Classification of Symmetry-Integrable Evolution Equations

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Abstract. This paper describes some recent developments which have made it possible to effectively classify homogeneous systems having infinitely many generalized symmetries, both in the commutative and the noncommutative case. It discusses the program that has to be carried out next to come to an automatic classification mechanism.

1. Introduction

The theory of integrable systems has developed in many directions, and although the interconnections between the different subjects are clearly suggested by the similarity of the results, they are not always so easy to prove or even formulate. The history of the subject experienced two developmental periods. In the first, following the discovery of the Korteweg–de Vries equation, a surprisingly large number of other integrable hierarchies, including mKdV, Sawada-Kotera, Kaup-Kupershmidt, was soon found. However, the second period was more disappointing, as the integrable well quickly dried up, at least in the most basic case of scalar, polynomial evolution equations linear in the highest order derivative. This led to the conjecture that all integrable systems of this particular form had been found. In this paper we do not discuss the well-known classification results as reviewed in [**SS84, Fok87, MSS91**]. We remark however that there one classifies equations of fixed order, but allows for much bigger equivalence classes. We only work with homogeneous equations and transformations that do not change the weight of the equation. More explicitly, we describe rigorous classification results for both commutative and noncommutative systems, [**SW98b, ?, ?**], including a proof of this particular conjecture, and a discussion of the general methods by which such complete classification results are established.

Of the various methods used to characterize integrable differential equations, including existence of infinitely many symmetries and/or conservation laws, soliton solutions, linearization by inverse scattering or differential substitution, Bäcklund transformation, Painlevé property, biHamiltonian structure, recursion operator, formal symmetry of infinite rank, etc., [**Zak91**], the most fruitful for systematic classification and discovery of new systems has been the characterization of integrable systems by the existence of a sufficient number of higher order symmetries. The

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higher order symmetry conditions can be effectively and, in certain cases, completely analyzed by an adaptation of the symbolic method of classical invariant theory, [**Olv99**], after which powerful number-theoretic results on factorizability of polynomials based on Diophantine approximation theory, [**Beu97**], are applied to complete the classification.

In [**SW98b, Wan98**], it was shown that the integrability of an evolution equation

$$u_t = u_n + f(u, \cdots, u_{n-1}), \qquad \text{where} \qquad u_n = D_x^n u, \tag{1}$$

with f a polynomial starting with terms that are at least quadratic, is determined by the existence of one nontrivial higher order symmetry. This led to the proof of a long-standing conjecture that "in all known cases the existence of one generalized symmetry implies the existence of infinitely many", [**Fok80**], under fairly relaxed conditions. In particular, for homogeneous scalar evolution equations, to prove the integrability of an equation of order 2 we need a symmetry of order 3, for an equation of order 3 we need a symmetry of order 5, for an equation of order 5 we need a symmetry of order 7, and for an equation of order 7 we need a symmetry of order 13, which enable us to give the complete list of integrable homogeneous equations.

The results were quickly generalized to noncommutative polynomial evolution equations of the form (1) in which the field variable u takes its values in an associative, non-commutative algebra, [**?**]. In this manner, it was rigorously proved that the list of integrable evolution equation in [**OS98a**] is complete. These equations can be regarded as quantizing classical integrable systems; see [**FC95**], where the authors treated the Korteweg–de Vries equation.

The classification method consists of several parts: first one has to determine whether a system has an symmetry, or for which parameter values a family of systems has a nontrivial generalized symmetry. Our theorem can then be invoked to show that such system is integrable. The check on the conditions of the theorem involves theoretical results and cannot be done (so far) using only computations. The application of the theorem leads to more calculations, but these involve only lower order terms and can therefore be done effectively for all orders using the symbolic method.

In this paper we will sketch the recent developments of the classification. We will not give the details nor the proofs of the theorems. The classification papers [**?, ?**] may serve as a model for the classification of other types of equations. we will give the lists of (equivalence classes of) integrable equations obtained so far, referring to the relevant papers for the details of how to effectively use these tables.

