On Recursion Operators

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Abstract

We observe that application of a recursion operator of Burgers equation does not produce the expected symmetries. This is explained by the incorrect assumption that $D_x^{-1}D_x = 1$. We then proceed to give a method to compute the symmetries using the recursion operator as a first order approximation.

Key words: Recursion operator, cylindrical Korteweg-de Vries equation (cKdV), (co)symmetries

1 Introduction

If we look at the historical development of the theory of integrable systems (with the KdV equation as the most famous example), we see that equations and symmetries, which did not explicitly depend on the time or space variables, were first studied. Based on this assumption a theoretical framework was developed. When later on examples were found of both equations and symmetries with explicit time or space dependency, the theoretical framework was slightly adapted to cover the new situation, but by that time the way things were calculated was already somewhat fixed, and these new features were not analyzed in any fundamental way.

In this paper we first describe some of the difficulties one has applying the results in the literature to explicitly time-dependent situations. These difficulties

[1] J. P. Wang gratefully acknowledges the support from NWO for this research.

came as quite a surprise to us, since they appear at such an elementary level, but they seem to be common to problems related to nonlocal variables (for details see [OSW00]). We then proceed to lay the foundations of a recursion operator formalism in the t-dependent case.

The goal of this paper is not to compute the symmetries of time-dependent equations; for instance one can derive the symmetries of cKdV from those of KdV using the transformation given in [Fuc93]. The point is that we want foolproof algorithms so that we can compute symmetries given any equation and its recursion operator.

To illustrate the problem we consider the well-known (Cf [Olv93, p.315]) Burgers equation

$$u_t = K = u_2 + uu_1 \qquad \left(u_t = \frac{\partial u}{\partial t}, \quad u_i = \frac{\partial^i u}{\partial x^i}\right),$$

which has a time-dependent recursion operator

$$\Re = t(D_x + \frac{1}{2}u + \frac{1}{2}u_1 D_x^{-1}) + \frac{1}{2}x + \frac{1}{2}D_x^{-1}.$$

Applying this operator repeatedly to a symmetry like $1+tu_1$, it is supposed to produce a hierarchy of symmetries. There is, however, one problem with this elementary fact: while $\Re(1+tu_1)$ is indeed a symmetry, $\Re^2(1+tu_1)$ is not!

Let L_K denote the Lie derivative. If Q is an evolutionary vectorfield, $L_K Q$ can be defined as

$$L_K Q = \frac{\partial Q}{\partial t} + D_Q K - D_K Q,$$

and we say that Q is a symmetry if $L_K Q = 0$. Since by definition, $(L_K\Re)Q = L_K(\Re Q) - \Re(L_K Q)$, a sufficient condition for an operator to map a symmetry to a new symmetry is that $L_K\Re = 0$. Thus one often defines a recursion operator by the property that $L_K\Re = 0$. It is easily checked that the above operator satisfies this condition for the Burgers equation. This seems a fairly paradoxical situation until one realizes how the proof that $L_K\Re = 0$ works: in the computation one uses the relation $D_x^{-1}D_x = 1$. But this relation is obviously not true [2], since it is wrong on $\ker D_x$, e.g., $D_x^{-1}D_x t = 0 \neq t$. Instead, we should read $D_x^{-1}D_x = 1 - \Pi$, where Π is the projection on $\ker D_x$ (For example $\Pi(t + t^2 u_1) = t$).

[2] A similar problem was observed by Guthrie [Gut94]. He states that if one only considers constants when applying D_x^{-1} without allowing time–dependent functions, this may lead to what he calls ‘bogus symmetries’.

Let us now compute $L_K\Re$ for the Burgers equation. We find that

$$L_K\Re = \frac{\partial \Re}{\partial t} + D_{\Re}[K] - D_K\Re + \Re D_K = -\frac{1}{2}(1+tu_1)\Pi D_x.$$

The combination ΠD_x is only nonzero on symmetries that contain a term of the form $g(t)x$. If we now compute $\Re(1+tu_1)$, we see that it equals

$$t^2u_2 + t^2uu_1 + tu + tu_1x + x$$

and there is indeed a term of the form $g(t)x$, namely x. This explains why $\Re^2(1+tu_1)$ is not a symmetry.

Having located the source of the problem, the next question is: how to adapt the definition of recursion operator so that it will also handle this case correctly? After giving the necessary definitions in section 2, we answer this question in section 3. In section 4 we explicitly compute the u-independent term of all the symmetries of the cKdV equation generated by the corrected recursion operator.

2 Recursion operators

In order to understand the behavior of recursion operators, it helps to have a good formal setup so that we know clearly in which spaces the objects we study live. Our approach is based on [GD79,Dor93] and is described in detail in [Wan98]. We only use one variable u to keep the notation simple, the generalization to systems being fairly obvious.

