Multilinear Hirota operators, modular forms and the Heisenberg algebra

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ABSTRACT. We use the imbedding of the total differential operator $\mathcal{D}$ into a Heisenberg algebra to give a method to generate the transvectants and their multilinear generalizations using the coherent state method. This leads to tensor product decompositions where all the components play an equal role.

1. The Heisenberg algebra

Consider the total differential operator $\mathcal{D}$, given on the generators by $\mathcal{D}u_k = u_{k+1}$ and obeying the Leibniz rule[1] $\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g)$ on polynomials $f, g \in P[u, u_1, \cdots]$, where we denote $\frac{\partial^k u}{\partial x^k}$ by u_k. This defines $\mathcal{D}$ on $P[u, u_1, \cdots]$. We now want to solve the following problem. Given a nonconstant $f \in P[u, u_1, \cdots]$, can we find $f^0, f^1 \in P[u, u_1, \cdots]$ such that

$$f = f^0 + \mathcal{D}f^1,$$

with f^0 in a direct summand of $\operatorname{Im}\mathcal{D}$?

We solve this problem by constructing a derivation $\mathcal{F}$, such that $\operatorname{Ker}\mathcal{F}$ is a direct summand of $\operatorname{Im}\mathcal{D}$ and such that $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{E} = [\mathcal{F}, \mathcal{D}]$ form a Heisenberg algebra (cf. [**SR94, SW97**]). This last property allows us to find an algorithm to do the splitting over

$$\operatorname{Ker}\mathcal{E} \oplus \operatorname{Ker}\mathcal{F} \oplus \operatorname{Im}\mathcal{D}$$

in concrete cases.

Since $P[u, u_1, \cdots]$ is generated by $u, u_1, \cdots$ and $u_1, u_2, \cdots \in \operatorname{Im}\mathcal{D}$, it seems natural to require $\mathcal{F}u = 0$ and $Fu_k \neq 0$. Let us try $Fu_k = ku_{k-1}$ on the generators, and extend $\mathcal{F}$ by requiring it to be a derivation just like $\mathcal{D}$, i.e. $\mathcal{F}(fg) = \mathcal{F}(f)g + f\mathcal{F}(g)$.

Here one should remark that this is not just trial and error. The guess is inspired by looking at the standard finite dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{R})$ and then taking the limit of $n \to \infty$, where n is the dimension of the representation space. This limiting behavior will explain the connection with classical invariant theory and modular functions, to be discussed in sections 5 and 6.

[1] See however section 8 for a quantized rule

Then one has on the generators for $\mathcal{E} = [\mathcal{F}, \mathcal{D}]$ the following relation.

$$\mathcal{E}u_k = [\mathcal{F}, \mathcal{D}]u_k = \mathcal{F}\mathcal{D}u_k - \mathcal{D}\mathcal{F}u_k = \mathcal{F}u_{k+1} - k\mathcal{D}u_{k-1} = (k+1)u_k - ku_k = u_k.$$

Since the commutator of two derivations is again a derivation, $\mathcal{E}$ is a derivation, measuring the degree of a monomial in $P[u, u_1, \cdots]$. Thus $P[u, u_1, \cdots] = \operatorname{Ker}\mathcal{E} \oplus \operatorname{Im}\mathcal{E}$, where $\operatorname{Ker}\mathcal{E}$ consists of the constants.

Example 1.1. *We let $f = uu_2 - u_1^2$ and apply the Heisenberg algebra to it.*

$$\begin{aligned} \mathcal{F}(uu_2 - u_1^2) &= 0, \\ \mathcal{E}(uu_2 - u_1^2) &= 2(uu_2 - u_1^2), \\ \mathcal{D}(uu_2 - u_1^2) &= uu_3 - u_1u_2. \end{aligned}$$

We can write the operators $\mathcal{F}, \mathcal{E}$ and $\mathcal{D}$ as differential operators as follows.

$$\begin{aligned} \mathcal{F} &= \sum_{i=1}^{\infty} iu_{i-1}\frac{\partial}{\partial u_i}, \\ \mathcal{E} &= \sum_{i=0}^{\infty} u_i\frac{\partial}{\partial u_i}, \\ \mathcal{D} &= \sum_{i=0}^{\infty} u_{i+1}\frac{\partial}{\partial u_i}. \end{aligned}$$

We find that

$$[\mathcal{F}, \mathcal{D}] = \mathcal{E},\ [\mathcal{E}, \mathcal{F}] = [\mathcal{E}, \mathcal{D}] = 0.$$

Thus we have embedded $\mathcal{D}$ in a Heisenberg algebra. One should remark that although the role of $\mathcal{F}$ and $\mathcal{D}$ seems to be quite symmetric in this Lie algebra, it is not so in the concrete representation, since $\mathcal{F}$ is (locally) nilpotent on $P[u, u_1, \cdots]$, (i.e. for any given $f \in P[u, u_1, \cdots]$ one can find $k \in \mathbb{N}$ such that $\mathcal{F}^k f = 0$) but $\mathcal{D}$ is not.

In the next section we answer the question: How do we use this fact to obtain a splitting

$$f = f^0 + \mathcal{D}f^1$$

for any $f \in P[u, u_1, \cdots]$?

2. The decomposition algorithm

In this section we shall prove that $P[u, u_1, \cdots]$ has a decomposition $P[u, u_1, \cdots] = \operatorname{Ker}\mathcal{F} \oplus \operatorname{Im}\mathcal{D}$. We will do this by, first of all, providing an algorithm, that for a given polynomial $v \in P[u, u_1, \cdots]$, finds a $f \in \operatorname{Ker}\mathcal{F}$ and $g \in P[u, u_1, \cdots]$ such that $v = f + \mathcal{D}g$, and secondly, prove that the given decomposition is a direct sum decomposition. Both components are based on the following lemma.

