ON INTEGRABLE SYSTEMS IN 3-DIMENSIONAL RIEMANNIAN GEOMETRY

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ABSTRACT. In this paper we introduce a new infinite dimensional pencil of Hamiltonian structures. These Poisson tensors appear naturally as the ones governing the evolution of the curvatures of certain flow of curves in three dimensional Riemannian manifolds with constant curvature. The curves themselves are evolving following arc-length preserving geometric evolutions for which the variation of the curve is an invariant combination of the tangent, normal and binormal vectors. Under very natural conditions, the evolution of the curvatures will be Hamiltonian and, in some instances, biHamiltonian and completely integrable.

1. Introduction

The theory of integrable systems has traditionally made use of geometrical concepts and procedures. In particular, the majority of completely integrable PDEs, or systems of PDEs, are connected to the existence of two compatible Hamiltonian structures with respect to which the systems are Hamiltonian. When that happens we call the system a biHamiltonian system. If one of the compatible Hamiltonian structures is nondegenerate, a recursion operator can be defined, which will generate a family of preserved quantities for the flow, effectively integrating the system. The field of Poisson geometry, or geometry of Hamiltonian evolutions, is thus a fundamental part in the study of completely integrable systems.

Recently, a close relationship with traditional differential geometry has appeared for some known Hamiltonian structures. It has been shown ([MB99, MB00b, MB00a]) that completely integrable systems like KdV equations, generalized KdV systems, complexly coupled KdVs, etc, have an underlying geometrical origin. For example, if the initial conditions of a generalized KdV system of PDEs are taken to be differential invariants of a projective parametrized curve (the reader can read projective curvatures instead of differential invariants for a better intuition), a flow of the KdV system is associated to a flow of projective curves that are solutions of an invariant projective evolution. Each dependent variable in the KdV flow will be the differential invariant of the corresponding dependent variable in the flow of curves. And, vice versa, if a flow of projective curves is solution of a certain evolution, invariant under the projective group, their differential invariants follow a KdV generalized system of PDEs. This is true in all dimensions. Some natural conditions similar to the ones needed in this paper are also needed there. We can

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give it an even stronger geometric interpretation when we look at the Hamiltonian structure rather than to the completely integrable system. In fact, the second Poisson structure for generalized KdV (the Adler–Gel'fand–Dikii bracket) is defined simply with the knowledge of an invariant frame along a projective curve and its differential invariants. Thus *the Adler–Gel'fand–Dikii bracket is defined by the projective geometry of parametrized projective curves*. This close relationship between invariant evolutions and the Hamiltonian structures of integrable systems also holds for plane curves under the action of $O(3,1)$, ([MB00a]) and, to some extend, for projective reparametrizations of the projective plane ([MB00b]).

These are instances of parametrized curves or surfaces. The Euclidean plane is one of the examples for which there seems to be no such a connection in the parametrized case. Still, if one considers unparametrized curves (that is, with arc–length parametrization) in the Euclidean plane following an invariant evolution which is arc–length preserving, the resulting evolution of the curvature would be, in fact, Hamiltonian with respect to a nondegenerate operator. Modified KdV is known to be obtained in this way ([LP91]). The situation in Euclidean space (and partially in spheres and the hyperbolic plane) was studied in connection with the vortex filament equation and its relationship to the nonlinear Schrödinger equation. Indeed, the vortex filament system in its various versions are known to be biHamiltonian and a Poisson map (the Hasimoto map) exists from its second Hamiltonian structure to the corresponding one for the nonlinear Schrödinger equation. See [MW83, LP91, DS94, YS98, Cal00] and references therein.

In this note we present a triplet of compatible Hamiltonian structures that arises in this natural way from the geometric arc–length preserving evolution of curves in any given 3–dimensional Riemannian manifold with constant curvature. We arrived at these structures from two different ways. First, one of the authors was asked whether a certain system of PDEs, which is currently being studied by Fels and Ivey (see [Ive01] for an announcement), has a recursion operator. She found this recursion operator and then proceeded to find the Hamiltonian pair, producing this way two of the Poisson tensors presented here, the ones we call $\mathcal{E}$ and $\mathcal{D}$. The other route is described in this paper, where we present the geometrical origin of the above mentioned Hamiltonian pair, as generating the first Hamiltonian structure in a pencil indexed by the curvature of the manifold. We denote the pencil by $\mathcal{E}+\mathcal{D}+\varkappa\mathcal{C}$. The second tensor in the pencil, $\mathcal{C}$, is compatible with the initial two, forming this way a Hamiltonian triplet. Indeed we show that if a flow of curves in a 3–dimensional Riemannian manifold with constant curvature $\varkappa$ follows an arc–length preserving geometric evolution, the evolution of its Riemannian curvatures is always, under natural conditions, a Hamiltonian flow with respect to the element of the pencil corresponding to the value $\varkappa$. [1] The close geometric relationship remains here: the triplet can be obtained solely from the intrinsic geometry of curves on 3–dimensional Riemannian manifolds with constant curvature.