Let $\mathcal{C}$ be the space of C^∞-functions of t and $\mathcal{P}$ the space of **local functions**, that is, formal power series $x, u, u_1, \cdots u_k$, for some $k < \infty$ with coefficients in $\mathcal{C}$. We now define the representation space $\mathcal{A}$ as $\mathcal{P}$ divided out by the image of D_x, that is, $\mathcal{P}/\text{im } D_x$, where

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1}\frac{\partial}{\partial u_k}.$$

The equivalence class of a local function $g \in \mathcal{P}$ is called a **functional** and denoted by $\int g$, the idea being that one can do *partial integration*, since

$$0 \equiv \int D_x(gh) = \int hD_xg + \int gD_xh.$$

Next the Lie algebra $\mathfrak{h}$ is defined as follows: Consider the space of vertical vectorfields of the form $\sum_{k=0}^{\infty} f_k \frac{\partial}{\partial u_k}$, $f_k \in \mathcal{P}$. Divide out the image of the Lie derivative of D_x, where the Lie derivative is given by the commutator of the vectorfields. An immediate consequence of this construction is that the elements in $\mathfrak{h}$ commute with D_x and that if $\sum_{k=0}^{\infty} f_k \frac{\partial}{\partial u_k} \in \mathfrak{h}$, then $f_k = D_x^k f_0$. This implies that the elements in the Lie algebra $\mathfrak{h}$ are determined by f_0 and we simply write this as $f_0 \in \mathfrak{h}$. Under this identification the Lie bracket now looks like

$$[k_0, f_0] = D_{f_0}[k_0] - D_{k_0}[f_0],$$

where $D_{f_0}[k_0]$ is the Fréchet derivative

$$D_{f_0}[k_0] = \sum_{k=0}^{\infty} \frac{\partial f_0}{\partial u_k} D_x^k k_0.$$

One associates the evolution equation

$$u_t = K_0$$

with the vectorfield $K = \frac{\partial}{\partial t} + K_0$, where $K_0 \in \mathfrak{h}$. The Lie derivative of the evolution equation on $\mathfrak{h}$ is given by the commutator. One verifies that this defines an action $L_K : \mathfrak{h} \to \mathfrak{h}$ and

$$L_K S = \frac{\partial S}{\partial t} + D_S[K_0] - D_{K_0}[S], \quad S \in \mathfrak{h}. \tag{1}$$

Note that the equation does not live in the same space as its symmetries. One does not see this difference when the symmetries are time–independent.

For any $c(t) \in \mathcal{C}$ and $k_0 \in \mathfrak{h}$, we have $c(t)k_0 \in \mathfrak{h}$ since $D_x c(t) = 0$. Therefore $\mathcal{C}$ is the natural ring of coefficients for $\mathfrak{h}$. We now have a ring of coefficients $\mathcal{C}$, a Lie algebra $\mathfrak{h}$ and a representation space $\mathcal{A}$, with the Lie derivative as the representation. From here on we can construct a Lie algebra complex. Let us give the first steps here, since we do not need the general theory.

We denote the space of $\mathcal{C}$-multilinear n-forms, taking their values in $\mathcal{A}$, by Ω^n. In particular, $\Omega^0 = \mathcal{A}$. Since $L_K : \mathfrak{h} \to \mathfrak{h}$, the action of the Lie derivative of $L_K : \Omega^n \to \Omega^n$ makes sense.

For any functional $\int g \in \Omega^0$, we define $L_K \int g = \int \left(\frac{\partial g}{\partial t} + D_g[K_0] \right)$. We now define and compute the action of L_K on Ω^1. Taking $\omega \in \Omega^1$, for any $f_0 \in \mathfrak{h}$ we have

$$(L_K\omega)(f_0) = L_K\omega(f_0) - \omega(L_K f_0)$$
$$= \int \left(\frac{\partial(\omega \cdot f_0)}{\partial t} + D_{\omega\cdot f_0}[K_0] - \omega \cdot (\frac{\partial f_0}{\partial t} + D_{f_0}[K_0] - D_{K_0}[f_0]) \right)$$
$$= \int \left(\frac{\partial \omega}{\partial t} + D_\omega[K_0] + D^\star_{K_0}[\omega] \right) \cdot f_0,$$

where $D^\star_{K_0}$ is the dual operator of D_{K_0} defined by

$$\int \omega \cdot D_{K_0}[f_0] = \int D^\star_{K_0}[\omega] \cdot f_0, \quad \omega \in \Omega^1, \quad f_0 \in \mathfrak{h}.$$

Since the pairing $\Omega^1 \times \mathfrak{h} \to \mathcal{A}$ is nondegenerate (Cf. [Dor93]), this leads to

$$L_K\omega = \frac{\partial \omega}{\partial t} + D_\omega[K_0] + D^\star_{K_0}[\omega] \tag{2}$$