Lemma 2.1. *In $U(\mathfrak{n}_3(\mathbb{R}))$ (the universal enveloping algebra of the Heisenberg algebra $\mathfrak{n}_3(\mathbb{R})$) the following relation holds*

$$\mathcal{F}\mathcal{D}^k = \mathcal{D}^k\mathcal{F} + k\mathcal{D}^{k-1}\mathcal{E} \qquad (k \geq 1).$$

PROOF. By induction on k. For $k = 1$ this follows directly from the relations of $\mathfrak{n}_3(\mathbb{R})$. For $k > 1$ one has

$$\begin{aligned} \mathcal{F}\mathcal{D}^k &= \mathcal{F}\mathcal{D}^{k-1}\mathcal{D} = \mathcal{D}^{k-1}\mathcal{F}\mathcal{D} + (k-1)\mathcal{D}^{k-2}\mathcal{E}\mathcal{D} \\ &= \mathcal{D}^{k-1}(\mathcal{D}\mathcal{F} + \mathcal{E}) + (k-1)\mathcal{D}^{k-1}\mathcal{E} \\ &= \mathcal{D}^k\mathcal{F} + k\mathcal{D}^{k-1}\mathcal{E} \end{aligned}$$

and this was to be shown. □

Using this lemma we can directly prove the following two corollaries.

COROLLARY 2.2. *On* Ker $\mathcal{F}$ *the following relation holds*

$$\mathcal{F}^k\mathcal{D}^k = k!\mathcal{E}^k.$$

PROOF. By induction on k. For $k = 1$ this is exactly the above lemma. For $k > 1$ we find

$$\mathcal{F}^k\mathcal{D}^k = \mathcal{F}^{k-1}\mathcal{F}\mathcal{D}^k = \mathcal{F}^{k-1}k\mathcal{D}^{k-1}\mathcal{E} = k!\mathcal{E}^k.$$

□

COROLLARY 2.3. *On* Ker $\mathcal{F}\mathcal{D}$ *the following relation holds*

$$\mathcal{D}^k\mathcal{F}^k = (-1)^k k!\mathcal{E}^k.$$

PROOF. By induction on k. For $k = 1$ this is exactly the above lemma. For $k > 1$ we find

$$\begin{aligned} \mathcal{E}^{k+1} &= -\mathcal{D}\mathcal{F}\mathcal{E}^k \\ &= (-1)^{k+1}\frac{1}{k!}\mathcal{D}\mathcal{F}\mathcal{D}^k\mathcal{F}^k \\ &= (-1)^{k+1}\frac{1}{k!}\mathcal{D}(\mathcal{D}^k\mathcal{F} + k\mathcal{D}^{k-1}\mathcal{E})\mathcal{F}^k \\ &= (-1)^{k+1}\frac{1}{k!}\mathcal{D}^{k+1}\mathcal{F}^{k+1} - k(-1)^k\frac{1}{k!}\mathcal{D}^k\mathcal{F}^k\mathcal{E} \\ &= (-1)^{k+1}\frac{1}{k!}\mathcal{D}^{k+1}\mathcal{F}^{k+1} - k\mathcal{E}^{k+1}, \end{aligned}$$

whence it follows that

$$(-1)^{k+1}(k+1)!\mathcal{E}^{(k+1)} = \mathcal{D}^{k+1}\mathcal{F}^{k+1}.$$

This concludes the induction proof. □

The following algorithm gives, for any f, its decomposition $f = f^0 + \mathcal{D}f^1$ where $f^0 \in \operatorname{Ker}\mathcal{F}$[**SR94**]. If efficiency is not at stake, one might be interested in the following formulae for this decomposition, cf. [**Ros99**].

$$\begin{aligned} f^0 &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\mathcal{E}^{-k}\mathcal{D}^k\mathcal{F}^k f \\ f^1 &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\mathcal{E}^{-k}\mathcal{D}^{k-1}\mathcal{F}^k f, \end{aligned}$$

where the implicit assumption is that the infinite sums make sense. The proof is easy (using lemma 2.1) and left to the reader. It might be an interesting exercise to show that the following algorithm does indeed implement the above formulae.

Algorithm 2.4. *Let $\mathfrak{k}$ be a subring of* $\operatorname{Ker}\mathcal{F}\cap P[[u,u_1,\cdots]]$ *and suppose that* $f\in\mathfrak{k}[u_1,u_2,\cdots]\cap\operatorname{Im}\mathcal{E}$.

1. *Put* $\alpha^{(0)}=f$ *and* $\beta^{(0)}=0$. *Let* $m=0$.
2. **while** $\mathcal{F}\alpha^{(m)}\neq 0$ **do**
   (a) *Determine* k_m *such that* $\mathcal{F}^{k_m+1}\alpha^{(m)}=0$ *and* $\mathcal{F}^{k_m}\alpha^{(m)}\neq 0$.
   (b) *Put* $\beta^{(m+1)}=\frac{1}{k_m!}\mathcal{E}^{-k_m}\mathcal{D}^{k_m-1}\mathcal{F}^{k_m}\alpha^{(m)}$.
   (c) *Put* $\alpha^{(m+1)}=\alpha^{(m)}-\mathcal{D}\beta^{(m+1)}$.
   (d) *Let* $m=m+1$.
   od
3. *Put* $f^0=\alpha^{(m)}$ *and* $f^1=\sum_{j=0}^{m}\beta^{(j)}$.

Theorem 2.5. *Algorithm 2.4 terminates and* $f=f^0+\mathcal{D}f^1$ *with* $f^0\in\operatorname{Ker}\mathcal{F}$.