Belonging to the pencil we find two nondegenerate structures, $\mathcal{D}$ and $\mathcal{C}$. We show that, while $\mathcal{E}+\mathcal{D}+\varkappa\mathcal{C}$ and $\mathcal{C}$ are the ones used to integrate the best known versions of the vortex filament equation on constant curvature manifolds, $\mathcal{E}+\mathcal{D}+\varkappa\mathcal{C}$ and $\mathcal{D}$ are used to integrate the system studied by Ivey and its generalizations to manifolds with nonzero constant curvature. Neither system can be written as a

[1] *The study in* [YS98] *suggests that* $\varkappa$ *plays the role of a spectral parameter. But that is another story.*

Hamiltonian with respect to the remaining third element in the triplet, so that one can deduce that $\{\mathcal{E}, \mathcal{D}, \mathcal{C}\}$ is a true triplet of compatible Hamiltonian structures, in the sense that neither of them lies on the hierarchy generated by the other two. The integrability of all the integrable systems associated to the triplet follow from their geometric origin and their special choice of Hamiltonian functional. The vortex filament equations and the system studied by Ivey form in fact the two canonical hierarchies of completely integrable systems related to the 3–dimensional Riemannian geometry of curves. Finally, we describe the effect of the Hasimoto transformation on the Hamiltonian pair associated to the vortex filament flow. As it is already known, the Hasimoto transform is a Poisson map from the vortex filament flow to the nonlinear Schrödinger equation. We then show that the hodographic transformation used by Ivey in this context for the Euclidean case has an role analogous to the Hasimoto transformation for the second hierarchy. The hodographic transformation is a Poisson map which, in the nonflat case, takes the second system in the hierarchy to a system of decoupled modified potential KdV equations. The decoupling does not hold in the flat case, degenerate in that sense. When the curvature of the manifold is zero, as the hodographic transformation is applied to the Hamiltonian pair, the transformed system suddenly lies in the integrable hierarchy of very simple equations. This relation was not at all clear prior to the application of the transformation.

Section 2 introduces all concepts of Riemannian geometry needed. Since the note relates two somehow separated subjects, we found necessary to include definitions that some of the readers might not be too familiar with, while others might find them very basic. We have done so in the simplest possible way including only the necessary concepts. Section 3 contains two theorems: Theorem 2 describes the evolution followed by the Riemannian curvatures of a flow of curves solution of an arc–length preserving geometric evolution. The calculation of this evolution can be carried out in many different ways, as one can see in the Euclidean case in [Cal00] and in the case of spheres in [DS94]. Although the most generalizable way (and also the simplest) would involve the use of Cartans definition of connection, we chose to describe it the way we think to be easier for a reader who is not familiar with Riemannian geometry and its Cartan interpretation. It is a longer procedure but perhaps easier to understand. Theorem 3 shows that the tensor defining the curvature evolution is a pencil of Hamiltonian structures indexed by the curvature of the manifold. We show that the pencil if formed by three compatible Poisson tensors. The proof of Theorem 3 needs many specialized definitions and formulae that we have preferred not to include here, since they are only used in the proof. We refer the reader to [Olv93] for the material needed. In section 4 we study the two canonical evolution equations, their hereditary operators and their integrable hierarchies. We also describe their associated transformation and how they lead to their simplification. The last section contains comments about the implications of this result as well as further open problems.

2. Definitions

2.1. **Definitions in Riemannian Geometry.** In this subsection we present all concepts about Riemannian manifolds that we need to use in the next section. Definitions and notations are mostly as in [Hic65] and [Pet98]. A manifold will be a C^∞–manifold.

Definition 1. (i) A *connection* on a manifold M is an operator ∇ which assigns to two C^∞ vector fields X and Y with domain Ω, a third C^∞ vector field denoted by $\nabla_X Y$ with the same domain Ω, in such a way that the following properties are satisfied

(1) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
(2) $\nabla_{X+W} Y = \nabla_X Y + \nabla_W Y$
(3) $\nabla_{fX} Y = f\nabla_X Y$
(4) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$

for any X, W vectors at $p \in M$, Y, Z smooth fields and f a smooth function defined in a neighborhood of p.