The method allows not only the classification of integrability with respect to symmetries, but also with respect to other objects like conservation laws. This however turns out to be more complicated than the symmetry analysis. The existence of a nontrivial symmetry is still required, even if one is looking for conservation laws; indeed, one often encounters the equations with only a finite number of conservation laws and no generalized symmetries.

2. Complete Classification Results

In all interesting integrable evolution equations, the right-hand side of equation is a homogeneous differential polynomial under a suitable weighting scheme. The differential equation (1) is said to be λ–*homogeneous* of *weight* μ if it admits the one-parameter group of scaling symmetries

$$(x, t, u) \longmapsto (a^{-1}x, a^{-\mu}t, a^{\lambda}u), \qquad a \in \mathbb{R}^{+}.$$

For example, the Korteweg–de Vries equation $u_t = u_{xxx} + uu_x$ is homogeneous of weight 3 for $\lambda = 2$.

Two evolution equations $u_t = K$ and $u_t = Q$ are symmetries of each other if and only if, [**Olv93**],

$$[K, Q] = 0. \tag{2}$$

An equation is called *integrable* if it has at infinitely many higher order symmetries.

2.1. Commutative Case.

In this section, we list all integrable hierarchies which are λ–homogeneous, with $\lambda \geq 0$. The classification theorem states that every λ–homogeneous evolution equation with linear leading term is equivalent, modulo homogeneous transformations in u, to an equation lying in one of the following hierarchies. For $\lambda > 0$ the equivalence transformations are just scalings $u \mapsto \alpha u$, while for $\lambda = 0$ we allow arbitrary change of variables $u \mapsto h(u)$.

2.1.1. $\lambda = 2$.

Korteweg–de Vries

$$u_t = u_3 + uu_1$$

Kaup-Kupershmidt

$$u_t = u_5 + 10uu_3 + 25u_1u_2 + 20u^2u_1$$

Sawada-Kotera

$$u_t = u_5 + 10uu_3 + 10u_1u_2 + 20u^2u_1$$

2.1.2. $\lambda = 1$.

Burgers'

$$u_t = u_2 + uu_1$$

Potential Korteweg–de Vries

$$u_t = u_3 + u_1^2$$

Modified Korteweg–de Vries

$$u_t = u_3 + u^2u_1$$

Potential Kaup-Kupershmidt

$$u_t = u_5 + 10u_1u_3 + \frac{15}{2}u_2^2 + \frac{20}{3}u_1^3$$

Potential Sawada-Kotera

$$u_t = u_5 + 10u_1u_3 + \frac{20}{3}u_1^3$$

Kupershmidt Equation ([4.2.6] in [MSS91])

$$u_t = u_5 + 5u_1u_3 + 5u_2^2 - 5u^2u_3 - 20\,uu_1u_2 - 5u_1^3 + 5u^4u_1$$

2.1.3. $\lambda = \frac{1}{2}$.

Ibragimov-Shabat [Cal87]

$$u_t = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1$$

2.1.4. $\lambda = 0$.

Potential Burgers'/Heat Equation

$$u_t = u_2 \quad \sim \quad u_t = u_2 + u_1^2$$

Potential modified Korteweg–de Vries

$$u_t = u_3 + u_1^3$$

Potential Kupershmidt equation

$$u_t = u_5 + 5u_2u_3 - 5u_1^2u_3 - 5u_1u_2^2 + u_1^5$$

2.2. Non–Commutative Case.

Recently, the analysis of integrable evolution equations in which the field variable u takes its values in an associative, non-commutative algebra, such as matrix, operator, Clifford, and group algebras, has attracted attention. A complete classification for $\lambda > 0$ homogeneous equations with linear leading term was established in [**?**]. (The case $\lambda = 0$ poses considerable technical difficulties.) There are only five non-commutative hierarchies, each generalizing one of the preceding commutative hierarchies. Interestingly, whereas the mKdV has two inequivalent non-commutative versions, there is no noncommutative generalization of the Sawada–Kotera, Kaup–Kupershmidt, Kupershmidt, or Ibragimov–Shabat hierarchies.