Proof.
1. The relation
$$\alpha=\alpha^{(m)}+\mathcal{D}\sum_{j=0}^{m}\beta^{(j)}$$
is an invariant of the **while**-loop of algorithm 2.4. This is a direct consequence of step 2.
2. $k_{m+1}<k_m$. This follows from the fact that
$$\mathcal{F}^{k_m}\alpha^{(m+1)}=\mathcal{F}^{k_m}\alpha^{(m)}-\frac{1}{k_m!}\mathcal{E}^{-k_m}\mathcal{F}^{k_m}\mathcal{D}^{k_m}\mathcal{F}^{k_m}\alpha^{(m)}=0$$
since $\mathcal{F}^{k_m}\alpha^{(m)}\in\operatorname{Ker}\mathcal{F}$ and Corollary 2.2 holds. This guarantees that algorithm 2.4 terminates.
3. The termination condition of the **while**-loop exactly states that $f^0=\alpha^{(m)}\in\operatorname{Ker}\mathcal{F}$.

This concludes the proof. □

Theorem 2.5 tells us that $\operatorname{Im}\mathcal{E}=\operatorname{Ker}\mathcal{F}+\operatorname{Im}\mathcal{D}$. However, we need the stronger statement

Theorem 2.6. $\operatorname{Im}\mathcal{E}$ *admits a direct sum decomposition* $\operatorname{Im}\mathcal{E}=\operatorname{Ker}\mathcal{F}\oplus\operatorname{Im}\mathcal{D}$.

Proof. Let $f\in\operatorname{Ker}\mathcal{F}\cap\operatorname{Im}\mathcal{D}$, say $f=\mathcal{D}w$. Since $\mathcal{F}f=\mathcal{F}\mathcal{D}w=0$, we find that $w\in\operatorname{Ker}\mathcal{F}\mathcal{D}$ and we can apply corollary 2.3 to find that for any k
$$w=(-1)^k\frac{1}{k!}\mathcal{E}^{-k}\mathcal{D}^k\mathcal{F}^k w.$$
This proves the statement, since $\mathcal{F}$ is a nilpotent operator on $\mathfrak{k}[u_1,u_2,\cdots]$, hence we see, for suitable k, that w and further f are 0. □

The following algorithm does the same things as 2.4, but it is more efficient in that it goes down only once applying $\mathcal{F}$ until the result is zero and then it goes up applying $\mathcal{D}$ until it is back where it started, all the while bookkeeping so that the final result comes out right. Using formal integration, this bookkeeping can be done uniformly[**SW97**]. It should be remarked that both the proof and the implementation can be done recursively in a very natural way. We use the notation $\mu^{\mathcal{E}}$ to denote the scaling $u_k\mapsto\mu u_k$. For instance, $\mu^{\mathcal{E}}u^n=\mu^n u^n$.

Algorithm 2.7. *Let* $f\in\operatorname{Im}\mathcal{E}\cap\mathfrak{k}[u_1,u_2,\cdots]$.

1. *Put* $\alpha^{(0)} = f$*, and let* $m = 0$.
2. **while** $\alpha^{(m)} \neq 0$ **do**
   (a) *Put* $\alpha^{(m+1)} = \mathcal{F}\alpha^{(m)}$.
   (b) *Let* $m = m + 1$.
   od
3. *Put* $\beta^{(m)} = 0$.
4. **while** $m \neq 0$ **do**
   (a) *Put* $\gamma^{(m-1)} = \int\int \beta^{(m)} \mu^{-1} d\mu d\lambda$.
   (b) *Put* $\beta^{(m-1)} = \mathcal{D}\gamma^{(m-1)} + \mu^{\mathcal{E}}(\alpha^{(m-1)} - \mathcal{D}\gamma^{(m-1)} \mid_{\lambda=1,\mu=1})$.
   (c) *Let* $m = m - 1$.
   od
5. *Put* $f^0 = \beta^{(0)} \mid_{\lambda=0,\mu=1}$ *and* $f^1 = \gamma^{(0)} \mid_{\lambda=1,\mu=1}$.

Then $f = f^0 + Df^1 \in \operatorname{Ker}\mathcal{F} \oplus \operatorname{Im}\mathcal{D}$.

Both algorithms may not always be very efficient. For instance, apply them to u_5. Therefore a good strategy might be to use another method first to split of some part in $\operatorname{Im}\mathcal{D}$, and then apply one of these algorithms to the remaining term. This way one can have the best of both worlds, being efficient *and* systematic.

3. Coherent states

In classical invariant theory the method to find new covariants is by taking transvectants of the known covariants, starting with the groundform(s).

In terms of the Clebsch-Gordan decomposition this means that one has explicit maps

$$\tau_n : V_p \otimes V_q \to V_{p+q-2n}$$

where V_p is the $p+1$-dimensional irreducible representation of $\mathfrak{sl}(s,\mathbb{R})$, or, equivalently, the groundform of degree n. We will now develop a method to find maps from

$$\bigotimes_{i=0}^{k} \operatorname{Ker}\mathcal{F} \to \operatorname{Ker}\mathcal{F}$$

which are multilinear generalizations of these transvectants and which correspond to multilinear Hirota operators. The basic idea is to consider the so called **coherent states** (Cf Appendix 7.A in [**GSW88**]).