(ii) We say an n–dimensional manifold M is a *Riemannian manifold* if M is endowed with a symmetric and positive definite 2–covariant tensor field $\langle , \rangle$. The tensor $\langle , \rangle$ is called the *Riemannian metric* of the manifold and it allows us to define distances, length, angles, orthogonality, etc, in the natural way. In particular, the *length* of a vector X is defined as

$$|X| = \sqrt{\langle X, X\rangle}$$

The simplest example of a Riemannian manifold is, of course, $\mathbb{R}^n$ with the usual dot product.

(iii) A *Riemannian connection* on a Riemannian manifold M is a connection ∇ on M such that

(5) $\nabla_X Y - \nabla_Y X = [X, Y]$ (the connection has zero *torsion tensor*),
(6) $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$,
for all fields X, Y, Z with a common domain.

The fundamental theorem of Riemannian manifolds states that on any Riemannian manifold there exists a unique Riemannian connection. Riemannian manifolds are thus the natural generalization of Euclidean spaces and the Riemannian connections the natural generalization of covariant (or directional) differentiation.

(iv) The *curvature tensor* of a connection ∇ is a tensor R that assigns to each pair of vectors X, Y at a point p a linear transformation $R(X, Y)$ of the tangent space to p, T_pM, into itself. If one imbedds X, Y and Z in smooth fields about p, $R(X, Y)Z$ is defined via the relation

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z \tag{2.1}$$

(v) The *Riemann–Christoffel curvature tensor* (of type $(0, 4)$) is the 4-covariant tensor

$$K(X, Y, Z, W) = \langle R(Z, W)Y, X\rangle$$

for any X, Y, Z, W tangent vectors at p.

Apart from being tensors (and thus multilinear with respect to $C^\infty(M)$ in all their components) the following is a summary of the best known properties of the curvature tensors.

Theorem 1. *The following relations are true*

(1) $R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$ *(first* Bianchi identity*),*
(2) $\nabla_Z R(X, Y)W + \nabla_X R(Y, Z)W + \nabla_Y R(Z, X)W = 0$ *(second* Bianchi identity*),*

(3) $K(X,Y,Z,W) = -K(Y,X,Z,W) = -K(X,Y,W,Z)$,
(4) $K(X,Y,Z,W) = K(Z,W,X,Y)$.

Finally we give the last group of definitions.

Definition 2. (i) Given two independent vectors X,Y in T_pM, the normalized quadratic form

$$\sec(X,Y) = \frac{K(X,Y,X,Y)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$$

is called *the sectional curvature* of X,Y. It can be easily checked that $\sec(X,Y)$ depends only on the plane π spanned by X and Y, and so the sectional curvature is also called $K(\pi)$, *the Riemannian curvature of the plane section* π.

(ii) A Riemannian manifold M is said to have *constant Riemannian curvature* $\varkappa$ if the Riemannian curvature of all plane sections is the constant $\varkappa$.

(iii) If S is a $(0,r)$ tensor, one can define the *covariant derivative of the tensor* along the vector field X by ensuring that Leibniz rule holds. That is $\nabla_X(S)$ is determined by the relation

$$\nabla_X\left(S(Y_1,\dots,Y_r)\right) = (\nabla_X S)(Y_1,\dots,Y_r) + S(\nabla_X Y_1,\dots,Y_r) + \dots + S(Y_1,\dots,\nabla_X Y_r)$$

to be held by any vectors $Y_1,\dots,Y_r$ in T_pM.

This proposition can be found in [Pet98].

Proposition 1. *The following properties are equivalent:*
(1) $\sec(\pi) = \varkappa$ *for all 2–planes in* T_pM.
(2) $R(X,Y)Z = \varkappa\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y\right)$ *for any* X,Y,Z *in* T_pM.

Corollary 1. *Assume the manifold M has constant Riemannian curvature. Then*
(a) $\nabla_X R = 0$ *along any direction determined by the vector field X. That is, the Riemann curvature tensor is* parallel.
(b) *If Z is orthogonal to X and Y, then $R(X,Y)Z = 0$.*
(c) *If W is orthogonal to X and Y, than $K(W,Z,X,Y) = 0$ for any Z.*

What follows is the description of the existence of a Frenet frame and Frenet formulae for any smooth curve on a Riemannian manifold.