2.2.1. $\lambda = 2$.

Korteweg–de Vries

$$u_t = u_3 + uu_1 + u_1u$$

2.2.2. $\lambda = 1$.

Burgers'

$$u_t = u_2 + uu_1 \quad u_t = u_2 + u_1u$$

Potential Korteweg–de Vries

$$u_t = u_3 + u_1^2$$

Modified Korteweg–de Vries I

$$u_t = u_3 + u^2u_1 + u_1u^2$$

Modified Korteweg–de Vries II

$$u_t = u_3 + uu_2 - u_2u - \frac{2}{3}uu_1u$$

3. The Symbolic Method

The symbolic method was first introduced by Gel'fand and Dikii, [**GD75**]. It was generalized by Shakiban, [**Sha81, Sha82**], who used it to apply the invariant theory of finite groups to the study of conservation laws of evolution equations, and Ball, Currie, and Olver, [**BCO81, Olv83**], to classify null Lagrangians arising in nonlinear elasticity. In [**Olv83**] the connections with the symbolic method of classical invariant theory were first recognized; see [**Olv99**] for the full details.

The basic idea of the symbolic method is simply to replace u_i, where i is an index — in our case counting the number of derivatives — by ξ^i, where ξ is now a symbol. We see that the basic operation of differentiation, i.e. replacing u_i by u_{i+1}, is now replaced by multiplication with ξ, as is the case in Fourier transform theory. For higher degree terms with multiple u's, one uses different symbols to denote differentiation; for example, the noncommutative binomial $u_i u_j$ has symbolic form $\xi_1^i \xi_2^j$. In the commutative case, one needs to average over permutations of the differentiation symbols so that $u_i u_j$ and $u_j u_i$ have the same symbolic form. However, in the noncommutative case, this is no longer necessary. In other words, the noncommutative symbolic method works with general polynomials, while in the commutative case one restricts to (multi)-symmetric polynomials.

A differential monomial takes the form $u_I = u_{i_1} u_{i_2} \cdots u_{i_k}$. We call k the *degree* of the monomial, $\#I = i_1 + \cdots + i_k$ the *index*, and $\max(i_j,\ j = 1, \cdots, k)$ the *order*. We let $\mathcal{U}_n^k$ denote the set of differential polynomials of degree $k+1$ and index n. Let $\mathcal{U}^k = \bigoplus_n \mathcal{U}_n^k$, and $\mathcal{U} = \bigoplus_{n,k\geq 0} \mathcal{U}_n^k$, the algebra of all differential polynomials. The *order* of a differential polynomial is the maximum of the orders of its constituent monomials.

The *transform* or *symbolic form* defines a linear isomorphism between the space $\mathcal{U}^k$ of (non)-commutative differential polynomials of degree $k+1$ and the space $\mathcal{A}^k = \mathbb{R}[\xi_1, \ldots, \xi_{k+1}]$ of algebraic polynomials in $k+1$ variables. It is uniquely defined by its action on monomials.

Definition 1. The *symbolic form* of a differential monomial is defined as

$$u_{i_1} u_{i_2} \cdots u_{i_k} \quad \longmapsto \quad \begin{cases} \xi_1^{i_1} \xi_2^{i_2} \cdots \xi_k^{i_k} & \text{(noncommutative);} \\ \displaystyle\sum_{\pi \in \mathbb{S}^k} \xi_{\pi(1)}^{i_1} \xi_{\pi(2)}^{i_2} \cdots \xi_{\pi(k)}^{i_k} & \text{(commutative).} \end{cases}$$

