma \begin{pmatrix} \alpha^{[N-h]} & -Y^{2[N-h]} \\ X^{2[N-h]} & -\alpha^{[N-h]} \end{pmatrix}.\end{aligned}$$

By the invariance argument given at the beginning of this Section, we have the inclusion

$$\mathbb{C}\langle \mathbf{e}_0, \mathbf{e}_+, \mathbf{e}_- \rangle \otimes \mathbb{C}[\alpha, \beta] \subset (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[X, Y])^{\mathbb{D}^N}$$

but the selling point of this method is the following: $\text{Ad}(T)$ preserves the bracket, and forms move through the bracket, hence one immediately finds that it is in fact a subalgebra with the “ $\mathfrak{sl}_2(\mathbb{C})$ -like” brackets given by

$$\begin{aligned}[\mathbf{e}_0, \mathbf{e}_\pm] &= \pm 2\gamma^2 \mathbf{e}_\pm \\ [\mathbf{e}_+, \mathbf{e}_-] &= \gamma^2 \mathbf{e}_0\end{aligned}$$

Recall that $\gamma^2 = \beta^2 - \alpha^N$. Moreover, this can be done for any finite group G .

6.1.1 Degree zero

In this section we will multiply the algebra elements by quotients of invariant forms in order to get matrices of degree zero, making them functions of $\lambda := \frac{X}{Y}$. This introduces poles, the set of which forms a collection of orbits of the group, by invariance. One can choose to restrict the poles to any such collection and we choose a single degenerate orbit: the zeros of α . The introduction of the homogeneous variable λ is motivated by applications; for instance, this would be in fact the standard set up in the context of integrable systems (see e.g. [6]).

Let $\bar{M} = \frac{\beta^n}{\alpha^m} M$ where n and m are the smallest nonnegative integers solving

$$n \deg(\beta) + \deg(M) = m \deg(\alpha).$$

The requirement that n and m be as small as possible introduces a distinction between the cases N odd and N even. Define the automorphic functions $\mathbb{I} := \frac{\beta^2}{\alpha^N}$ and $\mathbb{J} := \frac{\gamma^2}{\alpha^N} = \mathbb{I} - 1$.

CASE 1: N **odd**. One finds

$$\bar{\mathbf{e}}_0 = \frac{1}{\alpha^N} \mathbf{e}_0, \quad \bar{\mathbf{e}}_+ = \frac{\beta}{\alpha^{N+[h]}} \mathbf{e}_+, \quad \bar{\mathbf{e}}_- = \frac{\beta}{\alpha^{N+[N-h]}} \mathbf{e}_-,$$

resulting in the brackets

$$[\bar{\mathbf{e}}_0, \bar{\mathbf{e}}_{\pm}] = \pm 2\mathbb{J}\bar{\mathbf{e}}_{\pm}$$

$$[\bar{\mathbf{e}}_+, \bar{\mathbf{e}}_-] = \mathbb{J}\bar{\mathbf{e}}_0$$

CASE 2: N **even**. One finds

$$\bar{\mathbf{e}}_0 = \frac{1}{\alpha^N} \mathbf{e}_0, \quad \bar{\mathbf{e}}_+ = \frac{1}{\alpha^{N/2+[h]}} \mathbf{e}_+, \quad \bar{\mathbf{e}}_- = \frac{1}{\alpha^{N/2+[N-h]}} \mathbf{e}_-,$$

resulting in the brackets

$$[\bar{\mathbf{e}}_0, \bar{\mathbf{e}}_{\pm}] = \pm 2\mathbb{J}\bar{\mathbf{e}}_{\pm}$$