Let $\gamma : U \subset \mathbb{R} \to M$ be a smooth curve on a Riemannian manifold M with Riemannian connection ∇. From now on we will assume that all vector fields are defined on some common open subset of U. Let $V(x)$ be the tangent field at x obtained by differentiation with respect to x (also called the *velocity vector*). We will naturally say that γ is *parametrized by arc-length* whenever $|V(x)| = 1$ for all x in the domain of γ. Assume that γ is nondegenerate, that is, $V(x) \neq 0$ for all $x \in U$. We can then define the first vector in the Frenet frame, the *unit tangent vector*, as

$$\mathbf{e}_1(x) = \frac{V(x)}{|V(x)|}$$

Define the *geodesic curvature* (or first curvature) of γ to be the length of the field $\nabla_{\mathbf{e}_1}\mathbf{e}_1$, that is, $k_1 = |\nabla_{\mathbf{e}_1}\mathbf{e}_1|$.

From property (6) of the Riemannian connection one immediately sees that the vector $\nabla_{\mathbf{e}_1}\mathbf{e}_1$ must be orthogonal to $\mathbf{e}_1$ with respect to the Riemannian metric. In the case for which $k_1(x) \neq 0$ we can define the *first normal* to γ at x to be the unit vector $\mathbf{e}_2(x)$ in the direction of $\nabla_{\mathbf{e}_1}\mathbf{e}_1(x)$, so that

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1 = k_1\mathbf{e}_2.$$

Also using property (6) we see that

$$0 = \langle \nabla_{\mathbf{e}_1}\mathbf{e}_2, \mathbf{e}_1\rangle + \langle \mathbf{e}_2, \nabla_{\mathbf{e}_1}\mathbf{e}_1\rangle = \langle \nabla_{\mathbf{e}_1}\mathbf{e}_2, \mathbf{e}_1\rangle + k_1$$

so that

$$\langle \nabla_{\mathbf{e}_1}\mathbf{e}_2 + k_1\mathbf{e}_1, \mathbf{e}_1\rangle = \langle \nabla_{\mathbf{e}_1}\mathbf{e}_2 + k_1\mathbf{e}_1, \mathbf{e}_2\rangle = 0.$$

We call $k_2 = |\nabla_{\mathbf{e}_1}\mathbf{e}_2 + k_1\mathbf{e}_1|$ the *torsion* of γ (or second curvature).

Whenever $k_2 \neq 0$ we can define *the second normal* to γ to be the unit vector $\mathbf{e}_3$ in the direction of $\nabla_{\mathbf{e}_1}\mathbf{e}_2 + k_1\mathbf{e}_1$, so that

$$\nabla_{\mathbf{e}_1}\mathbf{e}_2 = k_2\mathbf{e}_3 - k_1\mathbf{e}_1.$$

The process above can be continued to define k_3, the third curvature and whenever $k_3 \neq 0$ we can define $\mathbf{e}_4$, the third normal. Etc.

Definition 3. The orthonormal vectors $\mathbf{e}_i$, $i = 1, \dots, n$ are called the *Frenet vectors* or *Frenet frame*. Equations

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= k_1\mathbf{e}_2 \\ \nabla_{\mathbf{e}_1}\mathbf{e}_i &= k_i\mathbf{e}_{i+1} - k_{i-1}\mathbf{e}_{i-1},\ i = 2, \dots, n-1 \\ \nabla_{\mathbf{e}_1}\mathbf{e}_n &= -k_{n-1}\mathbf{e}_{n-1} \end{aligned} \tag{2.2}$$

are called the *Frenet formulae.*

2.2. **Definitions in the Theory of Integrable Systems.** In this subsection we present the concepts about bi-Hamiltonian integrable system we need to use in the next section. Definitions and notations are mostly as in [Olv93]. Another good introduction is [Dor93].

Let $M \subset X \times U$ be an open subset of the space of independent and dependent variables $x = (x^1, \cdots, x^p)$ and $u = (u^1, \cdots, u^q)$. The algebra of differential functions $P[u]$ over M is denoted by $\mathcal{A}$, and its quotient space under the image of the total divergence is the space $\mathcal{F}$ of functionals $\mathcal{P} = \int P dx$. We take the schematic view here, so we won't make any reference to the fact that u^i are actual functions of x living in well-defined function spaces.

For a linear differential operator $\mathcal{D} : \mathcal{A}^q \to \mathcal{A}^q$, which we can think as $q \times q$ matrix differential operator depending on x, u and derivatives of u, we define a bracket on $\mathcal{F}$ as follows:

$$\{\mathcal{P}, \mathcal{Q}\} = \int \delta\mathcal{P} \cdot \mathcal{D}\delta\mathcal{Q}\, dx, \tag{2.3}$$

where $\delta\mathcal{P}$ is variational derivative of functional $\mathcal{P}$ and where by $\cdot$ we denote the usual dot product in $\mathbb{R}^q$.

