On Integrability of Systems of Evolution Equations

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We prove the conjecture, formulated in [BSW98], that almost all systems in the family

$$\mathcal{B}_m[a]: \quad \left\{ \begin{array}{ll} u_t = au_m + v^2 & \quad m \in \mathbb{N}_{\geq 2} \\ v_t = v_m & \quad a \in \mathbb{C} \backslash \{0\} \end{array} \right.$$

have at most finitely many symmetries by using number theory. We list the nine exceptional cases when the systems do have infinitively many symmetries. For such systems, we give the recursive operators to generate their symmetries. We treat both the commutative and the noncommutative (or quantum) case. This is the first example of a class of equations where such a classification has been possible.

1. INTRODUCTION

It was shown how to prove the classification with respect to the existence of infinitely many symmetries in the case of homogeneous scalar evolution equations in [SW98], using the symbolic method in combination with results from diophantine approximation theory (cf [Beu97]). The obvious program now is to generalize this in several directions. One direction is to assume that the field variable u takes its values in an associative, noncommutative algebra. This has been carried out successfully for the scalar case in [OW00]. Another direction is to classify systems of evolution equations. It is well known that even finding one single generalized symmetry

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for general systems is a difficult problem, and one therefore has to restrict ones attention to suitable subcases. In this paper we have completely (except for some singular values, which can be analyzed with slightly different methods) classified a class of systems introduced and studied by Bakirov [Bak91]. Since Bakirov's paper only appeared as a preprint, we refer the reader to [Olv93], where Exercise 5.16 asks the reader to prove that the system

$$\begin{cases} u_t = \frac{1}{2}u_2 + \frac{1}{4}v^2 & u_k = \frac{\partial^k u}{\partial x^k} \\ v_t = v_2 & v_k = \frac{\partial^k v}{\partial x^k}. \end{cases} \tag{1}$$

has a generalized symmetry and to find a recursion operator. The present paper can be considered as a generalization of this exercise, in that it determines the symmetry-integrable equations in the family of evolution systems

$$\mathcal{B}_m[a]: \qquad \begin{cases} u_t = au_m + v^2 & m \in \mathbb{N}_{\geq 2} \\ v_t = v_m & a \in \mathbb{C}\backslash\{0\} \end{cases} \tag{2}$$

We work over $\mathbb{N}_{\geq 2}$ here since, although the $\mathcal{B}_1[a]$ family is clearly integrable, the symmetries do not always start with linear terms. Therefore, finding all its symmetries is a problem that falls somewhat outside the scope of this paper.

Symmetry–integrable (or integrable for short) means that there exist infinitely many nontrivial independent symmetries of an evolution equation. We remark that those equations that are not integrable can theoretically have a positive number of generalized symmetries, cf. [BSW98, vdKS99].

The motivation for this paper is that this family of evolution equations is on the one hand rich enough to contain both equations which have exactly one symmetry ($\mathcal{B}_4[5]$ is the first example of these) and integrable equations ($\mathcal{B}_2[1]$, Diffusion system, [Oev84]; Eq. (1) can be transformed into $\mathcal{B}_2[1/2]$), and on the other hand is simple enough to do the complete classification. This is the first classification result for a family of systems up till infinite order with respect to integrability that we are aware of. The difficulty of the classification of systems comparing to the scalar case lies in the linear part, that is, determining the value of a in (2). For the classification of second order systems we refer to [MSS91].

Here we do not attempt to completely analyze all the possible cases, in that we only make the distinction between integrable and non-integrable. One could go for the following goal: determine for all $m \in \mathbb{N}$ and $a \in \mathbb{C}$ all the symmetries. We do not know whether such a classification is possible with the present techniques, although [BSW98] shows that it is (in principle) possible to do this for any given choice of m and a (see also [vdKS99]).

In this paper, we show that the only symmetry–integrable systems in the family of type (2) are: $\mathcal{B}_2[a]$, $\mathcal{B}_3[a]$, $\mathcal{B}_4[-1]$, $\mathcal{B}_4[-3]$, $\mathcal{B}_5[-\frac{1}{4}]$, $\mathcal{B}_5[\frac{-13\pm5\sqrt{5}}{2}]$, $\mathcal{B}_5[1]$, and $\mathcal{B}_7[1]$, that is,