Definition 1 *Let $u_t = K_0$ be an evolution equation. If a vectorfield $S \in \mathfrak{h}$ (a 1-form $\omega \in \Omega^1$) is invariant under $K = \frac{\partial}{\partial t} + K_0$, $K_0 \in \mathfrak{h}$, i.e., in the kernel of L_K, we call S (ω) a symmetry (cosymmetry) of the equation.*

An operator $\Re : \mathfrak{h} \to \mathfrak{h}$ is called a **recursion operator** if it maps symmetries to new symmetries. If $\Re S$ is well-defined, we know it is a symmetry when $S \in \mathfrak{h}$ is a symmetry and $(L_K\Re)S = 0$ due to the definition of $L_K\Re$, i.e.,

$$L_K(\Re S) = (L_K\Re)S + \Re L_K S.$$

Similarly, an operator $\Im : \Omega^1 \to \Omega^1$ is called a **conjugate recursion operator** if it maps cosymmetries to new cosymmetries.

3 The corrected recursion operator

We call an operator $\hat{\Re} : \mathfrak{h} \to \mathfrak{h}$ a **weak recursion operator** of the equation $u_t = K_0$ if it satisfies that $\hat{L}_K\hat{\Re} = \frac{\partial \hat{\Re}}{\partial t} + D_{\hat{\Re}}[K_0] - D_{K_0}\hat{\Re} + \hat{\Re}D_{K_0} = 0$ using the rule that $D_x^{-1}D_x = 1$. In this section, we give the corrected recursion operator $\Re$ based on a weak recursion operator $\hat{\Re}$.

Definition 2 *We define a projection $\Pi : \mathcal{P} \to \mathcal{C}$ as follows. Given any $f(t, x, u, \cdots, u_k)$ we put $(\Pi f)(t) = f(t, 0, 0, \cdots, 0)$.*

For example, let $f = t\cos(u)$, then $\Pi f = t$, while with $f = t\sin(u)$, $\Pi f = 0$.

Definition 3 *We denote all the terms with the projection operator Π in an operator $\cdot$ by $[[\cdot]]$. For example, we have $[[D_x^{-1}D_x]] = -\Pi$.*

Lemma 4 *For any local function $P \in \text{im } D_x$ and any $K_0 \in \mathfrak{h}$, we have*

$$D_{D_x^{-1}P}[K_0] = D_x^{-1} D_P[K_0] - [[D_x^{-1} D_P]] K_0.$$

PROOF. Let $D_x F = P$. We know $D_{D_x F}[K_0] = D_x D_F[K_0]$. Then

$$\begin{aligned} D_{D_x^{-1}P}[K_0] &= D_{(1-\Pi)F}[K_0] = D_F[K_0] = D_x^{-1} D_P[K_0] + \Pi D_F[K_0] \\ &= D_x^{-1} D_P[K_0] - [[D_x^{-1} D_P]] K_0, \end{aligned}$$

using $D_x^{-1} D_x = 1 - \Pi$ and $D_{\Pi F}[K_0] = 0$.

This lemma implies that $D_{\Re S}[K_0] = D_{\Re}[K_0] S + \Re D_S[K_0] - [[\Re D_S]] K_0$ if $\Re S$ is well defined, where the operator $\Re : \mathfrak{h} \to \mathfrak{h}$ and $S, K_0 \in \mathfrak{h}$.

Lemma 5 *Assume that $\Re S$ is well defined, where $\Re : \mathfrak{h} \to \mathfrak{h}$ is an operator and $S \in \mathfrak{h}$. Then, for $K = \frac{\partial}{\partial t} + K_0$, $K_0 \in \mathfrak{h}$, we have*

$$(L_K \Re) S = \left(\frac{\partial \Re}{\partial t} + D_{\Re}[K_0] - D_{K_0} \Re + \Re D_{K_0} \right) S - [[\Re D_S]] K_0. \tag{3}$$

PROOF. By the definition of the Lie derivative and formula (1), we have

$$\begin{aligned} & (L_K \Re) S = L_K(\Re S) - \Re L_K S \\ = & \frac{\partial (\Re S)}{\partial t} + D_{\Re S}[K_0] - D_{K_0}[\Re S] - \Re \left(\frac{\partial S}{\partial t} + D_S[K_0] - D_{K_0}[S] \right) \\ = & \left(\frac{\partial \Re}{\partial t} + D_{\Re}[K_0] - D_{K_0} \Re + \Re D_{K_0} \right) S - [[\Re D_S]] K_0. \end{aligned}$$

Note that the Lie derivative of an operator depends also on the object it acts on, i.e., it can not be written as some (differential) operator.