Definition 4. A linear operator $\mathcal{D} : \mathcal{A}^q \to \mathcal{A}^q$ is called Hamiltonian if the bracket (2.3) satisfies the conditions of skew-symmetry

$$\{\mathcal{P}, \mathcal{Q}\} = -\{\mathcal{Q}, \mathcal{P}\}, \tag{2.4}$$

and Jacobi identity

$$\{\{\mathcal{P},\mathcal{Q}\},\mathcal{S}\}+\{\{\mathcal{S},\mathcal{P}\},\mathcal{Q}\}+\{\{\mathcal{Q},\mathcal{S}\},\mathcal{P}\}=0, \tag{2.5}$$

for all functionals $\mathcal{P},\mathcal{Q},\mathcal{S}\in\mathcal{F}$. The bracket (2.3) is called *Poisson bracket*.

We say two Hamiltonian operators $\mathcal{D}$ and $\mathcal{E}$ form a *Hamiltonian pair* or are *compatible* if every linear combination $a\mathcal{D}+b\mathcal{E}$, $a,b\in\mathbb{R}$ is a Hamiltonian operator. In fact, we only need to check whether $\mathcal{D}+\mathcal{E}$ is a Hamiltonian operator (Lemma 7.20 in [Olv93]) to prove this. We say that three Hamiltonian operators form a *Hamiltonian triplet*, or are compatible, if any two of them are compatible.

An evolution system is a Hamiltonian system if for a Hamiltonian operator $\mathcal{D}$, there exist a functional $\mathcal{H}$, called Hamiltonian, such that

$$u_t=K[u]=\mathcal{D}\delta\mathcal{H},\quad K[u]\in\mathcal{A}^q.$$

If for a Hamiltonian pair $\mathcal{D}$ and $\mathcal{E}$, there exists corresponding Hamiltonian functionals $\mathcal{H}_1$ and $\mathcal{H}_0$ such that

$$u_t=K[u]=\mathcal{D}\delta\mathcal{H}_1=\mathcal{E}\delta\mathcal{H}_0, \tag{2.6}$$

we say the evolution system is a *bi-Hamiltonian system.*

Definition 5. A differential operator $\mathcal{D}:\mathcal{A}^q\to\mathcal{A}^q$ is *degenerate* if there is a nonzero differential operator $\tilde{\mathcal{D}}:\mathcal{A}^q\to\mathcal{A}$ such that $\tilde{\mathcal{D}}\cdot\mathcal{D}=0$.

In the field of nonlinear evolution equations, one important question to answer is whether a given equation is integrable, in the sense that it has infinitely many symmetries.

Definition 6. We say $Q[u]\in\mathcal{A}^q$ is a *symmetry* of $u_t=K[u]$ if and only if

$$[K,Q]=D_Q[K]-D_K[Q]=0,$$

where $D_Q[K]$ is the Fréchet derivative of Q in the direction of K. If

$$D_Q[K]+D_K^\star[Q]=0,$$

then $Q[u]$ is a *cosymmetry* of the equation.

For system (2.6), all variational derivatives of its Hamiltonian functionals are cosymmetries.

Definition 7. A linear differential operator $\mathfrak{R}:\mathcal{A}^q\to\mathcal{A}^q$ is a *recursion operator* of $u_t=K[u]$ if it maps a symmetry to a new symmetry.

It follows that $\mathfrak{R}$ is a recursion operator of $u_t=K[u]$ if and only if

$$\mathfrak{R}_t=[D_K,\mathfrak{R}].$$

Related to the concept of recursion operator is that of *hereditary operator.* An operator $\mathfrak{R}$ is said to be hereditary or Nijenhuis if

$$[\mathfrak{R}X,\mathfrak{R}Y]-\mathfrak{R}[\mathfrak{R}X,Y]-\mathfrak{R}[X,\mathfrak{R}Y]+\mathfrak{R}^2[X,Y]=0,\ \text{for all } X,Y\in dom\ \mathfrak{R}$$

All known recursion operators are hereditary, but one should notice that the definition of hereditary does not need any specific equation, it is a geometric property of the operator defining a structure on the space. Given an Hamiltonian pair $(\mathcal{D},\mathcal{E})$, *one constructs an hereditary operator by taking* $\mathfrak{R}=\mathcal{E}\mathcal{D}^{-1}$ *if* $\mathcal{D}$ *is nondegenerate* (cf. Theorem 7.24 in [Olv93]).