Suppose $\lambda \in \operatorname{Ker}\mathcal{F}$. We say that $f \in P[[u, u_1, \cdots]]$ is a **coherent state** if $\mathcal{F}f = \lambda f$. Clearly a coherent state cannot be a polynomial unless $\lambda = 0$. Denoting the space of coherent states with eigenvalue λ by V_λ, we see that $V_\lambda \cdot V_\mu = V_{\lambda+\mu}$. This enables us to construct elements in $\operatorname{Ker}\mathcal{F}$ from coherent states, since $V_\lambda \cdot V_{-\lambda} \subset V_0 = \operatorname{Ker}\mathcal{F}$. Observe that for $f \in V_\lambda$, $e^{\mu\mathcal{F}} f \in V_{e^{\mu}\lambda}$.

Example 3.1. *Let*

$$f = exp(u_1) = \sum_{n=0}^{\infty} \frac{1}{n!} u_1^n.$$

Then $\mathcal{F}f = uf$ *and* $f \in V_u$*. This shows that there exists a nontrivial coherent state.*

First of all, we have to construct elements in V_λ, given some $\lambda \in \operatorname{Ker}\mathcal{F}$. This is done as follows.

Definition 3.2. *Suppose λ commutes with $\mathcal{D},\mathcal{E}$ and $\mathcal{F}$ (as a multiplication operator). For $g \in \operatorname{Ker}\mathcal{F}$ let $T_\lambda(g) = \sum_{n=0}^\infty \frac{\lambda^n}{n!}\left(\frac{\mathcal{D}}{\mathcal{E}}\right)^n g$.*

Proposition 3.3. *One has, with $g, \lambda \in \operatorname{Ker}\mathcal{F}$,*

$$\mathcal{F}T_\lambda(g) = \lambda T_\lambda(g).$$

Proof.

$$\begin{aligned}
\mathcal{F}T_\lambda(g) &= \mathcal{F}\sum_{n=0}^\infty \frac{\lambda^n}{n!}\left(\frac{\mathcal{D}}{\mathcal{E}}\right)^n g \\
&= \sum_{n=0}^\infty \frac{\lambda^n}{n!}\mathcal{F}\left(\frac{\mathcal{D}}{\mathcal{E}}\right)^n g \\
&= \sum_{n=1}^\infty \frac{\lambda^n}{(n-1)!}\left(\frac{\mathcal{D}}{\mathcal{E}}\right)^{n-1} g \\
&= \lambda\sum_{n=1}^\infty \frac{\lambda^{n-1}}{(n-1)!}\left(\frac{\mathcal{D}}{\mathcal{E}}\right)^{n-1} g \\
&= \lambda T_\lambda(g).
\end{aligned}$$

If we do not require $g \in \operatorname{Ker}\mathcal{F}$, the result would be $\mathcal{F}T_\lambda = T_\lambda\mathcal{F} + \lambda T_\lambda$. □

Definition 3.4. *Fix $n \in \mathbb{N}$. Let $\lambda_i, f_i \in \operatorname{Ker}\mathcal{F}$ be such that $\sum_{i=0}^n \lambda_i = 0$. Then define*

$$\mu_n(f_0, \cdots, f_n) = \prod_{i=0}^n T_{\lambda_i}(f_i).$$

Corollary 3.5. *It follows from proposition 3.3 that μ_n is a multilinear map from $\otimes_{i=0}^n \operatorname{Ker}\mathcal{F}$ to $\operatorname{Ker}\mathcal{F}$.*

Thus we can start with $f_i, \lambda_i \in \operatorname{Ker}\mathcal{F}, i = 0, \cdots, n$, $\sum_{i=0}^n \lambda_i = 0$. We now express the $n+1$ functions λ_i in n functions μ_i in such a way that the sum of the λ_i's is automatically zero. There are many ways to do this. We make two choices. The first choice is to take $\lambda_0 = -\sum_{i=1}^n \mu_i$ and $\lambda_i = \mu_i$ for $i > 0$. This choice has the advantage that the multilinear transvectant formulae are easy to compute. It has the disadvantage that in the symmetric case the result is not invariant under the action of S_n on the μ_i. We write f_l^k for $\frac{1}{k!}(\frac{\mathcal{D}}{\mathcal{E}})^k f_l$.

$$\begin{aligned}
\mu_n(f_0,\cdots,f_n) &= \prod_{i=0}^n T_{\lambda_i}(f_i) = \\
&= T_{-\mu_1-\cdots-\mu_n}(f_0)\prod_{i=1}^n T_{\mu_i}(f_i) \\
&= \sum_{n_0=0}^\infty (-1)^{n_0}(\mu_1+\cdots+\mu_n)^{n_0} f_0^{n_0}\prod_{i=1}^n\sum_{n_i=0}^\infty \mu_i^{n_i} f_i^{n_i} \\
&= \sum_{n_0=0}^\infty (-1)^{n_0}\sum_{k_1+\cdots+k_n=n_0}\binom{n_0}{k_1,\cdots,k_n}\mu_1^{k_1}\cdots\mu_n^{k_n} f_0^{n_0}\prod_{i=1}^n\sum_{n_i=0}^\infty \mu_i^{n_i} f_i^{n_i} \\
&= \sum_{n_0,\cdots,n_n=0}^\infty (-1)^{n_0}\sum_{k_1+\cdots+k_n=n_0}\binom{n_0}{k_1,\cdots,k_n}\mu_1^{k_1+n_1}\cdots\mu_n^{k_n+n_n} f_0^{n_0}\cdots f_n^{n_n} \\
&= \sum_{n_1\cdots,n_n=0}^\infty\sum_{k_1=0}^{n_1}\cdots\sum_{k_n=0}^{n_n}(-1)^{k_1+\cdots+k_n}\binom{\sum_{i=1}^n k_i}{k_1,\cdots,k_n}\mu_1^{n_1}\cdots\mu_n^{n_n} f_0^{k_1+\cdots+k_n} f_1^{n_1-k_1}\cdots f_n^{n_n-k_n}
\end{aligned}$$

and we let

$$\begin{aligned}
&\tau_{n_1,\cdots,n_n}(f_0,\cdots,f_n) = \\
&\quad= \sum_{k_1=0}^{n_1}\cdots\sum_{k_n=0}^{n_n}(-1)^{k_1+\cdots+k_n}\binom{k_1+\cdots+k_n}{k_1,\cdots,k_n} f_0^{k_1+\cdots+k_n} f_1^{n_1-k_1}\cdots f_n^{n_n-k_n}.
\end{aligned}$$