$$\left\{\begin{array}{rl} u_t = & au_2 + v^2 \\ v_t = & v_2 \end{array}\right. \quad \left\{\begin{array}{rl} u_t = & au_3 + v^2 \\ v_t = & v_3 \end{array}\right. \quad \left\{\begin{array}{rl} u_t = & -u_4 + v^2 \\ v_t = & v_4 \end{array}\right.$$
$$\left\{\begin{array}{rl} u_t = & -3u_4 + v^2 \\ v_t = & v_4 \end{array}\right. \quad \left\{\begin{array}{rl} u_t = & -\frac{1}{4}u_5 + v^2 \\ v_t = & v_5 \end{array}\right. \quad \left\{\begin{array}{rl} u_t = & \frac{-13+5\sqrt{5}}{2}u_5 + v^2 \\ v_t = & v_5 \end{array}\right.$$
$$\left\{\begin{array}{rl} u_t = & \frac{-13-5\sqrt{5}}{2}u_5 + v^2 \\ v_t = & v_5 \end{array}\right. \quad \left\{\begin{array}{rl} u_t = & u_5 + v^2 \\ v_t = & v_5 \end{array}\right. \quad \left\{\begin{array}{rl} u_t = & u_7 + v^2 \\ v_t = & v_7 \end{array}\right.$$

and give their recursive operators to generate all the local nontrivial symmetries. The result is obtained by use of the symbolic method, introduced in [GD75], which allows us to translate differential polynomials into algebraic polynomials. This converts the problem of finding a symmetry into a division problem of polynomials. This can then be solved using number theory.

One can also combine the two directions of extension (i.e. noncommutative and system). The equations listed above are also non–commutative symmetry-integrable, [OS98, OW00]. The fact that the noncommutative case is so similar to the commutative is simply caused by the special form of the systems under consideration here. This form causes the symmetries to be of the same type as the system itself, ie., $\begin{pmatrix} bu_n + f[v] \\ v_n \end{pmatrix}$, where f is quadratic in v and its derivatives. This implies that the symmetries can be determined after the division problem. The only difference is interpreting the symbols back. For the symbolic expression $\frac{1}{2}\left(y_1^2y_2 + y_1y_2^2\right)v^2$, we interpret into v_1v_2 for the commutative case and $\frac{1}{2}(v_1v_2 + v_2v_1)$ for the noncommutative case.

The results in this paper also shed some light on the complete classification for systems. For a given system, if it consists of equation (2) and a suitable perturbation, the necessary condition of integrability is that the linear part is one of the eight cases and the orders of the symmetries are also compatible. The fact that one finds a 1-dimensional family, parametrized by a when $m = 2,\ 3$ gives a first order explanation as to why there are so many integrable systems of order 2 and 3.

2. THE SYMBOLIC METHOD

In the symbolic method one replaces derivatives u_k, v_k by powers $x^k u, y^k v$ (Usually one replaces u_k by x^k, but this leads to confusion in nonhomogeneous problems and in the more-variable case, since distinction between u and v disappears if there are no derivatives). When there are more u's or v's involved we add more symbols, one for every u or v. These will be denoted by x_i's or y_i's. E.g. v_1v_2 becomes $\frac{1}{2}\left(y_1^2y_2+y_1y_2^2\right)v^2$ for the commutative case and just $y_1y_2^2v^2$ for the noncommutative case. Formally this can be expressed by saying that we average the symbols over a given group and we write $< y_1y_2^2 > v^2$ for both expressions where we let the group vary to get the right answer. In the commutative case the group is Σ_n, the permutation group of n symbols, and in the noncommutative case the group is trivial. For linear expressions we usually drop the symbol for the average, since the group is trivial in any case. Also, when the expression is Σ_n-symmetric by itself, as in v^2, we drop the $<\cdot>$. This is also the way in which the expression $< v_1v_2 >$ should be read, but in an inverse way, that is, in the commutative case the group is trivial, and in the noncommutative case the group is Σ_n. Of course the details of the proofs vary from group to group and have to be checked carefully.

For the one-variable case, all definitions and proofs for the commutative and noncommutative case can be found in [SW98] and [OW00] respectly. The generalization to the more–variable case is straightforward. Since the specific equation we will be working on is very simple, we just write out the method for this case without giving the general theory.

Consider the system (2) and rewrite it as $(au_m+v^2)\frac{\partial}{\partial u}+v_m\frac{\partial}{\partial v}$. Its symbolic form is

$$\left(ax_1^m u+v^2\right)\frac{\partial}{\partial u}+y_1^m v\frac{\partial}{\partial v}.$$

In order to compute the symmetries of the equation (2) we need the commutator of the linear part of the equation with an arbitrary homogeneous vectorfield:

$$\begin{aligned}
&[ax_1^m u\frac{\partial}{\partial u}+y_1^m v\frac{\partial}{\partial v}, A(x[r],y[s])u^rv^s\frac{\partial}{\partial u}+B(x[k],y[l])u^kv^l\frac{\partial}{\partial v}]=\\
=&\left(a(\sum_{i=1}^r x_i+\sum_{i=1}^s y_i)^m-a\sum_{i=1}^r x_i^m-\sum_{i=1}^s y_i^m\right)A(x[r],y[s])u^rv^s\frac{\partial}{\partial u}\\
+&\left((\sum_{i=1}^k x_i+\sum_{i=1}^l y_i)^m-a\sum_{i=1}^k x_i^m-\sum_{i=1}^l y_i^m\right)B(x[k],y[l])u^kv^l\frac{\partial}{\partial v},
\end{aligned}$$