In general, in analogy with Fourier transforms, we denote the symbolic form of $P \in \mathcal{U}^k$, whether it is commutative or not, by $\widehat{P} \in \mathcal{A}^k$. The transform has two basic properties:

$$\widehat{D_x P}(\xi_1, \ldots, \xi_{k+1}) = (\xi_1 + \cdots + \xi_{k+1})\, \widehat{P}(\xi_1, \ldots, \xi_{k+1}),$$
$$\widehat{\frac{\partial P}{\partial u_i}}(\xi_1, \cdots, \xi_k) = \frac{1}{i!} \sum_{j=1}^{k+1} \frac{\partial^i \widehat{P}}{(\partial \xi_j)^i}\, (\xi_1, \cdots, \xi_{j-1}, 0, \xi_j, \cdots, \xi_k).$$

The following key result is a consequence of these formulae.

Proposition 2. Let $K \in \mathcal{U}^m$ and $Q \in \mathcal{U}^n$. Then $D_K(Q) \in \mathcal{U}^{m+n}$, where D_K is the Fréchet derivative of K, and when K and Q are noncommutative,

$$\widehat{D_K[Q]} = \sum_{\tau=1}^{m+1} \widehat{K}\left(\xi_1, \ldots, \xi_{\tau-1}, \sum_{\kappa=0}^{n} \xi_{\tau+\kappa}, \xi_{\tau+n+1}, \ldots, \xi_{m+n}\right) \widehat{Q}(\xi_\tau, \ldots, \xi_{\tau+n});$$

when K and Q are commutative, the right-hand side is needed to be symmetrized.

The following polynomials play a critical role in the analysis.

Definition 3. The G-functions are the (commutative) polynomials

$$G_k^{(m)} = \xi_1^k + \cdots + \xi_{m+1}^k - \left(\xi_1 + \cdots + \xi_{m+1}\right)^k.$$

The key fact is the following formula for the bracket of a differential polynomial with a linear differential polynomial:

$$\widehat{[u_k, Q]} = G_k^{(m)}\, \widehat{Q}, \qquad \text{whenever} \qquad Q \in \mathcal{U}^m. \tag{3}$$

This follows directly from Proposition 2 and the fact that u_k has symbolic form $\widehat{u_k} = \xi_1^k$. An immediate application is the known result that the space of the symmetries of linear evolution equations $u_t = u_n$ with $n > 1$ is $\mathcal{U}^0$.

The crucial step is the following result, [**Beu97, SW98b**], on the divisibility properties of the G-functions. The proof relies on sophisticated techniques from Diophantine analysis.

PROPOSITION 4. We have $G_k^{(m)} = T_k^m H_k^{(m)}$, where $(H_k^{(m)}, H_l^{(m)}) = 1$ for all $2 \le k < l$, and T_k^m is one of the following polynomials:

- $m = 1$:
  - $k = 0 \pmod 2$: $\xi_1\xi_2$
  - $k = 3 \pmod 6$: $\xi_1\xi_2(\xi_1+\xi_2)$
  - $k = 5 \pmod 6$: $\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)$
  - $k = 1 \pmod 6$: $\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)^2$
- $m = 2$:
  - $k = 0 \pmod 2$: 1
  - $k = 1 \pmod 2$: $(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)$
- $m > 2$: 1

4. Symmetries of λ-Homogeneous Equations

In section 5 we illustrate the general method given in this section by an elementary example.

Any λ–homogeneous n^{th} order evolution equation can be broken up into its homogeneous components, and so takes the form

$$u_t = K = K^0[u] + K^1[u] + K^2[u] + \cdots, \qquad \text{where} \qquad K^i \in \mathcal{U}^i. \tag{4}$$

We assume that $K^0[u] = u_n$ and $0 < \lambda \in \mathbb{Q}$. Note that the index of $K^i[u]$ is $n - i\lambda \ge 0$. When $i\lambda \notin \mathbb{N}$, $K^i[u] = 0$. This reduces the number of relevant λ to a finite set.