For instance, $\tau_{2,1}(f_0,f_1,f_2) = 1/2 f_0 f_1'' f_2' - f_0' f_1' f_2' + 1/2 f_0'' f_1 f_2' - 1/2 f_0' f_1'' f_2 + f_0'' f_1' f_2 - 1/2 f_0''' f_1 f_2$, where $'$ stands for the application of $\mathcal{D}\mathcal{E}^{-1}$.

Due to the independence of the monomials in μ_i, the transvectants $\tau_{n_1,\cdots,n_n}$ map into $\operatorname{Ker}\mathcal{F}$.

The second method goes as follows. Let $\omega\in\mathbb{C}$ be such that $\omega^{n+1}=1, \omega^p\neq 1$ for any $0<p<n+1$. Put $\lambda_i=\sum_{j=1}^n \omega^{ij}\mu_j$, with $\mu_j\in\operatorname{Ker}\mathcal{F}$ for $j=1,\cdots,n$, to obtain $T_{\lambda_i}(f_i)\in V_{\lambda_i}$, $i=0,\cdots,n$. Then $\mu_n(f_0,\cdots,f_n)\in\operatorname{Ker}\mathcal{F}$. For $n=1$ this is usually denoted by $f_0\star_{\mu_1} f_1$.

First of all, let us check that $\sum_{i=0}^n\lambda_i = 0$, as it should be. $\sum_{i=0}^n\lambda_i = \sum_{i=0}^n\sum_{j=1}^n\omega^{ij}\mu_j = \sum_{j=1}^n n\delta(j)\mu_j = 0$. We can view the λ_i as Discrete Fourier Transforms of the μ_i, where we take $\mu_0=0$ from the start.

It has the disadvantage that the formulae for the transvectants are relatively complicated, but the advantage that the result is symmetric in the symmetric case.

Let us write the formal expansion out. We take $\omega=-1$.

$$\begin{aligned}
f_0\star_{\mu_1} f_1 &= T_{\mu_1}(f_0)\cdot T_{-\mu_1}(f_1) \\
&= \sum_{r=0}^\infty \frac{\mu_1^r}{r!}\left(\frac{\mathcal{D}}{\mathcal{E}}\right)^r(f_0)\sum_{s=0}^\infty\frac{(-)^s\mu_1^s}{s!}\left(\frac{\mathcal{D}}{\mathcal{E}}\right)^s(f_1) \\
&= \sum_{n=0}^\infty \mu_1^n\sum_{r+s=n}(-1)^s\frac{1}{r!s!}\left(\frac{\mathcal{D}}{\mathcal{E}}\right)^r(f_0)\left(\frac{\mathcal{D}}{\mathcal{E}}\right)^s(f_1) \\
&= \sum_{n=0}^\infty\frac{\mu_1^n}{n!}\tau_n(f_0,f_1)
\end{aligned}$$

with

$$\tau_n(f_0, f_1) \quad = \sum_{r+s=n} (-1)^s \binom{n}{r} \left(\frac{\mathcal{D}}{\mathcal{E}}\right)^r (f_0) \left(\frac{\mathcal{D}}{\mathcal{E}}\right)^s (f_1). \tag{3.1}$$

When $\mathcal{E}(f_i) = f_i, i = 0, 1$, the τ_n are known as **Hirota operators**, and usually written as $\tau_n(f_0, f_1) = \mathbb{D}^n f_0 \cdot f_1$.

In the multilinear case one gets analogous formulae which are labeled by monomials in $\mu_1, \cdots, \mu_n$. If there are symmetries involved, for instance when one takes $f_0 = f_1$, the transvectants are labeled by the corresponding monomials in the invariants of the corresponding action of the (subgroup of the) symmetric group S_n on the μ_i. For instance, when $n = 1$, only the even powers of μ_1 survive, so we label the transvectants by the powers of $c_2 = \mu_1^2$. The invariants of S_n can be produced by considering

$$c_k = \sum_{i=0}^{n} \lambda_i^k, \qquad k > 1.$$

This procedure basically solves the problem of generalizing the Hirota operator [**Hir85, Hir83, Hir82**] to the multilinear case in a natural way, cf. [**GRH94, Hie96, Hie97b, Hie97a**]. It would be nice to have explicit formulae like (3.1) for these higher order transvectants. One may suspect that special functions would be involved [**Ros99**].

4. The Poisson algebra of functionals

One can identify $\operatorname{Ker}\mathcal{F}$ with the space of functionals, usually denoted by $\int f$, since functionals are equivalence classes *mod* $\operatorname{Im}\mathcal{D}$. With this identification we can view functionals as a Poisson algebra by the following lemma.

LEMMA 4.1. $\operatorname{Ker}\mathcal{F}$ *is a Poisson algebra with pointwise multiplication and Lie bracket* $[f, g] = \mathcal{D}(f)\mathcal{E}(g) - \mathcal{E}(f)\mathcal{D}(g)$.