$$[\bar{\mathbf{e}}_+, \bar{\mathbf{e}}_-] = \mathbb{J}\bar{\mathbf{e}}_0$$

6.2 The full algebra

Let χ_V denote the character of a G -module V . The identities

$$\chi_{V \oplus W} = \chi_V + \chi_W \tag{8}$$

$$\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g) \tag{9}$$

are handy when proving

$$\chi_{\mathfrak{gl}_k(\mathbb{C})}(g) = \chi_{\tau}(g)^* \chi_{\tau}(g)$$

$$\chi_{\mathfrak{sl}_k(\mathbb{C})}(g) = \chi_{\tau}(g)^* \chi_{\tau}(g) - 1$$

where χ_{τ} is the character of τ . The first equality follows from the isomorphism $\mathfrak{gl}_k(\mathbb{C}) \cong \mathbb{C}^k \otimes (\mathbb{C}^k)^*$ of G -modules, where $\text{Ad}(M)$ corresponds to left multiplication by M on the first factor and right multiplication by M^{-1} on the second. We note that $\text{Trace}(M^{-1}) = \sum \mu^{-1} = \sum \mu^* = (\sum \mu)^* = \text{Trace}(M)^*$ and apply (9) to find the expression for $\chi_{\mathfrak{gl}_k(\mathbb{C})}$.

The second equation follows from the observation that $\mathfrak{gl}_k(\mathbb{C}) = \mathbb{C}\text{Id} \oplus \mathfrak{sl}_k(\mathbb{C})$ and both terms are G -submodules. Indeed, $\text{Ad}(M)$ preserves trace and acts trivially on $\mathbb{C}\text{Id}$ (that is, $\chi_{\mathbb{C}\text{Id}}(g) = 1$). The statement now follows from (8).

Any character has a unique decomposition into irreducible characters. For instance, if $G = \mathbb{D}_N$ and τ has character ψ_{hj} , one finds

$$\chi_{\mathfrak{sl}_2(\mathbb{C})}(g) = \psi_{hj}(g)^* \psi_{hj}(g) - 1 = \psi_{2hj}(g) + \chi_2(g), \quad \forall g \in G. \quad (10)$$

Theorem 3 (Invariant matrices). *Suppose the characters of σ and τ are ψ_j and ψ_{hj} respectively, where $j, h \in \mathbb{Z}$, $\gcd(j, N) = 1$ and $2h \notin N/2\mathbb{Z}$. With the standard choice of matrices given in (2), the spaces of covariant matrices are given by*

$$\begin{aligned} (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\mathbb{D}_N} &= \left\langle \begin{pmatrix} 0 & X^{[2h]} \\ Y^{[2h]} & 0 \end{pmatrix}, \begin{pmatrix} 0 & Y^{[N-2h]} \\ X^{[N-2h]} & 0 \end{pmatrix}, \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta], \\ (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_2} &= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & X^{[2h]} \\ -Y^{[2h]} & 0 \end{pmatrix}, \begin{pmatrix} 0 & Y^{[N-2h]} \\ -X^{[N-2h]} & 0 \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta], \end{aligned}$$

and, if N is even,

$$\begin{aligned} (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_3} &= \left\langle \begin{pmatrix} 0 & Y^{[N/2-2h]} \\ X^{[N/2-2h]} & 0 \end{pmatrix}, \begin{pmatrix} \gamma_{N/2} & 0 \\ 0 & -\gamma_{N/2} \end{pmatrix}, \begin{pmatrix} 0 & X^{[N/2+2h]} \\ Y^{[N/2+2h]} & 0 \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta], \\ (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_4} &= \left\langle \begin{pmatrix} 0 & Y^{[N/2-2h]} \\ -X^{[N/2-2h]} & 0 \end{pmatrix}, \begin{pmatrix} \beta_{N/2} & 0 \\ 0 & -\beta_{N/2} \end{pmatrix}, \begin{pmatrix} 0 & X^{[N/2+2h]} \\ -Y^{[N/2+2h]} & 0 \end{pmatrix} \right\rangle \otimes \mathbb{C}[\alpha, \beta]. \end{aligned}$$

Moreover, the matrices between the brackets are independent over the ring $\mathbb{C}[\alpha, \beta]$.