Similarly, the formula for the Lie derivative of an operator $\Im : \Omega^1 \to \Omega^1$ is as follows:

Lemma 6 *Assume that $\Im \omega$ is well defined, where $\Im : \Omega^1 \to \Omega^1$ is an operator and $\omega \in \Omega^1$. Then, for $K = \frac{\partial}{\partial t} + K_0$, $K_0 \in \mathfrak{h}$, we have*

$$(L_K \Im) \omega = \left(\frac{\partial \Im}{\partial t} + D_{\Im}[K_0] + D_{K_0}^* \Im - \Im D_{K_0}^* \right) \omega - [[\Im D_\omega]] K_0. \tag{4}$$

If $\Pi D_x^m K_0 = 0$ for all $m \in \mathbb{N}$, i.e., the expression K_0 contains no terms of the form of $c(t)x^j$, $j \in \mathbb{N}$, we have the usual formula for the Lie derivative:

$$L_K \Re = \frac{\partial \Re}{\partial t} + D_{\Re}[K_0] - D_{K_0}\Re + \Re D_{K_0}, \quad \Re : \mathfrak{h} \to \mathfrak{h}. \tag{5}$$

$$L_K \Im = \frac{\partial \Im}{\partial t} + D_{\Im}[K_0] + D_{K_0}^* \Im - \Im D_{K_0}^*, \quad \Im : \Omega^1 \to \Omega^1. \tag{6}$$

Theorem 7 *Consider the equation $u_t = K_0$. Assume that it has a weak recursion operator $\hat{\Re}$ of the form $\sum_{i=0}^n f^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)}$, where Γ is some index set such that the $h^{(j)} \in \mathfrak{h}$ are symmetries and the $\xi^{(j)} \in \Omega^1$ are cosymmetries of the equation. Then*

$$[[\hat{\Re} D_{K_0}]] = -\sum_{j\in\Gamma}\sum_{l\geq 1} h^{(j)} \Pi D_x^{l-1} \xi^{(j)} \frac{\partial K_0}{\partial u_l},$$

$$[[\hat{\Re}^* D_{K_0}^*]] = \sum_{j\in\Gamma}\sum_{l\geq 1} (-1)^l \xi^{(j)} \Pi D_x^{l-1} h^{(j)} \frac{\partial K_0}{\partial u_l}.$$

If S is a symmetry of the equation, the next symmetry is

$$\left(\hat{\Re} - [[\hat{\Re} D_{K_0}]]\Big|_{\Pi=\int^t \Pi}\right) S + [[\hat{\Re} D_S]]\Big|_{\Pi=\int^t \Pi} K_0. \tag{7}$$

If ω is a cosymmetry of the equation, the next cosymmetry is

$$\left(\hat{\Re}^* + [[\hat{\Re}^* D_{K_0}^*]]\Big|_{\Pi=\int^t \Pi}\right) \omega + [[\hat{\Re}^* D_\omega]]\Big|_{\Pi=\int^t \Pi} K_0. \tag{8}$$

By the notation $\Big|_{\Pi=\int^t \Pi}$ we mean the following: we insert before the Π the integration with respect to explicit time t. And we do not suppose that $D_x^{-1} f(t) = f(t) D_x^{-1}$. If we would have allowed D_x^{-1} to commute with elements in $\ker D_x$, the result would depend on where we would have put these elements relative to Π. For instance, if we have $t D_x^{-1}$, this would lead to $t \int^t \mathrm{d}\tau \Pi$, but if we start with $D_x^{-1} t$ this leads to $\int^t \tau \, \mathrm{d}\tau \Pi$.

Corollary 8 *If, moreover, $\Pi D_x^m K_0 = 0$ for all $m \in \mathbb{N}$, the recursion operator reads*

$$\Re = \hat{\Re} + \sum_{j\in\Gamma}\sum_{l\geq 1} h^{(j)} \int^t \Pi D_x^{l-1} \xi^{(j)} \frac{\partial K_0}{\partial u_l} = \hat{\Re} - L_K \hat{\Re}\Big|_{\Pi=\int^t \Pi}.$$

and the conjugate recursion operator reads

$$\Re^* = \hat{\Re}^* + \sum_{j\in\Gamma}\sum_{l\geq 1}(-1)^l \xi^{(j)} \int^t \Pi D_x^{l-1} h^{(j)} \frac{\partial K_0}{\partial u_l} = \hat{\Re}^* - L_K \hat{\Re}^* \Big|_{\Pi=\int^t \Pi} .$$

Note that the correction preserves D_x^{-1} terms. This implies that the usual results of the “recursion operators” $\hat{\Re}$ are still valid, such as the structure of their non-local terms and the statement that $\hat{\Re}^k S$ are local if $\hat{L}_S \hat{\Re} = 0$, cf. [Wan98,Bil93].