PROOF. The proof is straightforward and relies on the fact that $\mathcal{F}, \mathcal{E}$ and D are derivations. □

5. Classical invariant theory

PROPOSITION 5.1. *Let*

$$\mathcal{D}_n = \sum_{k=0}^{n-1} (n-k) u_{k+1} \frac{\partial}{\partial u_k}.$$

If $f = f(u, u_1, \cdots, u_r) \in \operatorname{Ker}\mathcal{F}$ *then*

$$\sigma_n(f) = \sum_{i=0}^{s} \frac{1}{i!} D_s^i f X^i Y^{r-i}$$

is a covariant for $s \geq r$*, considering the* u_i *as coefficients of the groundform (generated by taking* $f = u$*). In the other direction, if*

$$Q = \sum_{i=0}^{s} \binom{s}{i} Q_i X^i Y^{r-i}$$

is a covariant, then $Q_0 \in \operatorname{Ker}\mathcal{F}$*. Thus the space of covariants projects on the kernel of* $\mathcal{F}$ *by taking* $\pi(Q) = Q_0$*, and there are infinitely many sections* σ_n *to* *this projection from* Ker $\mathcal{F}$ *to the space of covariants, one for every order of the groundform once this order is* $\geq$ *than the order of (that is, the highest derivative in)* f.

PROPOSITION 5.2. *Let*

$$\mathcal{H}_n = \sum_{k=0}^{n} (n-2k) u_k \frac{\partial}{\partial u_k}$$

and

$$\mathcal{F}_n = \sum_{k=1}^{n} k u_{k-1} \frac{\partial}{\partial u_k}.$$

Remark that $\mathcal{F}_n$ *is just* $\mathcal{F}$ *restricted to the polynomial (or formal power series) ring* $\mathcal{R}^n$ *spanned by* $u, \cdots, u_n$ *and that this ring is left invariant under* $\mathcal{F}, \mathcal{H}_n$ *and* $\mathcal{D}_n$. *Then*

$$[\mathcal{F}_n, \mathcal{D}_n] = \mathcal{H}_n, \quad [\mathcal{H}_n, \mathcal{F}_n] = 2\mathcal{F}_n, \quad [\mathcal{H}_n, \mathcal{D}_n] = -2\mathcal{D}_n,$$

that is, $\mathcal{F}_n, \mathcal{D}_n$ *and* $\mathcal{H}_n$ *are a representation of* $\mathfrak{sl}_2$.

PROOF. Completely standard in $\mathfrak{sl}_2$-representation theory. □

6. Rankin-Cohen bilinear operators

We recall the definition of the Rankin-Cohen bracket of two modular forms f and g with weight $w(f)$ and $w(g)$ on some group $\Gamma \subset PSL(2,\mathbb{R})$. The connection between the Rankin-Cohen bracket and transvectants of $\mathfrak{sl}(2,\mathbb{R})$ has been noted and discussed in [**Zag94, Olv99**], see also [**CMZ97, EI98**]. Let $\mathcal{D} = \frac{d}{d\tau}$ and let $f^{(i)} = \mathcal{D}^i f$. The n^{th} Rankin-Cohen bracket of f and g is defined by the formula

$$[f,g]_n(\tau) = \sum_{r+s=n} (-1)^r \binom{n+w(f)-1}{s} \binom{n+w(g)-1}{r} f^{(r)}(\tau) g^{(s)}(\tau).$$

Let us now define for f and g the symbols $\hat{f}$ and $\hat{g}$ and put

$$\hat{f}_i = \frac{1}{\Gamma(w(f)+i)} f^{(i)}.$$

We then see that

$$\begin{aligned} [f,g]_n(\tau) &= \sum_{r+s=n} (-1)^r \binom{n+w(f)-1}{s} \binom{n+w(g)-1}{r} f^{(r)}(\tau) g^{(s)}(\tau) \\ &= \frac{\Gamma(n+w(f))\Gamma(n+w(g))}{n!} \sum_{r+s=n} (-1)^r \binom{n}{r} \hat{f}_r \hat{g}_s \\ &= \Gamma(n+w(f))\Gamma(n+w(g)) \tau_n(\hat{f},\hat{g}) \\ &= \frac{\Gamma(n+w(f))\Gamma(n+w(g))}{\Gamma(2n+w(f)+w(g))} \tau_n(f,g) \end{aligned}$$

where we put $\mathcal{E}(\hat{f}) = \hat{f}$ and $\mathcal{E}(\hat{g}) = \hat{g}$.

REMARK 6.1. *One might be inclined to put* $\mathcal{E}(f) = w(f) f$. *This however makes the commutator of* $\mathcal{E}$ *and* $\mathcal{D}$ *nonzero. This shows the main difference between the*

Heisenberg and the $\mathfrak{sl}_2$ *representations. The last show much more detail, but nevertheless all the information can be recovered from the Heisenberg representation by restriction. Observe that one can define* $\mathcal{H}u_k = ku_k$*, and one then has*

$$[\mathcal{H}, \mathcal{E}] = 0, \qquad [\mathcal{H}, \mathcal{F}] = -\mathcal{F} \qquad [\mathcal{H}, \mathcal{D}] = \mathcal{D}.$$