Proof. We will use the previous results to find the Poincaré series of the spaces of covariants. Then we will show that the spaces given on the right hand sides in the theorem consists of the suggested type of covariants, and that they have the correct Poincaré series, which then proves the theorem.

By Molien's Theorem: Theorem 1, the lemma on covariant forms: Lemma 2, the theorem on covariant vectors: Theorem 2 and the expression for the character of $\mathfrak{sl}_2(\mathbb{C})$: equation (10),

it follows that

$$\begin{aligned}
P\left((\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N}, t\right) &= \frac{1}{|G|} \sum_{g \in G} \frac{\chi_{\mathfrak{sl}_2(\mathbb{C})}(g)}{\det(1 - \sigma_g^{-1}t)} \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{\psi_{2hj}(g) + \chi_2(g)}{\det(1 - \sigma_g^{-1}t)} \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{\psi_{2hj}(g)}{\det(1 - \sigma_g^{-1}t)} + \frac{1}{|G|} \sum_{g \in G} \frac{\chi_2(g)}{\det(1 - \sigma_g^{-1}t)} \\
&= P\left((V^{\psi_{2hj}} \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N}, t\right) + P(\mathbb{C}[X, Y]_{\mathbb{D}_N}^{\chi_2}, t) \\
&= \frac{t^{[2h]} + t^{[N-2h]}}{(1-t^2)(1-t^N)} + \frac{t^N}{(1-t^2)(1-t^N)} \\
&= \frac{t^{[2h]} + t^{[N-2h]} + t^N}{(1-t^2)(1-t^N)}.
\end{aligned}$$

Before calculating the Poincaré series of the χ_2 -covariant matrices, note that $\chi_2^*(g)\chi_2(g) = 1$ and $\chi_2^*(g)\psi_{2hj}(g) = \psi_{2hj}(g)$. It follows then that

$$P\left((\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_2}, t\right) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi_2^*(g)\chi_{\mathfrak{sl}_2(\mathbb{C})}(g)}{\det(1 - \sigma_g^{-1}t)} = \frac{1 + t^{[2h]} + t^{[N-2h]}}{(1-t^2)(1-t^N)}.$$

For the next type of covariants, we check that $\chi_3^*(g)\chi_2(g) = \chi_4(g)$. It takes a bit more effort to see that, if N is even, $\chi_3^*(g)\psi_{2hj}(g) = \psi_{(2h+N/2)j}(g)$, so let us do the calculation in detail. An element $g \in \mathbb{D}_N$ is either of the form r^m or sr^m . In the latter case the claim holds, since $\psi_j(sr^m) = 0$ for all j . Before plugging in $g = r^m$ note that j is odd since it is coprime to the even number N . One has $\chi_3^*(r^m)\psi_{2hj}(r^m) = (-1)^m \omega^{m(2hj)} + (-1)^m \omega^{N-m(2hj)} = (\omega^{jN/2})^m \omega^{m(2hj)} + (\omega^{-jN/2})^m \omega^{N-m(2hj)} = \omega^{mj(N/2+2h)} + \omega^{N-mj(N/2+2h)} = \psi_{(N/2+2h)j}(r^m)$. This is enough preparation to find the remaining Poincaré series:

$$P\left((\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[X, Y])_{\mathbb{D}_N}^{\chi_3}, t\right) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi_3^*(g)\chi_{\mathfrak{sl}_2(\mathbb{C})}(g)}{\det(1 - \sigma_g^{-1}t)} = \frac{t^{[N/2-2h]} + t^{N/2} + t^{[N/2+2h]}}{(1-t^2)(1-t^N)}.$$

By Corollary 1, the space of χ_4 -covariants has the same Poincaré series.