This approach to correct recursion operators removes Guthrie’s objection “the ‘obvious’ definitions of $D_x^{-1}(Q)$ do not survive simple coordinate changes”. We remark that the example in [Gut94] does not lead to any obstructions, only illustrates the freedom of choice.

For example, consider the **cylindrical KdV equation (cKdV)**

$$u_t = K_0 = u_3 + uu_1 - \frac{u}{2t}. \tag{9}$$

The operator $\hat{\Re} = t(D_x^2 + \frac{2}{3}u) + \frac{1}{3}x + (\frac{\sqrt{t}u_1}{3} + \frac{1}{6\sqrt{t}})D_x^{-1}\sqrt{t}$ satisfies $\hat{L}_K \hat{\Re} = 0$ and $\hat{L}_K \hat{\Re}^* = 0$, cf. [ZC86]. Since $\Pi D_x^m K_0 = 0$ for all $m \in \mathbb{N}$, by formula (5) and (6), we have

$$L_K \hat{\Re} = -(\frac{\sqrt{t}u_1}{3} + \frac{1}{6\sqrt{t}})\Pi D_x^2 \sqrt{t}; \qquad L_K \hat{\Re}^* = \sqrt{t}\Pi D_x^2 \frac{1}{6\sqrt{t}}.$$

Using the notation $\Big|_{\Pi=\int^t \Pi}$, the recursion operator now reads

$$\Re = \hat{\Re} + (\frac{\sqrt{t}u_1}{3} + \frac{1}{6\sqrt{t}}) \int^t d\tau \sqrt{\tau}\Pi D_x^2$$

and the conjugate recursion operator now reads

$$\Re^* = \hat{\Re}^* - \sqrt{t} \int^t d\tau \frac{1}{6\sqrt{\tau}} \Pi D_x^2.$$

When S is any symmetry of the equation, we see that

$$L_K \left(L_K \hat{\Re} \Big|_{\Pi=\int^t \Pi} \right) S = (L_K \hat{\Re}) S - \left(L_K (\frac{\sqrt{t}u_1}{3} + \frac{1}{6\sqrt{t}}) \right) \int^t d\tau \sqrt{\tau}\Pi D_x^2 S.$$

Table 1
Symmetries of Burgers: $u_t = u_2 + uu_1$

0	$tu_1 + 1$
1	$t^2(u_2 + uu_1) + t(u + u_1 x) + x$
2	$t^3(u_3 + \frac{3}{2}u_1^2 + \frac{3}{2}uu_2 + \frac{3}{4}u_1 u^2) + t^2(\frac{3}{2}xu_2 + \frac{3}{2}xuu_1 + 3u_1 + \frac{3}{4}u^2)$ $+t(\frac{3}{2}xu + \frac{3}{4}x^2 u_1 + \frac{3}{2}) + \frac{3}{4}x^2$
3	$t^4(u_4 + 2u_3 u + \frac{3}{2}u^2 u_2 + 5u_1 u_2 + 3u_1^2 u + \frac{1}{2}u_1 u^3)$ $+t^3(2xu_3 + 3xuu_2 + 6u_1 u + \frac{1}{2}u^3 + \frac{3}{2}xu_1 u^2 + 3xu_1^2 + 5u_2)$ $+t^2(\frac{3}{2}x^2 u_2 + \frac{3}{2}x^2 uu_1 + 6xu_1 + \frac{3}{2}xu^2 + 3u)$ $+t(\frac{1}{2}x^3 u_1 + 3x + \frac{3}{2}x^2 u) + \frac{1}{2}x^3$

Table 2
Symmetries of cKdV: $u_t = u_3 + uu_1 - \frac{u}{2t}$

0	$\sqrt{t}u_1 + \frac{1}{2\sqrt{t}}$
1	$t^{3/2}(u_3 + uu_1) + \frac{\sqrt{t}}{2}(u + xu_1) + \frac{x}{4\sqrt{t}}$
2	$t^{5/2}(u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1 u_2 + \frac{5}{6}u_1 u^2)$ $+t^{3/2}(\frac{5}{6}xu_3 + \frac{5}{3}u_2 + \frac{5}{6}xuu_1 + \frac{5}{12}u^2) + \sqrt{t}(\frac{5}{12}xu + \frac{5}{24}x^2 u_1) + \frac{5x^2}{48\sqrt{t}}$
3	$t^{7/2}\left(u_7 + \frac{7}{3}uu_5 + 7u_1 u_4 + \frac{35}{18}u_3 u^2 + \frac{35}{3}u_3 u_2 + \frac{70}{9}uu_1 u_2\right.$ $\left.+\frac{35}{18}u_1^3 + \frac{35}{54}u_1 u^3\right) + t^{5/2}\left(\frac{7}{6}xu_5 + \frac{7}{2}u_4 + \frac{35}{18}xuu_3 + \frac{35}{9}xu_1 u_2\right.$ $\left.+\frac{35}{9}uu_2 + \frac{35}{12}u_1^2 + \frac{35}{36}xu_1 u^2 + \frac{35}{108}u^3\right)$ $+t^{3/2}(\frac{35}{72}x^2 u_3 + \frac{35}{18}xu_2 + \frac{35}{72}x^2 uu_1 + \frac{35}{24}u_1 + \frac{35}{72}xu^2)$ $+\sqrt{t}(\frac{35}{432}x^3 u_1 + \frac{35}{144}x^2 u + \frac{35}{144}) + \frac{35}{864}\frac{x^3}{\sqrt{t}}$