where $x[r]$ stands for $x_1,\cdots,x_r$. Putting this expression equal to zero, we obtain that either $A=0$ or $r=1$ and $s=0$; also $B=0$ or $k=0$ and $l=1$ if $a\neq 1$ and $m\neq 1$. So the linear part of the (potential) symmetry will be of the form, when $a\neq 1$ and $m\neq 1$,

$$\gamma x_1^p u\frac{\partial}{\partial u}+\delta y_1^q v\frac{\partial}{\partial v},\qquad \gamma,\delta\in\mathbb{C}.$$

Or, if we go back to our old notation,

$$\begin{cases} u_t &= \gamma u_p \\ v_t &= \delta v_q. \end{cases}$$

When looking for symmetries of a given order, we may as well take $q = p = n$ without loss of generality. Now computing the commutator of this linear part of the (potential) symmetry with the quadratic part of our equations, we obtain

$$[\gamma x_1^n u\frac{\partial}{\partial u} + \delta y_1^n v\frac{\partial}{\partial v}, v^2\frac{\partial}{\partial u}] = (\gamma(y_1+y_2)^n - \delta(y_1^n + y_2^n))\, v^2\frac{\partial}{\partial u}.$$

Defining $G_\lambda^n(y_1, y_2) = \lambda(y_1+y_2)^n - (y_1^n + y_2^n)$, we can now construct the quadratic terms of the symmetry as follows. First, we have, $\delta \neq 0$,

$$\begin{aligned} &[(ax_1^m u + v^2)\frac{\partial}{\partial u} + y_1^m v\frac{\partial}{\partial v}, (\gamma x_1^n u + \hat{A}v^2)\frac{\partial}{\partial u} + \delta y_1^n v\frac{\partial}{\partial v}] \\ = &\left(\hat{A}G_a^m(y_1,y_2) - \delta G_{\gamma/\delta}^n(y_1,y_2)\right) v^2\frac{\partial}{\partial u}. \end{aligned} \tag{3}$$

Let

$$\hat{A} = \delta G_{\gamma/\delta}^n(y_1,y_2)/G_a^m(y_1,y_2)$$

If $\hat{A}$ is polynomial in y_1, y_2, then $(\gamma x_1^n u + \hat{A}v^2)\frac{\partial}{\partial u} + \delta y_1^n v\frac{\partial}{\partial v}$ is a symmetry of system (2). Whether $\hat{A}$ is indeed polynomial is partly answered by the following results. A slightly weaker version had been proved in [BSW98].

Remark 2. 1. For any $n \in \mathbb{N}$, $\begin{pmatrix} v_n \\ 0 \end{pmatrix}$ is a symmetry of (2) when $a = 1$. We call them trivial linear symmetries. Its nontrivial symmetries can be obtained by taking $a = 1$ in the general results.

THEOREM 2.1. *Let $\mathcal{G}_a^m(X) = a(X+1)^m - X^m - 1,\ a \in \mathbb{C}\backslash\{0\},\ m \in \mathbb{N}_{\geq 2}$. Suppose that there are infinitely many pairs $b \in \mathbb{C},\ n \in \mathbb{N}$ such that $\mathcal{G}_a^m$ divides $\mathcal{G}_b^n$. Then we are in one of the following cases,*

$m = 2$. *Then $n \in \mathbb{N}_{\geq 2}$ arbitrary and $b = (\alpha^n+1)/(\alpha+1)^n$, where α is a zero of $\mathcal{G}_a^2$.*

$m = 3$. *Then $n \in \mathbb{N}_{\geq 3}$ odd and $b = (\alpha^n+1)/(\alpha+1)^n$, where α is a zero $\neq -1$ of $\mathcal{G}_a^3$.*

$m = 4$ *and* $a = -1$. *Then* $n \equiv 1 (\text{mod } 3)$ *and* $b = (-1)^{n-1}$

$m = 4$ *and* $a = -3$. *Then* $n \equiv 0 (\text{mod } 4)$ *and* $b = 1 + (1+i)^n$.

$m=5$ *and* $a=-1/4$. *Then* $n\equiv 1(\mathrm{mod}\ 4)$ *and* $b=(1+i)^{1-n}$.
$m=5$ *and* $a=(-13\pm 5\sqrt{5})/2$. *Then* $n\equiv 5(\mathrm{mod}\ 10)$, $b=(a+1)^{n/5}-1$.
$m=5$ *and* $a=1$. *Then* $n\equiv 1(\mathrm{mod}\ 6)$ *or* $n\equiv 5(\mathrm{mod}\ 6)$ *and* $b=1$.
$m=7$ *and* $a=1$. *Then* $n\equiv 1(\mathrm{mod}\ 6)$ *and* $b=1$.