The proof is completed if one checks that the proposed spaces consist of the correct type of covariants, and that they have the above calculated Poincaré series. The first check is straight forward and left to the reader. The latter requires a proof of independence. However, the equations are the same as in Theorem 2 on covariant vectors, only h is replaced by $2h$. Here we require $2h \notin N/2\mathbb{Z}$. Thus independence follows from Theorem 2. \square

Remark. If one allows σ to be non-faithful, the only covariant spaces that occur which

are not of the form shown above, are zero- or one-dimensional over the ring of invariant forms. Therefore they do not generate interesting Lie algebras.

6.2.1 Lie algebra structures

For convenience, let

$$A := \begin{pmatrix} 0 & X^{[2h]} \\ Y^{[2h]} & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & Y^{[N-2h]} \\ X^{[N-2h]} & 0 \end{pmatrix}, \quad C := \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix},$$

so that $(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[X, Y])^{\mathbb{D}^N} = \langle A, B, C \rangle \otimes \mathbb{C}[\alpha, \beta]$. The brackets are

$$\begin{aligned} [A, B] &= 2C, \\ [B, C] &= -2\alpha^{[N-2h]}A + 2\beta B, \\ [C, A] &= 2\beta A - 2\alpha^{[2h]}B. \end{aligned}$$

6.2.2 Degree zero

We follow the same argument as in section 6.1.1 to find homogeneous matrices and find there are three different cases. Let

$$\deg A = [2h], \quad \deg B = [N - 2h], \quad \deg C = N.$$

CASE 1: N **odd**, $[2h]$ **even**. Then $[N - 2h] = N - [2h]$ is odd.

$$\bar{A} = \frac{1}{\alpha^{[2h]/2}}A, \quad \bar{B} = \frac{\beta}{\alpha^{1/2(N+[N-2h])}}B, \quad \bar{C} = \frac{\beta}{\alpha^N}C.$$

Recall the automorphic function $\mathbb{I} := \frac{\beta^2}{\alpha^N}$; the brackets read then

$$\begin{aligned} [\bar{A}, \bar{B}] &= 2\bar{C}, \\ [\bar{B}, \bar{C}] &= -2\mathbb{I}\bar{A} + 2\mathbb{I}\bar{B}, \\ [\bar{C}, \bar{A}] &= 2\mathbb{I}\bar{A} - 2\bar{B}. \end{aligned}$$

CASE 2: N **odd**, $[2h]$ **odd**. Then $[N - 2h] = N - [2h]$ is even.

$$\bar{A} = \frac{\beta}{\alpha^{1/2(N+[2h])}}A, \quad \bar{B} = \frac{1}{\alpha^{1/2[N-2h]}}B, \quad \bar{C} = \frac{\beta}{\alpha^N}C.$$

In this case the brackets are

$$\begin{aligned} [\bar{A}, \bar{B}] &= 2\bar{C}, \\ [\bar{B}, \bar{C}] &= -2\bar{A} + 2\mathbb{I}\bar{B}, \\ [\bar{C}, \bar{A}] &= 2\mathbb{I}\bar{A} - 2\mathbb{I}\bar{B}. \end{aligned}$$

CASE 3: N **even**. $[2h] = 2h + lN$ for some integer l . Therefore $[2h]/2 \in \mathbb{N}$. Also $\frac{[N-2h]}{2} \in \mathbb{N}$.

$$\bar{A} = \frac{1}{\alpha^{[2h]/2}} A, \quad \bar{B} = \frac{1}{\alpha^{1/2(N-2h)}} B, \quad \bar{C} = \frac{1}{\alpha^{N/2}} C.$$

Define now the automorphic function as $I := \frac{\beta}{\alpha^{N/2}}$; the brackets then read

$$\begin{aligned} [\bar{A}, \bar{B}] &= 2\bar{C}, \\ [\bar{B}, \bar{C}] &= -2\bar{A} + 2I\bar{B}, \\ [\bar{C}, \bar{A}] &= 2I\bar{A} - 2\bar{B}. \end{aligned}$$

Notice that the structure constants do not depend on h . That is, they are independent of the chosen representation. This extends the uniformity result of Lombardo and Sanders [7] and Bury [1].