Since $\frac{\sqrt{t}u_1}{3} + \frac{1}{6\sqrt{t}}$ is a symmetry, this ensures that $(L_K \Re)S = 0$, and thus $\Re$ is the recursion operator. We list some of the lower order symmetries of Burgers and cKdV in table 1 and 2, respectively.

Similarly, we prove that $\Re^*$ is the conjugate recursion operator. We list some of the lower order cosymmetries cKdV in table 3.

Remark 9 *Assume $S_0 \in \mathfrak{h}$ is the starting symmetry of the recursive procedure (7). The correction term only appears when $\hat{\Re}$ is explicitly time–dependent and either $\Pi D_x^m S_0 \neq 0$ or $\Pi D_x^m K_0 \neq 0$ for some m. This explains why it does not happen so often though surprisingly it does for the Burgers equation.*

There exist integrable equations that do not satisfy the condition $\Pi D_x^m K_0 = 0$ for all $m \in \mathbb{N}$ in Corollary 8. In this case, we have to compute their symmetries and cosymmetries one by one according to formula (7) and (8) in Theorem 7.

Table 3
Cosymmetries of cKdV: $u_t = u_3 + uu_1 - \frac{u}{2t}$

0	$\sqrt{t}$
1	$t^{3/2}u + \frac{x}{2}\sqrt{t}$
2	$t^{5/2}(u_2 + \frac{1}{2}u^2) + \frac{x}{2}ut^{3/2} + \frac{x^2}{8}\sqrt{t}$
3	$t^{7/2}\left(u_4 + \frac{5}{3}uu_2 + \frac{5}{6}u_1^2 + \frac{5}{18}u^3\right) + \frac{5}{6}t^{5/2}(xu_2 + u_1 + \frac{1}{2}xu^2)$ $+\frac{5}{24}t^{3/2}(x^2u + 1) + \frac{5}{144}x^3\sqrt{t}$
4	$t^{9/2}\left(u_6 + \frac{7}{3}uu_4 + \frac{14}{3}u_1u_3 + \frac{7}{2}u_2^2 + \frac{35}{18}u^2u_2 + \frac{35}{18}uu_1^2 + \frac{35}{216}u^4\right)$ $+t^{7/2}\left(\frac{7}{6}xu_4 + \frac{7}{3}u_3 + \frac{35}{18}xuu_2 + \frac{35}{36}xu_1^2 + \frac{35}{18}uu_1 + \frac{35}{108}xu^3\right)$ $+t^{5/2}(\frac{35}{72}x^2u_2 + \frac{35}{36}xu_1 + \frac{35}{72}u + \frac{35}{144}x^2u^2) + t^{3/2}(\frac{35}{432}x^3u + \frac{35}{144}x)$ $+\frac{35}{3456}x^4\sqrt{t}$

Table 4
Symmetries of the equation $u_t + u_3 + 6uu_1 + \frac{3u}{t} - \frac{5x}{2t^2} = 0$

0	$t^3u_1 - \frac{1}{2}t^2$
1	$t^9(u_3 + 6uu_1) - t^8(3xu_1 + 3u) + \frac{3}{2}t^7x$
2	$t^{15}\left(u5 + 10uu_3 + 20u_1u_2 + 30u^2u_1\right) - t^{14}\left(5xu_3 + 10u_2 + 30xuu_1 + 15u^2\right)$ $+t^{13}(\frac{15}{2}x^2u_1 + 15xu) - \frac{15}{4}t^{12}x^2$
3	$t^{21}\left(u_7 + 14uu_5 + 42u_1u_4 + 70u_2u_3 + 70u^2u_3 + 280uu_1u_2 + 70u_1^3 + 140u^3u_1\right)$ $-t^{20}(7xu_5 + 21u_4 + 70xuu_3 + 140xu_1u_2 + 140uu_2$ $+105u_1^2 + 210xu^2u_1 + 70u^3)$ $+t^{19}\left(\frac{35}{2}x^2u_3 + 70xu_2 + 105x^2uu_1 + \frac{105}{2}u_1 + 105xu^2\right)$ $-t^{18}\left(\frac{35}{2}x^3u_1 + \frac{105}{2}x^2u + \frac{35}{4}\right) + \frac{35}{4}t^{17}x^3$