Let $S \in \mathcal{U}$ be an m^{th} order symmetry of the evolution equation (4). We break up the bracket condition $[S, K] = 0$ into its homogeneous summands, leading to the series of successive symmetry equations

$$\sum_{i+j=r} [S^j, K^i] = 0, \qquad \text{for} \qquad r = 0, 1, \ldots\ . \tag{5}$$

We know that S must have nontrivial linear term, $S^0 \neq 0$, and we can set $S^0 = u_m$ without loss of generality. Clearly we have $[S^0, K^0] = 0$. The next equation to be solved is

$$[u_m, K^1] + [S^1, u_n] = 0, \quad \textit{i.e.} \quad G_m^{(1)}\widehat{K^1} = G_n^{(1)}\widehat{S^1}. \tag{6}$$

This is trivially satisfied if K has no quadratic terms: $K^1 = 0$. Concentrating on $K^1 \neq 0$, we have the following theorem:

THEOREM 5. *Suppose the evolution equation* (4) *has a nonzero symmetry S of order $m \ge 2$. Suppose $Q^1 \in \mathcal{U}^1$ is non-zero with the same weight as u_k, where $k \neq m, n$, and k odd if n is odd, that satisfies $[u_n, Q^1] + [K^1, u_k] = 0$. Then there exists a unique symmetry of the form $Q = u_k + Q^1 + Q^2 + \cdots$. Moreover, the symmetries Q and S commute.*

We make a very interesting observation. Suppose Q is a nontrivial k^{th} odd order symmetry of (4) with odd n, whose quadratic terms have symbolic form:

$$\widehat{Q}^1 = \frac{\widehat{K}^1 G_k^{(1)}}{G_n^{(1)}} = \frac{\widehat{K}^1\ (\xi_1^2+\xi_1\xi_2+\xi_2^2)^{s-s'}\ H_k^{(1)}}{H_n^{(1)}}.$$

Proposition 4 implies that $\lambda \leq 3+2\min(s, s')$, where $s' = \frac{n-3}{2} \pmod 3$ and $s = \frac{k-3}{2} \pmod 3$. Then Theorem 5 implies that there is a symmetry $Q = u_{2s+3} + Q^1 + \cdots$ of the original equation. The evolution equations defined by Q and K have the same symmetries, so instead of considering K we may consider the equation given by Q, which is of order $q = 2s + 3$ for $s = 0, 1, 2$. It follows that we only need to find the symmetries of λ-homogeneous equations (with $\lambda \leq 7$) of order ≤ 7 in order to obtain the complete classification of symmetries of λ-homogeneous scalar polynomial equations starting with linear terms.

A similar observation can be made for even $n > 2$. Suppose we have found a nontrivial symmetry with quadratic term

$$\widehat{Q}^1 = \frac{\widehat{K}^1 G_k^{(1)}}{G_n^{(1)}} = \frac{\widehat{K}^1\, G_k^{(1)}}{\xi_1\xi_2\, H_n^{(1)}}$$

This immediately implies $\lambda \leq 2$. Then there is a symmetry $Q = u_2 + Q^1 + \cdots$ of the original equation. Therefore, we only need to find the symmetries of 2^{nd} order equations to get the complete classification of symmetries of λ-homogeneous scalar polynomial equations (with $\lambda \leq 2$) starting with an even linear term.