Proof. It is easy to check for $m=2,\ 3$. When $m\geq 4$, Lemma 2.1 tells us that if we are not in one of the exceptional cases listed, we can choose $\alpha,\beta\neq 0,-1$, two zeros of $\mathcal{G}_a^m$, in such a way that none of the pairs $\alpha\beta$, $(1+\alpha)/(1+1/\beta)$ or α/β, $(1+\alpha)/(1+\beta)$ consists of roots of unity. From the proof of Theorem 2.2 in [BSW98], it then follows that $\mathcal{G}_a^m|\mathcal{G}_b^n$ for at most finitely many pairs (b,n). ▮

Remark 2. 2. Comparing Theorem 2.1 with Theorem 2.2 of [BSW98], one may notice that the case $a=1, m=7$ was not excluded in [BSW98], where $f_{a,m}=\mathcal{G}_a^m$. We made the implicit assumption $a\neq 0,1$. This is an error we like to correct here.

There is one-to-one correspondence between G_a^m and $\mathcal{G}_a^m$ since

$$G_a^m(y_1,y_2)=a(y_1+y_2)^m-y_1^m-y_2^m=y_1^m\left(a(y_1/y_2+1)^m-(y_1/y_2)^m-1\right)$$

We now translate these results on $\mathcal{G}_a^m$ to those on G_a^m, and further back to those on symmetries of Eq. (2).

THEOREM 2.2. *System (2) has infinitely many symmetries in the form* $\begin{pmatrix} b_n u_n+Q_n \\ v_n\end{pmatrix}$ *iff the system is of the form*

$\mathcal{B}_2[a]$*–then* $b_n=b_{n-1}-\frac{1-a}{2}b_{n-2}$ *with* $b_1=1$, $b_2=a$;

$$Q_n=D_xQ_{n-1}-\frac{1-a}{2}D_x^2Q_{n-2}+<vv_{n-2}>\ \text{ with } Q_1=0, Q_2=v^2.$$

$\mathcal{B}_3[a]$*–then* $b_{2n+1}=\frac{2a+1}{3}b_{2n-1}-\frac{(1-a)^2}{9}b_{2n-3}$ *with* $b_1=1$, $b_3=a$; *and* $Q_1=0$, $Q_3=v^2$,

$$\begin{aligned} Q_{2n+1} &= \frac{2a+1}{3}D_x^2Q_{2n-1}-\frac{(1-a)^2}{9}D_x^4Q_{2n-3} \\ &- \frac{2}{9}D_x^{-1}<(a-1)v_2v_{2n-3}+(2a-5)v_1v_{2n-2}+(a-7)vv_{2n-1}>. \end{aligned}$$

$\mathcal{B}_4[-1]$*–then* $b_{3n+1}=-2b_{3n-2}-b_{3n-5}$ *with* $b_1=1$, $b_4=-1$; *and* $Q_1=0$, $Q_4=v^2$,

$$Q_{3n+1}=-2D_x^3Q_{3n-2}-D_x^6Q_{3n-5}+<v_2v_{3n-5}+4v_1v_{3n-4}+4vv_{3n-3}>.$$

$\mathcal{B}_4[-3]$*–then* $b_{4n} = -3b_{4n-4} + 4b_{4n-8}$ *with* $b_0 = 2$, $b_4 = -3$; *and* $Q_0 = 0$, $Q_4 = v^2$,

$$\begin{aligned} Q_{4n} &= 4D_x^8 Q_{4n-8} - 3D_x^4 Q_{4n-4} - 2 < v_4 v_{4n-8} > \\ & - < 10v_3 v_{4n-7} + 17v_2 v_{4n-6} + 10v_1 v_{4n-5} > . \end{aligned}$$

$\mathcal{B}_5[-\frac{1}{4}]$*–then* $b_{4n+1} = -\frac{1}{2}b_{4n-3} - \frac{1}{16}b_{4n-7}$ *with* $b_1 = 1$, $b_5 = -\frac{1}{4}$; *and* $Q_0 = 0$, $Q_5 = v^2$,

$$\begin{aligned} Q_{4n+1} &= -\frac{1}{2}D_x^4 Q_{4n-3} - \frac{1}{16}D_x^8 Q_{4n-8} + \frac{1}{10}D_x^{-1} < v_4 v_{4n-7} + 8v_3 v_{4n-6} > \\ &+ \frac{1}{10}D_x^{-1} < 26v_2 v_{4n-5} + 40v_1 v_{4n-4} + 25vv_{4n-3} > . \end{aligned}$$