The **generalized Korteweg–de Vries equation** [Cho87]

$$u_t + u_3 + 6uu_1 + 6fu - x(f' + 12f^2) = 0, \quad f \in \mathcal{C}$$

is integrable with the weak recursion operator

$$\hat{\Re} = g^2\left(D_x^2 + 4(u - xf)\right) + 2g(u_1 - f)D_x^{-1}g,$$

where $g = e^{\int^t 6f(\tau)d\tau}$. We have $\Pi D_x K_0 \neq 0$ when $f' + 12f^2 \neq 0$, i.e., $f \neq \frac{1}{C+12t}$. Let us take $f = \frac{1}{2t}$. Then the equation becomes $u_t + u_3 + 6uu_1 + \frac{3u}{t} - \frac{5x}{2t^2} = 0$ and $\hat{\Re} = t^6D_x^2 + 4t^6u - 2xt^5 + (2t^3u_1 - t^2)D_x^{-1}t^3$. We list some of lower order symmetries and cosymmetries of this equation in table 4 and 5, respectively.

Table 5
Cosymmetries of the equation $u_t + u_3 + 6uu_1 + \frac{3u}{t} - \frac{5x}{2t^2} = 0$

0	t^3
1	$t^9 u - \frac{1}{2} t^8 x$
2	$t^{15}(u_2 + 3u^2) - 3t^{14} xu + \frac{3}{4} t^{13} x^2$
3	$t^{21}\left(u_4 + 10uu_2 + 5u_1^2 + 10u^3\right) - t^{20}\left(5xu_2 + 5u_1 + 15xu^2\right)$ $+t^{19}(\frac{15}{2}x^2 u + \frac{5}{4}) - \frac{5}{4} t^{18} x^3$
4	$t^{27}\left(u_6 + 14uu_4 + 28u_1u_3 + 21u_2^2 + 70u^2u_2 + 70uu_1^2 + 35u^4\right)$ $-t^{26}\left(7xu_4 + 14u_3 + 70xuu_2 + 35xu_1^2 + 70uu_1 + 70xu^3\right)$ $+t^{25}\left(\frac{35}{2}x^2u_2 + 35xu_1 + \frac{105}{2}x^2u^2 + \frac{35}{2}u\right) - t^{24}\left(\frac{35}{2}x^3u + \frac{35}{4}x\right) + \frac{35}{16}t^{23}x^4$

4 More details on symmetries of cKdV

In this section we want to derive the u-independent part of all symmetries. If by K_0^0 we denote the linear part of the equation $u_t = K_0$, and by f the u-independent part of the symmetry, then f obeys

$$f_t = D_{K_0^0} f.$$

So for cKdV this implies that

$$f_t = f_3 - \frac{f}{2t}.$$

We find that if we try the homogeneous expression $f_{\lambda,j} = \sum_{p=0}^{\infty} a_p^{\lambda,j} x^{3p+j} t^{\lambda-p}$, $j = 0, 1, 2$, this gives us the recursion relation

$$(\lambda + \frac{1}{2} - p) a_p^{\lambda,j} = (3p + j + 1)(3p + j + 2)(3p + j + 3) a_{p+1}^{\lambda,j}.$$

This results in a divergent sequence[3] unless we make it finite by taking $\lambda = n - \frac{1}{2}, n \in \mathbb{N}$. We then obtain:

$$f_{n,j} = \sum_{p=0}^{n} \frac{\binom{n}{p}}{\binom{3p+j}{3p}} \frac{p!}{(3p)!} x^{3p+j} t^{n-p-\frac{1}{2}} b_{n,j},$$

[3] Observe that this is one of the very rare instances that topology plays a role in the otherwise completely formal and algebraic theory of integrable systems. It would be interesting to see whether this is a general phenomenon, that we usually miss because we assume expressions to be polynomial (or formal power series).