Finally, we must analyze the case when K has no quadratic terms. Assume that $K^i = 0$ for $i = 1, \ldots, j-1$, and $K^j \neq 0$ for some $j > 1$. In place of (6), we now need to solve the leading order equation

$$[u_m, K^j] + [S^j, u_n] = 0.$$

Using (3), the symbolic form of this condition is

$$\widehat{S}^j = \frac{\widehat{K}^j G_m^{(j)}}{G_n^{(j)}}. \tag{7}$$

Proposition 4 implies that this polynomial identity has no solutions when $j \geq 3$, or when $j = 2$ and n is even, since $G_m^{(j)}$ and $G_n^{(j)}$ have no common factors, and the degree of K^j is $n - j\lambda < n$, which is the degree of $G_n^{(j)}$. This implies that there are no symmetries for such equations. When $j = 2$ and n is odd, the equation can only have odd order symmetries, and we have the similar theorem as Theorem 5 in this case. Moreover, if the equation (7) can be solved for any m, it can also be solved for $m = 3$. By now, we have proved the following:

Theorem 6. *A nontrivial symmetry of a λ-homogeneous equation is part of a hierarchy starting at order* 2, 3, 5 *or* 7.

Only an equation with nonzero quadratic or cubic terms can have a nontrivial symmetry. For such λ, we must find a third order symmetry for a second order equation, a fifth order symmetry for a third order equation, a seventh order symmetry for a fifth order equation with quadratic terms, and the thirteenth order symmetry for a seventh order equation with quadratic terms. It remains to analyze each of these particular cases in detail. The last case can be easily reduced to the case of fifth order equations by determining the quadratic terms of the equation. A straightforward computation, done with the help of Maple, completes the proof of our lists. The details of the computation are completed as in the commutative case described in [?].

5. Computational example

To illustrate how the symbolic method works, we give the symbolic calculation for the fifth order symmetry of the Korteweg–de Vries equation. When one computes a symmetry, the natural approach is to do this degree by degree. So for instance, if we have as the equation

$$u_t = K = K_0 + K_1 = u_3 + uu_1 \qquad \text{(KdV)}$$

then we compute as a symmetry

$$S = S_0 + S_1 + \cdots = u_5 + a_1 uu_3 + a_2 u_1 u_2 + \cdots$$

We have to solve $[K_0, S_1] + [K_1, S_0] = 0$, i.e.,

$$\begin{aligned} & D_x^3 S_1 + uD_x S_0 + u_1 S_0 = \\ & = \; D_x^5 K_1 + a_1 u D_x^3 K_0 + a_1 u_3 K_0 + a_2 u_1 D_x^2 K_0 + a_2 u_2 D_x K_0. \end{aligned}$$

If we translate this to the symbolic method we obtain

$$(\xi_1 + \xi_2)^3 \hat{S}_1 + (\xi_1^5 + \xi_2^5)\hat{K}_1 = (\xi_1 + \xi_2)^5 \hat{K}_1 + (\xi_1^3 + \xi_2^3)\hat{S}_1.$$

Thus we can formally solve

$$\hat{S}_1 = \frac{(\xi_1 + \xi_2)^5 - \xi_1^5 - \xi_2^5}{(\xi_1 + \xi_2)^3 - \xi_1^3 - \xi_2^3}\hat{K}_1,$$

and this is a real solution if $\hat{S}_1$ turns out to be a polynomial. Thus we have translated our problem into the following question. If we let

$$G_n^{(1)}(\xi_1, \xi_2) = (\xi_1 + \xi_2)^n - \xi_1^n - \xi_2^n,$$

then which factors do $G_n^{(1)}$ and $G_m^{(1)}$ have in common? Using the results in Proposition 4, we can determine whether the symmetry we are looking for exists or not. In this case, the answer is simple, that is,

$$\hat{S}_1 = \frac{5}{3}(\xi_1^2 + \xi_1\xi_2 + \xi_2^2)\hat{K}_1 = \frac{5}{6}(\xi_1^3 + 2\xi_1^2\xi_2 + 2\xi_1\xi_2^2 + \xi_2^3)u^2.$$

Let us compute S_2 by solving $[S_1, K_1] + [S_2, K_0] = 0$. By Proposition 2, this leads to

$$\hat{S}_2 = \frac{5}{6}\frac{(\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_1 + \xi_3)(\xi_1 + \xi_2 + \xi_3)}{(\xi_1 + \xi_2 + \xi_3)^3 - \xi_1^3 - \xi_2^3 - \xi_3^3}u^3 = \frac{5}{18}(\xi_1 + \xi_2 + \xi_3)u^3.$$

Note that $[S_2, K_1] = 0$ next degree. Therefore, the fifth order symmetry is

$$S = S_0 + S_1 + S_2 = u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1$$

This illustrates both the simplification induced by the symbolic method as well as the role of the G-functions in the whole analysis.