The operator $\mathcal{H}$ *counts the number of derivatives in an expression. If* P_k^l *denotes the space of* $\mathcal{E}$*–eigenvectors with eigenvalue* l *and* $\mathcal{H}$*–eigenvectors with eigenvalue* k*, then*

$$\tau_n : P_k^l \otimes P_r^s \to P_{k+r+n}^{l+s}.$$

We can now define the Rankin-Cohen bracket as

$$[f, g]_n(\tau) \quad = \tfrac{1}{B(n+w(f), n+w(g))} \tau_n(f, g).$$

7. Intertwining operators

The following is inspired by [**Ath99**]. Let V be a representation space of the Heisenberg algebra. We know that $V = \operatorname{Ker}\mathcal{E} \oplus \operatorname{Ker}\mathcal{F} \oplus \operatorname{Im}\mathcal{D}$. Let $\hat{V} = \operatorname{Ker}\mathcal{F} \oplus \operatorname{Im}\mathcal{D}$. The elements in $\hat{V}$ can be written as sums of terms of the form $\mathcal{D}^i f_\alpha$, with $f_\alpha \in \operatorname{Ker}\mathcal{F}$. Consider now the n-fold tensor product of such $\hat{V}$'s. Let

$$W^n = \hat{V}^{\otimes^n}.$$

Notice that the representation lifts from $\hat{V}$ to W^n by the usual derivation rule. A typical monomial element in W^n looks like

$$(\frac{\mathcal{D}}{\mathcal{E}})^{\iota_1} f_{\alpha_1} \otimes \cdots \otimes (\frac{\mathcal{D}}{\mathcal{E}})^{\iota_n} f_{\alpha_n}, \qquad f_{\alpha_k} \in \operatorname{Ker}\mathcal{F}|\hat{V}.$$

We denote $(\frac{\mathcal{D}}{\mathcal{E}})^{\iota_k} f_{\alpha_k}$ by $f_{\alpha_k}^{\iota_k}$.

Let ω be such that $\omega^r = 1$ and $\omega^j \neq 1$ for $0 < j < r$, with $1 < r \leq n$. Then define for $0 < p < r$,

$$\begin{aligned} S^p_{s_1,\cdots,s_r}\{\cdot\} &= S^p_{s_1,\cdots,s_r} f^{\iota_1}_{\alpha_1} \otimes \cdots \otimes f^{\iota_n}_{\alpha_n} \\ &= \sum_{j=1}^r \omega^{jp} f^{\iota_1}_{\alpha_1} \otimes \cdots \otimes f^{\iota_{s_j}+1}_{\alpha_{s_j}} \otimes \cdots \otimes f^{\iota_n}_{\alpha_n} \\ &= \sum_{j=1}^r \omega^{jp} \{\iota_{s_j} + 1\}, \end{aligned}$$

where we have introduced the shorthand $\{\}$ notation.

Lemma 7.1. *The* $S^p_{s_1,\cdots,s_r}$ *are intertwining operators, that is, they commute with the action of the Heisenberg algebra.*

Proof. The commutator with $\mathcal{E}$ and $\mathcal{D}$ is immediately seen to be zero. The commutator with $\mathcal{F}$ vanishes because

$$\sum_{j=1}^r \omega^{jp} = 0.$$

The computations are left to the reader. □

In the typical case that the representation space $\hat{V}$ is an algebra, we can now apply the multiplication operator to obtain a map $\hat{S}^p_{s_1,\cdots,s_r} : \bigotimes^n \operatorname{Ker}\mathcal{F} \to \operatorname{Ker}\mathcal{F}$.

CONJECTURE 7.2. *The multilinear transvectants can be considered as compositions of the $S^p_{s_1,\cdots,s_n}$ maps and multiplication, cf.* [**Ath99**].

If this is true, then the transvectants themselves, now viewed as maps for W^n to itself, are also intertwining operators. We will show this by taking the λ_i in the definition of the transvectants by the coherent state method to be constant (or, abstractly, invariant under the action of the Heisenberg algebra). One finds that

$$\mathcal{F}T_\lambda = T_\lambda\mathcal{F} + \lambda T_\lambda.$$

Moreover,

$$\mathcal{D}T_\lambda = T_\lambda\mathcal{D}, \qquad \mathcal{E}T_\lambda = T_\lambda\mathcal{E}.$$

Thus

$$\begin{aligned} &\mathcal{F}\cdot(T_{\lambda_0}\otimes\cdots\otimes T_{\lambda_n})(g_0\otimes\cdots\otimes g_n) = \\ &= \mathcal{F}T_{\lambda_0}g_0\otimes\cdots\otimes T_{\lambda_n}g_n + \cdots + T_{\lambda_0}g_0\otimes\cdots\otimes\mathcal{F}T_{\lambda_n}g_n \\ &= T_{\lambda_0}\mathcal{F}g_0\otimes\cdots\otimes T_{\lambda_n}g_n + \lambda_0 T_{\lambda_0}g_0\otimes\cdots\otimes T_{\lambda_n}g_n \\ &\quad + \cdots + T_{\lambda_0}g_0\otimes\cdots\otimes T_{\lambda_n}\mathcal{F}g_n + T_{\lambda_0}g_0\otimes\cdots\otimes\lambda_n T_{\lambda_n}g_n \\ &= (T_{\lambda_0}\otimes\cdots\otimes T_{\lambda_n})(\mathcal{F}g_0\otimes\cdots\otimes g_n + \cdots + g_0\otimes\cdots\otimes\mathcal{F}g_n) \\ &\quad + \sum_{i=0}^n \lambda_i T_{\lambda_0}g_0\otimes\cdots\otimes T_{\lambda_n}g_n \\ &= (T_{\lambda_0}\otimes\cdots\otimes T_{\lambda_n})\mathcal{F}\cdot(g_0\otimes\cdots\otimes g_n), \end{aligned}$$

since $\sum_{i=0}^n \lambda_i = 0$. The computation for $\mathcal{D}$ and $\mathcal{E}$ is similar, but even simpler. This shows that the transvectants generated by the coherent state method are intertwining operators with the representation of the Heisenberg algebra on $W^n = \hat{V}^{\otimes^n}$.