where we put $b_{n,j} = a_0^{n-\frac{1}{2},j}$. Thus we obtain

$$\frac{f_{n,j}}{b_{n,j}} = x^j t^{n-1/2} \, {}_2F_3[-n,1; \frac{j+1}{3}, \frac{j+2}{3}, \frac{j+3}{3}; \frac{-x^3}{27\,t}]$$

where ${}_2F_3$ is a generalized hypergeometric function [Olv74]. Then

$$\begin{aligned}
L_K \hat{\Re} f_{n,j} \Big|_{\Pi=\int^t \Pi} &= -(\frac{\sqrt{t}u_1}{3} + \frac{1}{6\sqrt{t}})\Pi \int^t D_x^2 \sqrt{\tau} f_{n,j} \, \mathrm{d}\tau = \\
&= -(\frac{\sqrt{t}u_1}{3} + \frac{1}{6\sqrt{t}})\Pi \int^t D_x^2 \sum_{p=0}^{n} \frac{\binom{n}{p}}{\binom{3p+j}{3p}} \frac{p!}{(3p)!} x^{3p+j} \tau^{n-p} b_{n,j} \, \mathrm{d}\tau \\
&= -(\frac{\sqrt{t}u_1}{3} + \frac{1}{6\sqrt{t}})\Pi \int^t \sum_{p=0}^{n} j(j-1) \frac{\binom{n}{p}}{\binom{3p+j-2}{3p}} \frac{p!}{(3p)!} x^{3p+j-2} \tau^{n-p} b_{n,j} \, \mathrm{d}\tau \\
&= -\frac{2}{n+1} \delta_{j,2} (\frac{\sqrt{t}u_1}{3} + \frac{1}{6\sqrt{t}}) t^{n+1} b_{n,2}.
\end{aligned}$$

Observe that only the first term in the expression for $f_{n,j}$ is used in the answer. This allows us to compute explicitly the correction terms for all orders. The only problem we have is that we usually normalize the symmetries such that the term with the highest derivative and the same homogeneity degree as $f_{n,j}$ looks like

$$t^{3n+j+\frac{1}{2}} u_{6n+2j+1}.$$

Since the recursion operator does not change the coefficient of this term, we can compute $b_{n,j}$ be applying $\Re$ to $f_{n,j}, j = 0,1,2$. We find that

$$\begin{aligned}
\Pi D_x^{j+1} \Re \frac{f_{n,j}}{b_{n,j}} &= \frac{n + \frac{1}{3}j + \frac{1}{2}}{j+1} \Pi D_x^{j+1} \frac{f_{n,j+1}}{b_{n,j+1}}, \quad j = 0,1 \\
\Pi \Re \frac{f_{n,2}}{b_{n,2}} &= (2 + \frac{1}{3}\frac{1}{n+1})\Pi \frac{f_{n+1,0}}{b_{n+1,0}}, \quad j = 2.
\end{aligned}$$

In the second expression the $\frac{1}{3}\frac{1}{n+1}$ comes from the correction term. Thus we should define the $b_{n,j}$ by the recursion relation

$$\begin{aligned}
b_{n,j+1} &= \frac{n + \frac{1}{3}j + \frac{1}{2}}{j+1} b_{n,j}, \quad j = 0,1 \\
b_{n+1,0} &= 2\frac{n + \frac{2}{3}}{n+1} b_{n,2}, \quad j = 2.
\end{aligned}$$

We have $b_{0,0} = \frac{1}{2}$, since the first order symmetry is $\sqrt{t}u_1 + \frac{1}{2\sqrt{t}}$. We find $b_{0,1} = \frac{1}{4}$, $b_{0,2} = \frac{5}{48}$, $b_{1,0} = \frac{35}{144}$ and in general, since

$$b_{n+1,j} = \frac{(n + \frac{3+2j}{6})(n + \frac{5+2j}{6})(n + \frac{7+2j}{6})}{n+1} b_{n,j}, \quad j = 0, 1, 2,$$

we find that

$$b_{n,j} = (\frac{3+2j}{6})_n (\frac{5+2j}{6})_n (\frac{7+2j}{6})_n \frac{b_{0,j}}{n!}.$$

where $(a)_n$ is the Pochhammer symbol $\frac{\Gamma(a+n)}{\Gamma(a)}$. Thus we have

$$\begin{aligned} f_{1,j} &= \sqrt{t}\, x^j \frac{b_{1,j} \left(\binom{3+j}{3} + \frac{x^3}{6t} \right)}{\binom{3+j}{3}} \\ &= \sqrt{t}\, x^j \frac{(3+2j)(5+2j)(7+2j) \left(\binom{3+j}{3} + \frac{x^3}{6t} \right)}{6^3 \binom{3+j}{3}} b_{0,j} \end{aligned}$$

and we see that

$$f_{1,0} = \sqrt{t}\, \frac{35}{144} \left(1 + \frac{x^3}{6t} \right).$$

This agrees with the symmetry with leading term $t^{\frac{7}{2}} u_7$ as given in table 3.

The methods in this section can be used for other equations, and to do similar computations for cosymmetries.

5 Concluding remarks

We have seen the theory of recursion operators is not quite as well settled as we may have believed, but we have shown how the correction can be obtained by a combination of abstract methods and concrete calculations. Although the time–dependent symmetries and recursion operators are not all that common, this has clarified the way things should be defined.

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