6. Future developments

The reason that the basic technique works in both commutative and non-commutative situations is simple: in the linear term one does not see any difference between the two, cf. formula (6), and so the same polynomials arise in the symbolic versions. the differences between the two cases only arise during the final, detailed analysis that leads to the complete classification.

There are a number of obvious extensions to the results obtained thus far.

- Systems. There has been a lot of work done on classifying second order two–dimensional systems of evolution equations, both in the commutative, [**MSY87, MSS91, Fou00**], and non-commutative, [**OS98a, OS98b**], cases. Note that the G-functions will depend on the eigenvalues of the linear part of the system, and one has to generalize the results of F. Beukers, [**Beu97**]. Nevertheless, there is some progress in this direction too. In [**?**], it was rigorously proved that an example due Bakirov of a fourth order system of two coupled evolution equations only possesses one nontrivial symmetry, of order six. This shows that, for systems, one higher order symmetry does not necessarily imply infinitely many, and hence integrability (see [**?**] for further classification results for systems of this type). Bakirov's example does not violate the more refined version of the conjecture, [**Fok87**], that a system of m evolution equations requires m higher order symmetries in order to be integrable. However, in [**vdKS99**] an explicit, two-dimensional counterexample to this conjecture is given.
- Scalar case with $\lambda < 0$. Some results are known for $\lambda = -1$, which were obtained from the case $\lambda = 0$. Although there seem to be many more integrable systems for $\lambda \leq 0$, this is compensated for by the fact that there are also many more homogeneous transformations. The results for $\lambda = 0$ indicate that one might expect a finite list of symmetry-integrable equations.
- Other algebraic structures. One can think of cyclic commutativity [**Rot99**], where
$$D_{u^2u_1}[h] = huu_1 + uhu_1 + u^2h_1 \equiv uu_1h + u_1uh + u^2h_1.$$
This seems to be the right kind of differentiation for the nonpolynomial case, which otherwise seems next to impossible.
- Another direction is to allow coefficients in the equations which do not commute with the field variables.

References

[BCO81] J. M. Ball, J. C. Currie, and P. J. Olver. Null Lagrangians, weak continuity, and variational problems of arbitrary order. *J. Funct. Anal.*, 41(2):135–174, 1981.

[Beu97] Frits Beukers. On a sequence of polynomials. *J. Pure Appl. Algebra*, 117/118:97–103, 1997. Algorithms for algebra (Eindhoven, 1996).

[BSW98] Frits Beukers, Jan A. Sanders, and Jing Ping Wang. One symmetry does not imply integrability. *J. Differential Equations*, 146(1):251–260, 1998.

[BSW00] Frits Beukers, Jan A. Sanders, and Jing Ping Wang. On integrability of systems of evolution equations. *Journal of Differential Equations*, 167, 2000.

[Cal87] F. Calogero. The evolution partial differential equation $u_t = u_{xxx} + 3(u_{xx}u^2 + 3u_x^2u) + 3u_xu^4$. *J. Math. Phys.*, 28(3):538–555, 1987.

[FC95] Benno Fuchssteiner and A. Roy Chowdhury. A new approach to the quantum KdV. *Chaos Solitons Fractals*, 5(12):2345–2355, 1995. Solitons in science and engineering: theory and applications.

[Fok80] A. S. Fokas. A symmetry approach to exactly solvable evolution equations. *J. Math. Phys.*, 21(6):1318–1325, 1980.