The following is an important property: If X is a symmetry of the hereditary operator $\mathfrak{R}$, then for any k, l,

$$[\mathfrak{R}^k X, \mathfrak{R}^l X] = 0.$$

If there exists $\mathcal{H}_1 \in \mathcal{F}$ such that $\mathfrak{R}^\star \delta\mathcal{H}_0 = \delta\mathcal{H}_1$, then for any $n \in \mathcal{N}$, there exists $\mathcal{H}_n \in \mathcal{F}$ such that $\mathfrak{R}^{\star n}\delta\mathcal{H}_0 = \delta\mathcal{H}_n$. This explains how the infinitely many conserved densities (or Hamiltonians) arise, implying the integrability of the Hamiltonian system.

3. Hamiltonian triplet

In this section we assume that we are working on a 3–dimensional Riemannian manifold M with constant curvature $\varkappa$. We remark that there are obvious generalizations to n–dimensional Riemannian manifolds, but the exact connection with integrable systems needs a further study, which we plan to undertake. There are also generalizations to other homogeneous spaces. Many of these cases are still open.

Theorem 2. *Let $\gamma(x,t)$ be a family of curves on M satisfying a geometric evolution system of equations of the form*

$$\gamma_t = h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3 \tag{3.1}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the Frenet frame of γ, and where h_1, h_2, h_3 are arbitrary smooth functions of the curvatures k_1, k_2 and their derivatives with respect to x. Since we are in 3–dimensional space, from now on we use the notation κ, τ for k_1, k_2.

Assume that x is the arc-length parameter and that evolution (3.1) is arc-length preserving.

Then, the curvatures κ, τ satisfy the evolution

$$\begin{pmatrix} \kappa \\ \tau \end{pmatrix}_t = P \begin{pmatrix} h_3 \\ h_1 \end{pmatrix} \tag{3.2}$$

where, if we denote by D_x the total differentiation operator with respect to x,

$$\begin{aligned} P = &\begin{pmatrix} -\tau D_x - D_x\tau & D_x^2 \frac{1}{\kappa} D_x - \frac{\tau^2}{\kappa} D_x + D_x\kappa \\ D_x \frac{1}{\kappa} D_x^2 - D_x\frac{\tau^2}{\kappa} + \kappa D_x & D_x\left(\frac{\tau}{\kappa^2} D_x + D_x\frac{\tau}{\kappa^2}\right) D_x + \tau D_x + D_x\tau \end{pmatrix} \\ &+ \varkappa \begin{pmatrix} 0 & \frac{1}{\kappa} D_x \\ D_x \frac{1}{\kappa} & 0 \end{pmatrix}. \end{aligned} \tag{3.3}$$

$\varkappa$ denotes the curvature of the manifold.

Proof. A short comment on the calculations to follow: Let us denote by $T = \gamma_t$ and $\mathbf{e}_1 = \gamma_x$, assuming x to be arc-length. These vectors are defined as the pull–forward vectors of the vectors $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, tangent to the domain of γ, through γ. That is, if $\gamma : U \subset \mathbb{R}^2 \to M$, then $\gamma_t = \gamma_* \frac{\partial}{\partial t}$ acting on functions as $\gamma_t(f) = \frac{\partial}{\partial t} f(\gamma(t,x))$; likewise for x. Thus, by applying T or $\mathbf{e}_1$ to functions defined along γ we are indeed taking their derivatives with respect to t or x, respectively. If (3.1) is arc–length preserving, these two vectors will commute since $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ commute and γ_* preserves Lie brackets. The condition needed to guarantee that (3.1) is arc–length preserving will be given below.

In order to find the evolution of the curvatures κ and τ we will also find the evolution of the two first members of the frame, $\mathbf{e}_1$ and $\mathbf{e}_2$.

From property (5) of a Riemannian connection and the fact that t and x differentiation commute, if γ is a solution of (3.1) then

$$\nabla_T \mathbf{e}_1 = \nabla_{\mathbf{e}_1} T = \nabla_{\mathbf{e}_1} \left(h_1 \mathbf{e}_1 + h_2 \mathbf{e}_2 + h_3 \mathbf{e}_3\right). \tag{3.4}$$

Using property (4) of ∇ and Frenet formulae (2.2), evolution (3.4) can be rewritten as

$$\nabla_T \mathbf{e}_1 = (h_1' - \kappa h_2)\mathbf{e}_1 + (h_2' + h_1\kappa - \tau h_3)\mathbf{e}_2 + (h_3' + h_2\tau)\mathbf{e}_3 \tag{3.5}$$

where we denote by $'$ the total differentiation of the functions with respect to x. Equation (3.5) already shows which condition must be satisfied by (3.1) in order to be arc-length preserving. Indeed, it suffices to have $\nabla_T \mathbf{e}_1$ to be orthogonal to $\mathbf{e}_1$ in order to have a constant $|\mathbf{e}_1|$ along the flow of (3.1). This leads to

$$h_2 = \frac{h_1'}{\kappa}, \tag{3.6}$$

(see [Ive01]).