8. The q-Heisenberg algebra

Let us now try to mimic the construction of the bilinear transvectants in the quantum case. Following [**Maj95, Jan96**] we let

$$[q^\mathcal{C},\mathcal{D}] = 0, \qquad [q^\mathcal{C},\mathcal{F}] = 0, \qquad [\mathcal{F},\mathcal{D}] = \mathcal{E} = \frac{q^\mathcal{C} - q^{-\mathcal{C}}}{q - q^{-1}}.$$

Here q is a formal parameter which is supposed to take its values in the ring over which we take the tensor product, i.e. $q^m g_0\otimes g_1 = g_0\otimes q^m g_1$, $m\in\mathbb{Z}$. We then define the comultiplication on the Heisenberg algebra by

$$\begin{aligned} \Delta\mathcal{D} &= \mathcal{D}\otimes 1 + q^\mathcal{C}\otimes\mathcal{D} \\ \Delta\mathcal{F} &= \mathcal{F}\otimes q^{-\mathcal{C}} + 1\otimes\mathcal{F} \\ \Delta q^\mathcal{C} &= q^\mathcal{C}\otimes q^\mathcal{C}. \end{aligned}$$

Observe that in our favourite representation, $q^{\mathcal{C}}u = qu$ and therefore $\mathcal{E}u = u$. We define $T_\lambda = exp(\lambda\frac{\mathcal{D}}{\mathcal{E}})$. We now compute

$$\begin{aligned}\mathcal{F}T_\lambda &= \mathcal{F}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(\frac{\mathcal{D}}{\mathcal{E}})^n \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(\frac{\mathcal{D}}{\mathcal{E}})^n\mathcal{F} + \lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(\frac{\mathcal{D}}{\mathcal{E}})^n \\ &= T_\lambda\mathcal{F} + \lambda T_\lambda.\end{aligned}$$

As before, one has $q^{\mathcal{C}}T_\lambda = T_\lambda q^{\mathcal{C}}$ and $\mathcal{D}T_\lambda = T_\lambda\mathcal{D}$. Then one has, with $\lambda_0 = \mu, \lambda_1 = -q^{-\mathcal{C}}\mu$, μ a formal parameter in the ring over which the tensor product is taken,

$$\begin{aligned}&\Delta\mathcal{F}(T_{\lambda_0}\otimes T_{\lambda_1}) = \\ &= (\mathcal{F}T_{\lambda_0}\otimes q^{-\mathcal{C}}T_{\lambda_1}) + (T_{\lambda_0}\otimes\mathcal{F}T_{\lambda_1}) \\ &= (T_{\lambda_0}\mathcal{F}\otimes q^{-\mathcal{C}}T_{\lambda_1}) + (\lambda_0 T_{\lambda_0}\otimes q^{-\mathcal{C}}T_{\lambda_1}) \\ &\quad +(T_{\lambda_0}\otimes T_{\lambda_1}\mathcal{F}) + (T_{\lambda_0}\otimes\lambda_1 T_{\lambda_1}) \\ &= (T_{\lambda_0}\otimes T_{\lambda_1})(\mathcal{F}\otimes q^{-\mathcal{C}}) + (T_{\lambda_0}\otimes T_{\lambda_1})(\mu\otimes q^{-\mathcal{C}}) \\ &\quad +(T_{\lambda_0}\otimes T_{\lambda_1})(1\otimes\mathcal{F}) - (T_{\lambda_0}\otimes T_{\lambda_1})(1\otimes q^{-\mathcal{C}}\mu) \\ &= (T_{\lambda_0}\otimes T_{\lambda_1})\Delta\mathcal{F}.\end{aligned}$$

Similarly, one shows that $T_{\lambda_0}\otimes T_{\lambda_1}$ commutes with $\Delta\mathcal{D}$ and $\Delta q^{\mathcal{C}}$. We can now compute the first transvectant (cf. [**Lei98**])

$$\tau_1^q(g_0\otimes g_1) = \frac{\mathcal{D}}{\mathcal{E}}g_0\otimes g_1 - g_0\otimes q^{-\mathcal{C}}\frac{\mathcal{D}}{\mathcal{E}}g_1$$

and the second

$$\tau_2^q(g_0\otimes g_1) = \frac{1}{2}\frac{\mathcal{D}^2}{\mathcal{E}^2}g_0\otimes g_1 - \frac{\mathcal{D}}{\mathcal{E}}g_0\otimes q^{-\mathcal{C}}\frac{\mathcal{D}}{\mathcal{E}}g_1 + \frac{1}{2}g_0\otimes q^{-2\mathcal{C}}\frac{\mathcal{D}^2}{\mathcal{E}^2}g_1.$$

If we let $g_0 = g_1 = u$, we obtain

$$\tau_2^q(u\otimes u) = \frac{1}{2}(q^{-2}u\otimes u_2 + u_2\otimes u) - q^{-1}u_1\otimes u_1.$$

Here one defines implicitly:

$$\begin{aligned}\mathcal{E}u_k &= u_k \\ \mathcal{D}u_k &= u_{k+1} \\ \mathcal{F}u_k &= \mathcal{F}\mathcal{D}^k u = \mathcal{D}^k\mathcal{F}u + k\mathcal{E}\mathcal{D}^{k-1}u = ku_{k-1}.\end{aligned}$$

One verifies that

$$\begin{aligned}&\Delta\mathcal{F}\frac{1}{2}(q^{-2}u\otimes u_2 + u_2\otimes u) - \Delta\mathcal{F}q^{-1}u_1\otimes u_1 = \\ &= q^{-2}u\otimes u_1 + q^{-1}u_1\otimes u - q^{-2}u\otimes u_1 - q^{-1}u_1\otimes u \\ &= 0.\end{aligned}$$

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