$\mathcal{B}_5[-\frac{13}{2} \pm \frac{5\sqrt{5}}{2}]$*–then*

$$b_{10n+5} = -(9+11a)b_{10n-5} + (10+11a)b_{10n-15}$$

with $b_5 = a$, $b_{15} = -683 \pm 305\sqrt{5} = 110 + 122a$; *and* $Q_5 = v^2$,

$$\begin{aligned} Q_{15} &= -\frac{368+462a}{25} < vv_{10} > - \frac{4114+4816a}{25} < v_1 v_9 > \\ & - \frac{18282+22138a}{25} < v_2 v_8 > - \frac{48312+59638a}{25} < v_3 v_7 > \\ & - \frac{84084+104816a}{25} < v_4 v_6 > - \frac{50383+62962a}{25} < v_5 v_5 > \end{aligned}$$

and

$$\begin{aligned} & Q_{10n+5} = -(9+11a)D_x^{10} Q_{10n-5} + (10+11a)D_x^{20} Q_{10n-15} \\ & - D_x^{-1} < \frac{132+88a}{5} v_1 v_{10n} + \frac{1002+928a}{5} v_2 v_{10n-1} > \\ & - D_x^{-1} < \frac{4564+4806a}{5} v_3 v_{10n-2} + \frac{14752+16178a}{5} v_4 v_{10n-3} > \\ & - D_x^{-1} < \frac{177392+196578a}{25} v_5 v_{10n-4} + \frac{326188+362202a}{25} v_6 v_{10n-5} > \\ & - D_x^{-1} < \frac{466874+518576a}{25} v_7 v_{10n-6} + \frac{525546+583784a}{25} v_8 v_{10n-7} > \\ & - D_x^{-1} < \frac{466874+518576a}{25} v_9 v_{10n-8} + \frac{326226+362244a}{25} v_{10} v_{10n-9} > \\ & - D_x^{-1} < \frac{177354+196826a}{25} v_{11} v_{10n-10} + \frac{73496+81504a}{25} v_{12} v_{10n-11} > \\ & - D_x^{-1} < \frac{22424+24846a}{25} v_{13} v_{10n-12} + \frac{4746+5254a}{25} v_{14} v_{10n-13} > \\ & - D_x^{-1} < \frac{622+688a}{25} v_{15} v_{10n-14} + \frac{38+42a}{25} v_{16} v_{10n-15} > . \end{aligned}$$

$\mathcal{B}_5[1]$–*then* $b_{6n+7-2s} = 2b_{6n+1-2s} - b_{6n-5-2s}$ *with* $b_1 = b_5 = b_7 = b_{11} = 1$, *where* $s = 0,\ 1$*; and* $Q_1 = 0$, $Q_7 = \frac{7}{5} < 2vv_2 + v_1^2 >$, $Q_5 = v^2$, $Q_{11} = \frac{11}{5} < 2vv_6 + 6v_1v_5 + 14v_2v_4 + 9v_3^2 >$,

$$\begin{aligned} Q_{6n+7-2s} &= 2D_x^6 Q_{6n+1-2s} - D_x^{12} Q_{6n-5-2s} \\ &- \frac{2}{5} D_x^{-1} < v_9 v_{6n-6-2s} + 11 v_8 v_{6n-5-2s} + 54 v_7 v_{6n-4-2s} > \\ &- \frac{2}{5} D_x^{-1} < 155 v_6 v_{6n-3-2s} + 286 v_5 v_{6n-2-2s} + 351 v_4 v_{6n-1-2s} > \\ &- \frac{2}{5} D_x^{-1} < 285 v_3 v_{6n-2s} + 144 v_2 v_{6n+1-2s} + 36 v_1 v_{6n+2-2s} > . \end{aligned}$$

$\mathcal{B}_7[1]$–*then* $b_{6n+7} = 2b_{6n+1} - b_{6n-5}$ *with* $b_1 = b_7 = 1$*; and* $Q_1 = 0$, $Q_7 = v^2$,

$$\begin{aligned} Q_{6n+7} &= 2D_x^6 Q_{6n+1} - D_x^{12} Q_{6n-5} \\ &- \frac{2}{7} D_x^{-1} < v_7 v_{6n-6} + 10 v_6 v_{6n-5} + 43 v_5 v_{6n-4} > \\ &- \frac{2}{7} D_x^{-1} < 102 v_4 v_{6n-3} + 141 v_3 v_{6n-2} + 108 v_2 v_{6n-1} + 36 v_1 v_{6n} > . \end{aligned}$$

This concludes the list.