In the next Section we present the same result but we diagonalise the algebras, so to obtain a *normal form*.

6.2.3 Diagonal form

CASE 1: N **odd**, $[2h]$ **even**.

The matrix corresponding to $\text{ad}(\bar{A})$ acting on the $\mathbb{C}[\mathbb{I}]$ -module with basis $\{\bar{A}, \bar{B}, \bar{C}\}$ reads

$$\text{ad}(\bar{A}) \cong \begin{pmatrix} 0 & 0 & -2\mathbb{I} \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

This matrix is diagonalized to

$$\text{ad}(\bar{A}) \cong \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

when the basis is changed to

$$\bar{e}_0 := \bar{A}, \quad \bar{e}_+ := -\mathbb{I}/2\bar{A} + 1/2\bar{B} + 1/2\bar{C}, \quad \bar{e}_- := 1/2\bar{A} - 1/2\bar{B} + 1/2\bar{C}.$$

In this basis, the brackets are

$$[\bar{e}_0, \bar{e}_\pm] = \pm 2\bar{e}_\pm$$

$$[\bar{e}_+, \bar{e}_-] = \mathbb{I}\bar{e}_0$$

This transformation is invertible, for we have $\bar{A} = \bar{e}_0$, $\bar{B} = \bar{e}_+ - \bar{e}_- + \mathbb{I}\bar{e}_0$ and $\bar{C} = \bar{e}_+ + \bar{e}_-$.

CASE 2: N odd, $[2h]$ odd.

In this case $\text{ad}(\bar{B})$ has eigenvalues 0 and ± 2 . Diagonalization with respect to this operator gives a basis

$$\bar{e}_0 := \bar{B} \quad \bar{e}_+ := -1/2\bar{A} + 1/2\bar{B} + 1/2\bar{C} \quad \bar{e}_- := 1/2\bar{A} - 1/2\bar{B} + 1/2\bar{C}$$

and the same brackets:

$$[\bar{e}_0, \bar{e}_\pm] = \pm 2\bar{e}_\pm$$

$$[\bar{e}_+, \bar{e}_-] = \mathbb{I}\bar{e}_0$$

This basis transformation is invertible as well.

CASE 3: N even. The matrix for $\text{ad}(\bar{A})$ is exactly the same as for CASE 1, apart from \mathbb{I} being replaced by I . Hence one finds the same basis that diagonalizes it and calculates the brackets

$$[\bar{e}_0, \bar{e}_\pm] = \pm 2\bar{e}_\pm$$

$$[\bar{e}_+, \bar{e}_-] = \mathbb{J}\bar{e}_0$$

The following result holds

Theorem 4. *Let \mathbb{D}_N be the dihedral group and let its action on the space of 2×2 matrices and on the complex plane be defined by simultaneously acting on the matrices and on the complex plane with any two irreducible representations of \mathbb{D}_N . Then, the Automorphic Lie Algebras $(\mathfrak{sl}_2 \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N}$ are isomorphic as Lie algebras and the Lie algebra structure does not depend on the initially chosen representations.*

Note that here *isomorphic* means that when we identify the modular invariants (the automorphic functions) the structure constants of the Lie algebra in normal form are the same.

Furthermore, the linear transformation that transforms the Lie algebra into normal form should be invertible.

7 Dihedral rational maps and minimal energy Skyrmions

It is well known that rational maps can be used to understand the structure of solutions of the Skyrme model [3], known as Skyrmions (see, for example, [8] and references therein). In the *Rational Maps Ansatz* [8], a Skyrme field $U(r, z)$ with baryon number B is constructed from a degree B rational map between Riemann spheres. Briefly, let us consider a point \mathbf{x} in \mathbb{R}^3 by a pair (r, z) where r is the radial distance from the origin, and z is a Riemann sphere coordinate giving the point on the unit two sphere which intersects the half line through the origin and the point \mathbf{x} : that is $z = \tan(\theta/2)e^{i\phi}$, using spherical coordinates. Let now $R(z)$ be a degree B rational map between Riemann spheres