[Fok87] A.S. Fokas. Symmetries and integrability. *Studies in Applied Mathematics*, 77:253–299, 1987.

[Fou00] Mikhail V. Foursov. On integrable coupled KdV-type systems. *Inverse Problems*, 16:259–274, 2000.

[GD75] I. M. Gel'fand and L. A. Dikiĭ. Asymptotic properties of the resolvent of Sturm-Liouville equations, and the algebra of Korteweg-de Vries equations. *Uspehi Mat. Nauk*, 30(5(185)):67–100, 1975. English translation: Russian Math. Surveys 30 (1975), no. 5, 77–113.

[MSS91] A. V. Mikhaĭlov, A. B. Shabat, and V. V. Sokolov. The symmetry approach to classification of integrable equations. In *What is integrability?*, Springer Ser. Nonlinear Dynamics, pages 115–184. Springer, Berlin, 1991.

[MSY87] A. V. Mikhaĭlov, A. B. Shabat, and R. I. Yamilov. A symmetric approach to the classification of nonlinear equations. Complete lists of integrable systems. *Uspekhi Mat. Nauk*, 42(4(256)):3–53, 1987.

[Olv83] Peter J. Olver. Hyper-Jacobians, determinantal ideals and weak solutions to variational problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 95(3-4):317–340, 1983.

[Olv93] Peter J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[Olv99] Peter J. Olver. *Classical Invariant Theory*, volume 44 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.

[OS98a] Peter J. Olver and Vladimir V. Sokolov. Integrable evolution equations on associative algebras. *Comm. Math. Phys.*, 193(2):245–268, 1998.

[OS98b] P.J. Olver and V.V. Sokolov. Non–abelian integrable systems of the derivative nonlinear Schrödinger type. *Inverse Problems*, 14:L5–L8, 1998. Letter to the editor.

[OW99] Peter J. Olver and Jing Ping Wang. Classification of integrable one-component systems on associative algebras. Technical report, UMN, 1999. To be published in *Proceedings of the London Mathematical Society.*

[Rot99] Gian-Carlo Rota. Combinatorial snapshots. *Math. Intelligencer*, 21(2):8–14, 1999.

[Sha81] Chehrzad Shakiban. A resolution of the Euler operator. II. *Math. Proc. Cambridge Philos. Soc.*, 89(3):501–510, 1981.

[Sha82] Chehrzad Shakiban. An invariant theoretic characterization of conservation laws. *Amer. J. Math.*, 104(6):1127–1152, 1982.

[SS84] V. V. Sokolov and A. B. Shabat. Classification of integrable evolution equations. In *Mathematical physics reviews, Vol. 4*, volume 4 of *Soviet Sci. Rev. Sect. C: Math. Phys. Rev.*, pages 221–280. Harwood Academic Publ., Chur, 1984.

[SW98a] Jan A. Sanders and Jing Ping Wang. Combining Maple and Form to decide on integrability questions. *Comput. Phys. Comm.*, 115(2-3):447–459, 1998.

[SW98b] Jan A. Sanders and Jing Ping Wang. On the integrability of homogeneous scalar evolution equations. *J. Differential Equations*, 147(2):410–434, 1998.

[SW00] Jan A. Sanders and Jing Ping Wang. On the integrability of non-polynomial scalar evolution equations. *J. Differential Equations*, 166:132–150, 2000.

[vdKS99] Peter H. van der Kamp and Jan A. Sanders. Almost integrable evolution equations. Technical Report WS–534, Vrije Universiteit Amsterdam, 1999. Submitted to Selecta Mathematica.

[Wan98] Jing Ping Wang. *Symmetries and Conservation Laws of Evolution Equations*. PhD thesis, Vrije Universiteit, Amsterdam, 1998.

[Zak91] V. E. Zakharov, editor. *What is integrability?* Springer-Verlag, Berlin, 1991.

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