Proof. Notice that in each case the b_n can be explicitly determined as in Theorem 2.1. We can prove the statements by directly checking whether the recursive operators do produce new symmetries by induction. Here we give another proof by the symbolic method.
For $m = 2$, we have to determine when $G_a^2(y_1, y_2) \mid G_{b_n}^n(y_1, y_2)$ assuming $n > 2$. Since $G_a^2(y_1, y_2) = (a-1)(y_1+y_2)^2 + 2y_1y_2$, we can take $y_1y_2 = \frac{1-a}{2}(y_1+y_2)^2$. Therefore, $b_n = \frac{y_1^n + y_2^n}{(y_1+y_2)^n}|_{y_1y_2 = \frac{1-a}{2}(y_1+y_2)^2} = \frac{y_1^{n-1}+y_2^{n-1}}{(y_1+y_2)^{n-1}} - \frac{1-a}{2}\frac{y_1^{n-2}+y_2^{n-2}}{(y_1+y_2)^{n-2}} = b_{n-1} - \frac{1-a}{2} b_{n-2}$. We know that

$$\begin{aligned} \tilde{Q}_n &= \frac{G_{b_n}^n(y_1,y_2)}{G_a^2(y_1,y_2)} = \frac{b_{n-1}(y_1+y_2)^n - \frac{1-a}{2} b_{n-2}(y_1+y_2)^n - y_1^n - y_2^n}{G_a^2(y_1,y_2)} \\ &= \frac{(y_1+y_2)(b_{n-1}(y_1+y_2)^{n-1} - y_1^{n-1} - y_2^{n-1})}{G_a^2(y_1,y_2)} \\ &- \frac{1-a}{2} \frac{(y_1+y_2)^2(b_{n-2}(y_1+y_2)^{n-2} - y_1^{n-2} - y_2^{n-2})}{G_a^2(y_1,y_2)} \\ &+ \frac{(a-1)(y_1+y_2)^2(y_1^{n-2}+y_2^{n-2}) + 2y_1y_2(y_1^{n-2}+y_2^{n-2})}{2G_a^2(y_1,y_2)} \\ &= (y_1+y_2)\tilde{Q}_{n-1} - \frac{1-a}{2}(y_1+y_2)^2 \tilde{Q}_{n-2} + \frac{1}{2}(y_1^{n-2}+y_2^{n-2}). \end{aligned}$$

This corresponds to $Q_n = D_x Q_{n-1} - \frac{1-a}{2} D_x^2 Q_{n-2} + < vv_{n-2} >$. In the same way, we can prove the other cases. ▮

Corollary 2.1. *Under the same conditions as in Theorem 2.1, the system*

$$\begin{cases} u_t = \ au_m + \sum_{i,j<m} \beta_{i,j} v_i v_j & \beta_{ij} \in \mathbb{C}. \\ v_t = \ v_m & a \in \mathbb{C}\backslash\{0\}, \quad m \in \mathbb{N}_{\geq 2} \end{cases}$$

is (non–)commutative symmetry-integrable. The recursive operators for symmetries in Theorem 2.2 are valid with adapted initial values of Q_n.

Proof. The argument is simple: recalling formula (3), we only need to change v^2 into the symbolic form of $\sum_{i,j<m} \beta_{i,j} v_i v_j$ whenever it is commutative or not. Therefore, the division argument that is valid for v^2 is also valid for more general quadratic expressions. ▮

One can not draw the conclusion that the list is complete in this more general case. However, for a given system, this can be worked out. Let $g = \sum_{i,j<m} b_{i,j} v_i v_j$. We only need to check for any $\mathcal{G}_a^m$ whether its zeros are included the roots of $\hat{g}(X, 1)$, the symbolic form of g and a subset of the form of V_α in the proof of Lemma 2.1. Moreover, we can conclude that when $m > \max(i, j) + 7$, the system is not symmetry-integrable.

Corollary 2.2. *The systems:* $\begin{cases} u_t = \ au_m + v^l & m \in \mathbb{N}_{\geq 2}, \ l \in \mathbb{N}_{\geq 3} \\ v_t = \ v_m & a \in \mathbb{C}\backslash\{0\} \end{cases}$
are not symmetry-integrable except for $a = 1$ *and* $m = l = 3$.

Proof. Using the symbolic method, we need to analyze the factorizability of polynomials $a(y_1+y_2+y_3)^m - y_1^m - y_2^m - y_3^m$ for any $a \in \mathbb{C}, m \in \mathbb{N}_{\geq 2}$. Note that their factors give rise to the factors for $G_a^m(y_1, y_2) = a(y_1 + y_2)^m - y_1^m - y_2^m$ by taking $y_3 = 0$. Therefore, the corollary can be proved by going through the list in Theorem 2.1. ▮

Lemma 2.1. *Let* $a \in \mathbb{C}\backslash\{0\}, \ m \in \mathbb{N}$ *and* $\mathcal{G}_a^m(X) = a(X+1)^m - X^m - 1$. *Suppose* $m \geq 4$ *and*

$$(a, m) \neq (-1, 4), (-3, 4), (-1/4, 5), ((-13 \pm 5\sqrt{5})/2, 5), (1, 5), (1, 7).$$