$$R(z) = \frac{p(z)}{q(z)}$$

where p and q are polynomials in z such that $\max[\deg(p), \deg(q)] = B$ and p and q have no common factors. In terms of $R(z)$ the Ansatz for the Skyrme field reads

$$U(r, z) = \exp \left[\frac{if(r)}{1 + |R|^2} \begin{pmatrix} 1 - |R|^2 & 2\bar{R} \\ 2R & |R|^2 - 1 \end{pmatrix} \right] \quad (11)$$

where \bar{R} stands for the complex conjugate of R and where $f(r)$ is a real profile function satisfying $f(0) = \pi$ while the boundary value $U = 1$ at ∞ requires that $f(\infty) = 0$. Substituting the Skyrme field (11) into the Skyrme energy functional results in the following expressions for the energy

$$E = \frac{1}{3\pi} \int_0^\infty \left(r^2 (f'(z))^2 + 2B((f'(z))^2 + 1) \sin^2(f(z)) + \frac{\sin^4(f(z))}{r^2} \mathcal{I} \right) dr, \quad (12)$$

where \mathcal{I} denotes the integral

$$\mathcal{I} = \frac{1}{4\pi} \int \int \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2idz d\bar{z}}{(1 + |z|^2)^2}. \quad (13)$$

Thus, according to the Rational Map Ansatz, the problem of minimising the energy (12) reduces to the simpler problem of finding the rational map R which minimises the function \mathcal{I} . Then the profile function $f(z)$ minimising the energy E is found by solving a second order differential equation with B and \mathcal{I} as parameters.

In this paper we consider rational maps with dihedral symmetry and are interested in computing the corresponding \mathcal{I} at a given baryon number B . Rational maps with dihedral symmetry can be identified with an invariant vector in \mathbb{C}^2 and can be constructed by dividing the first component of the vector by the second. Since their degrees are equal we obtain a rational function in $\lambda = X/Y$, where X and Y are coordinates of \mathbb{C}^2 . Taking the invariant vectors of section 5 and identifying λ with z we find

$$R(z) = z^B, \quad B \in \mathbb{Z}.$$

It follows then that

$$\mathcal{I}(B) = \frac{B^4}{4\pi} \int \int \left(\frac{1 + |z|^2}{1 + |z|^{2B}} |z|^{B-1} \right)^4 \frac{2idzd\bar{z}}{(1 + |z|^2)^2} = \frac{|B|^3}{3} \left[1 + \left(1 - \frac{1}{B^2} \right) \frac{\pi/B}{\sin(\pi/B)} \right].$$

We check that $\mathcal{I}(1) = 1$ and $\mathcal{I}(B) = \frac{2}{3}|B|^3(1 + o(1))$ for $B \rightarrow \pm\infty$. Notice also that $\mathcal{I}(-B) = \mathcal{I}(B)$.

More generally, it is worth pointing out that it is possible to construct symmetric rational maps from homogeneous invariant (equivariant) vectors for all Platonic groups, that is, for the icosahedral, octahedral and tetrahedral groups. Thus, this would suggest that one can compute the possible symmetry groups at given Baryon numbers by group theoretical methods. This complements the research in energy minimising solutions which turn out to be symmetric.

8 Conclusions

With the aid of representation theory and classical invariant theory, we have constructed the spaces of dihedral-invariant vectors and matrices. In particular, in the matrix case we find Automorphic Lie Algebras, and we show that the Lie algebra structure does not depend on the initially chosen representations of the dihedral group, thereby extending the uniformity result of Lombardo and Sanders [7], and Bury [1]. In the last Section, we consider dihedral invariant vectors to compute symmetric rational maps related to the energy of Skyrmons with dihedral symmetry. Along with the rise of interest in the subject from an algebraic point of view, this work encourages further studies of possible applications of Automorphic Lie Algebras to high energy physics.

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