The evolution of κ can be found from here. On one hand we have

$$2\kappa\kappa_t = (\kappa^2)_t = 2\langle \nabla_T(\nabla_{\mathbf{e}_1}\mathbf{e}_1), \nabla_{\mathbf{e}_1}\mathbf{e}_1\rangle = 2\kappa\langle \nabla_T \nabla_{\mathbf{e}_1}\mathbf{e}_1, \mathbf{e}_2\rangle. \tag{3.7}$$

Meanwhile, from the definition of the curvature tensor

$$\nabla_T \nabla_{\mathbf{e}_1}\mathbf{e}_1 = \nabla^2_{\mathbf{e}_1} T + R(T, \mathbf{e}_1)\mathbf{e}_1 \tag{3.8}$$

where R is the curvature tensor of the manifold. Therefore

$$\kappa_t = \langle \nabla_{\mathbf{e}_1}\nabla_T \mathbf{e}_1, \mathbf{e}_2\rangle + K(\mathbf{e}_2, \mathbf{e}_1, T, \mathbf{e}_1) \tag{3.9}$$

where K is the Riemann–Christoffel curvature tensor of M. On the other hand, we can rewrite the application of $\nabla_{\mathbf{e}_1}$ to (3.5) as

$$\begin{aligned}\nabla_{\mathbf{e}_1}\nabla_T \mathbf{e}_1 = \nabla^2_{\mathbf{e}_1} T = {} & \left[(h_1' - \kappa h_2)' - \kappa(h_2' + h_1\kappa - \tau h_3)\right]\mathbf{e}_1 \\ & + \left[(h_2' + h_1\kappa - \tau h_3)' + \kappa(h_1' - \kappa h_2) - \tau(h_3' + h_2\tau)\right]\mathbf{e}_2 \\ & + \left[(h_3' + h_2\tau)' + \tau(h_2' + h_1\kappa - \tau h_3)\right]\mathbf{e}_3\end{aligned} \tag{3.10}$$

with the simple use of the Frenet formulae. From these two relations we obtain the evolution of κ as given by

$$\kappa_t = (h_2' + \kappa h_1 - \tau h_3)' + \kappa(h_1' - \kappa h_2) - \tau(h_3' + \tau h_2) + K(\mathbf{e}_2, \mathbf{e}_1, T, \mathbf{e}_1). \tag{3.11}$$

Applying the tensorial properties of K and Theorem 1, we obtain

$$\begin{aligned}\kappa_t = {} & (h_2' + \kappa h_1 - \tau h_3)' + \kappa(h_1' - \kappa h_2) - \tau(h_3' + \tau h_2) \\ & + h_2 K(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1) + h_3 K(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1),\end{aligned} \tag{3.12}$$

We can now find the evolution of $\mathbf{e}_2$ from the Frenet relationship

$$\mathbf{e}_2 = \frac{1}{\kappa}\nabla_{\mathbf{e}_1}\mathbf{e}_1. \tag{3.13}$$

Indeed, if we apply ∇_T to (3.13), and we use that the result should be orthogonal to $\mathbf{e}_2$ (since it is a unit vector), this leads to

$$\nabla_T \mathbf{e}_2 = \frac{1}{\kappa}\nabla_T \nabla_{\mathbf{e}_1}\mathbf{e}_1 - \frac{1}{\kappa}\langle \nabla_T \nabla_{\mathbf{e}_1}\mathbf{e}_1, \mathbf{e}_2\rangle \mathbf{e}_2. \tag{3.14}$$

Substituting (3.8) into it, we obtain

$$\nabla_T \mathbf{e}_2 = \frac{1}{\kappa}\nabla^2_{\mathbf{e}_1} T + \frac{1}{\kappa} R(T, \mathbf{e}_1)\mathbf{e}_1 - \frac{1}{\kappa}\langle \nabla^2_{\mathbf{e}_1} T, \mathbf{e}_2\rangle \mathbf{e}_2 - \frac{1}{\kappa} K(\mathbf{e}_2, \mathbf{e}_1, T, \mathbf{e}_1)\mathbf{e}_2. \tag{3.15}$$

We are now in position to find the evolution of τ. As we did in (3.7) for κ, it is very simple to see that

$$\tau_t = \langle \nabla_T \left(\nabla_{\mathbf{e}_1} \mathbf{e}_2 + \kappa \mathbf{e}_1 \right), \mathbf{e}_3 \rangle. \tag{3.16}$$