In that case $\mathcal{G}_a^m$ *has two zeros* $\alpha, \beta \neq 0, -1$ *such that none of the pairs* $\alpha/\beta, (1+\alpha)/(1+\beta)$ *or* $\alpha\beta, (1+\alpha)/(1+1/\beta)$ *consists of roots of unity.*

Proof. Let α, β be two zeros of $\mathcal{G}_a^m$ not equal to $0, -1$. Suppose that $\alpha/\beta, (1+\alpha)/(1+\beta)$ are roots of unity. Then we have $|\alpha| = |\beta|$ and

$|1+\alpha| = |1+\beta|$. Hence β lies on the intersection of the circles $|z| = |\alpha|$ and $|z+1| = |1+\alpha|$ which implies $\beta = \alpha$ or $\beta = \overline{\alpha}$. Similarly if $\alpha\beta$ and $(1+\alpha)/(1+1/\beta)$ are roots of unity then $\beta = 1/\alpha$ or $\beta = 1/\overline{\alpha}$. As a consequence the statement of the Lemma it is proved for any $\mathcal{G}_a^m$ whose zeros are not a subset of a set of the form $V_\alpha = \{0, -1, \alpha, 1/\alpha, \overline{\alpha}, 1/\overline{\alpha}\}$.
Suppose now that there exists an α such that the zeros of $\mathcal{G}_a^m$ form a subset of V_α. If $\mathcal{G}_a^m$ has multiple zeros then, according to Lemma 3.1 in [BSW98], the multiple zero is an $(m-1)^{\text{st}}$ root of unity which we may assume to be equal to α. Together with $1/\alpha$ these are the only multiple zeros and they have multiplicity two. Whether $\mathcal{G}_a^m$ has multiple zeros or not, its is clear that if $a \neq 1$ and $m \geq 6$ or if $a = 1$ and $m \geq 8$, then $\mathcal{G}_a^m$ has a zero outside V_α and the Lemma is true.
The cases $(a, m) = (1, 4), (1, 5), (1, 6), (1, 7)$ can now be checked by hand and we find that only $(a, m) = (1, 4), (1, 6)$ satisfies the conclusion of our Lemma. From now on we assume $a \neq 1$.
Suppose $m = 4$. In the case of double zeros we have $\alpha = 1, \zeta_3$ or ζ_3^2, where $\zeta_3 = e^{2\pi i/3}$. Note that $\alpha = 1$ implies $a = 1/8$ and $\mathcal{G}_{1/8}^4(X) = -(X-1)^2(7X^2 + 10X + 7)/8$. We verify by hand that our Lemma is true for this polynomial. Note that $\alpha = \zeta_3, \zeta_3^2$ implies $a = -1$, which case is excluded by our assumptions. Now suppose $\mathcal{G}_a^4$ has only simple zeros. Then $\mathcal{G}_a^4$ has, up to a constant factor, the shape

$$(X-\alpha)(X-1/\alpha)(X-\overline{\alpha})(X-1/\overline{\alpha}).$$

We also have

$$\frac{\mathcal{G}_a^4(X)}{a-1} = X^4 + \frac{4a}{a-1}X^3 + \frac{6a}{a-1}X^2 + \frac{4a}{a-1}X + 1, \quad a \neq 1$$

Comparison of the coefficients yields $b + \overline{b} = -4a/(a-1)$ and $b\overline{b} + 2 = 6a/(a-1)$ where $b = \alpha + 1/\alpha$. Hence $3(b+\overline{b}) + 2(2 + b\overline{b}) = 0$. Note that this implies $|b + 3/2| = 1/2$, hence $|\alpha + 1/\alpha + 3/2| = 1/2$. Let us take $\beta = \overline{\alpha}$. If $\alpha\beta = |\alpha|^2$ were a root of unity, this would be 1. Hence $|\alpha| = 1$ and together with $|\alpha + 1/\alpha + 3/2| = 1/2$ this yields ζ_3, ζ_3^2 since $\alpha \neq -1$. We have dealt with this case above. So $\alpha\beta$ is not a root of unity. According to Lemma 2.2 the condition $|\alpha + 1/\alpha + 3/2| = 1/2$ entails that $\alpha/\overline{\alpha}$ and $(1+\alpha)/(1+\overline{\alpha})$ cannot be both roots of unity unless $\alpha = 1, \zeta_3, \zeta_3^2, -1 \pm i$ or $(-1 \pm i)/2$. We already treated $1, \zeta_3, \zeta_3^2, (-1 \pm i)/2$. The cases $\alpha = -1 \pm i$ also yield $a = -3$ which we excluded from our assumptions.
Suppose $m = 5$. In the case of double zeros we have $\alpha = 1, \pm i$. Note that $\alpha = 1$ implies $a = 1/16$ and $\mathcal{G}_{1/16}^5(X) = -5(X+1)(X-1)^2(3X^2 + 2X + 3)/16$. The Lemma is true for this polynomial. Note that $\alpha = \pm i$ implies $a = -1/4$, which case is excluded by our assumptions. Now suppose $\mathcal{G}_a^5$

has only simple zeros. Then $\mathcal{G}_a^5$ has, up to a constant factor, the shape

$$(X+1)(X-\alpha)(X-1/\alpha)(X-\overline{\alpha})(X-1/\overline{\alpha}).$$

We also have

$$\frac{\mathcal{G}_a^5(X)}{(a-1)(X+1)} = X^4 + \frac{4a+1}{a-1}X^3 + \frac{6a-1}{a-1}X^2 + \frac{4a+1}{a-1}X + 1, \quad a \neq 1$$

Comparison of the coefficients yields $b+\overline{b} = -(4a+1)/(a-1)$ and $b\overline{b}+2 = (6a-1)/(a-1)$, where $b = \alpha + 1/\alpha$. Hence $(b+\overline{b}) + (2+b\overline{b}) = 2$. Note that this implies $|b+1| = 1$ hence $|\alpha+1/\alpha+1| = 1$. Let us take $\beta = \overline{\alpha}$. If $\alpha\beta = |\alpha|^2$ were a root of unity, then it is 1 and hence $|\alpha| = 1$. Together with $|\alpha+1/\alpha+1| = 1$ this implies $\alpha = \pm i$ since $\alpha \neq -1$. But we have dealt with these cases above. According to Lemma 2.3 the condition $|\alpha+1/\alpha+1| = 1$ entails that $\alpha/\overline{\alpha}$ and $(1+\alpha)/(1+\overline{\alpha})$ cannot be both roots of unity unless $\alpha = \pm i, -1-\zeta_5, -1-\zeta_5-\zeta_5^3$, where ζ_5 is any primitive fifth root of unity. The values $\pm i$ are already dealt with. Finally the values of α in the cyclotomic field $\mathbb{Q}(\zeta_5)$ give rise to $a = (-13 \pm 5\sqrt{5})/2$, which were also excluded. ▮

Lemma 2.2. *Let $z \in \mathbb{C}\setminus\{0,-1\}$ be such that $|z+1/z+3/2| = 1/2$ and such that $z/\overline{z}, (1+z)/(1+\overline{z})$ are both roots of unity. Then $z = -1\pm i$, $(-1\pm i)/2$ or $z^3 = 1$.*

Proof. Put $x = z/\overline{z}$ and $y = (1+z)/(1+\overline{z})$. If $x = y$, then $z = -1$, which excluded from the assumption. If $x \neq y$, a short calculation shows that $z = -x(y-1)/(y-x)$. Substituting this in the condition $|z+1/z+3/2| = 1/2$ gives us, after some calculation using Maple[CGG+91], and the fact that $\overline{x} = x^{-1}$, $\overline{y} = y^{-1}$,

$$2x^2y^2 - x^2y + x^2 - xy^3 - 2xy^2 - xy + 2y^2 - y^3 + y^4 = 0.$$

Using an algorithm by C.J. Smyth (cf [BS01]), we can solve this equation for roots of unity and find that

$$(x,y) = (1,\pm i), (\pm i, -1), (\pm i, \mp i), (\zeta_3, \zeta_3^2), (\zeta_3^2, \zeta_3)$$

where $\zeta_3 = e^{2\pi i/3}$. These pairs give rise to the values of z in our Lemma. ▮

Lemma 2.3. *Let $z \in \mathbb{C}\setminus\{0,-1\}$ be such that $|z+1/z+1| = 1$ and such that $z/\overline{z}, (1+z)/(1+\overline{z})$ are both roots of unity. Then $z = \pm i$ or $z = -1-\zeta_5, -1-\zeta_5-\zeta_5^3$, where ζ_5 is any primitive fifth root of unity.*

Proof. Put $x = z/\overline{z}$ and $y = (1+z)/(1+\overline{z})$. If $x = y$, then $z = -1$, which excluded from the assumption. If $x \neq y$, a short calculation shows that $z = -x(y-1)/(y-x)$. Substituting this in the condition $|z+1/z+1| = 1$ gives us, after some calculation using Maple, and the fact that $\overline{x} = x^{-1}$, $\overline{y} = y^{-1}$,

$$x^2y^2 - x^2y + x^2 - xy - xy^3 + y^2 - y^3 + y^4 = 0$$

Using the algorithm from [BS01] we can solve this equation for roots of unity and find that

$$(x, y) = (1, \pm i), (-1, \pm i), (\zeta_5, \zeta_5^2), (\zeta_5, \zeta_5^4)$$

where ζ_5 is any primitive fifth root of unity. These pairs give rise to the values of z in our Lemma. ∎

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