But

$$\nabla_T \nabla_{\mathbf{e}_1} \mathbf{e}_2 = \nabla_{\mathbf{e}_1} \nabla_T \mathbf{e}_2 + R(T, \mathbf{e}_1)\mathbf{e}_2$$

so that, applying $\nabla_{\mathbf{e}_1}$ to (3.15) we obtain, after some short calculations

$$\langle \nabla_{\mathbf{e}_1} \nabla_T \mathbf{e}_2, \mathbf{e}_3 \rangle = \left(\frac{1}{\kappa} \langle \nabla^2_{\mathbf{e}_1} T, \mathbf{e}_3 \rangle \right)' + \left(\frac{1}{\kappa} K(\mathbf{e}_3, \mathbf{e}_1, T, \mathbf{e}_1) \right)' \tag{3.17}$$

and from here we obtain

$$\begin{aligned} \tau_t = & \left[\frac{1}{\kappa}(h_3' + h_2 \tau)' + \frac{\tau}{\kappa}(h_2' + h_1 \kappa - \tau h_3) \right]' + \kappa (h_3' + h_2 \tau) \\ & + \left[\frac{h_2}{\kappa} K(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1) + \frac{h_3}{\kappa} \sec(\mathbf{e}_1, \mathbf{e}_3) \right]' \\ & + h_2 K(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1) + h_3 K(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1). \end{aligned} \tag{3.18}$$

If we finally impose arc-length preserving condition (3.6) to evolutions (3.11) and (3.18), we obtain that the evolution of κ and τ can be written as

$$\begin{pmatrix} \kappa \\ \tau \end{pmatrix}_t = \hat{P} \begin{pmatrix} h_3 \\ h_1 \end{pmatrix}$$

where, if D_x is the total derivative with respect to x, and we denote $K_{ijkr} = K(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_r)$,

$$\begin{aligned} \hat{P} = & \begin{pmatrix} -\tau D_x - D_x \tau & D_x^2 \frac{1}{\kappa} D_x - \frac{\tau^2}{\kappa} D_x + D_x \kappa \\ D_x \frac{1}{\kappa} D_x^2 - D_x \frac{\tau^2}{\kappa} + \kappa D_x & D_x \left(\frac{\tau}{\kappa^2} D_x + D_x \frac{\tau}{\kappa^2} \right) D_x + \tau D_x + D_x \tau \end{pmatrix} \\ & + \begin{pmatrix} K_{2131} & \frac{1}{\kappa} \sec(\mathbf{e}_1, \mathbf{e}_2) D_x \\ K_{3231} + D_x \frac{1}{\kappa} \sec(\mathbf{e}_1, \mathbf{e}_3) & \frac{1}{\kappa} K_{3221} D_x + D_x \frac{K_{3121}}{\kappa^2} D_x \end{pmatrix}. \end{aligned} \tag{3.19}$$

If the manifold M has constant curvature $\varkappa$, Proposition 1 and Corollary 1 provide the values

$$K(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1) = K(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2) = K(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3) = 0$$

and if $i \neq j$

$$\sec(\mathbf{e}_i, \mathbf{e}_j) = \varkappa$$

for any $i, j = 1, \ldots, 3$. We simply need to substitute the values in (3.19), to obtain the result of this theorem. □

The general scheme to obtain the evolution equations runs as follows. Let us define $k_0 = 0$. Then we use the following formulae to inductively compute all derivatives with respect to t.

$$\begin{aligned} \nabla_T \mathbf{e}_i = & \\ = & \begin{cases} \nabla_{\mathbf{e}_1} T & \text{if } i = 1 \\ (\nabla_{\mathbf{e}_1} \nabla_T \mathbf{e}_{i-1} - k_{i-1,t} \mathbf{e}_i + k_{i-2,t} \mathbf{e}_{i-2} & \\ + k_{i-2} \nabla_T \mathbf{e}_{i-2} + R(T, \mathbf{e}_1) \mathbf{e}_{i-1}) / k_{i-1} & \text{if } i > 1 \end{cases} \end{aligned}$$

and

$$k_{it} = \langle \nabla_{\mathbf{e}_1} \nabla_T \mathbf{e}_i + k_{i-1} \nabla_T \mathbf{e}_{i-1}, \mathbf{e}_{i+1} \rangle + K(\mathbf{e}_{i+1}, \mathbf{e}_i, T, \mathbf{e}_1).$$

We now turn back to our 3-dimensional problem.

Theorem 3. *The skew-symmetric operators*

$$\mathcal{C} = \begin{pmatrix} 0 & \frac{1}{\kappa}D_x \\ D_x\frac{1}{\kappa} & 0 \end{pmatrix} \tag{3.20}$$

$$\mathcal{D} = \begin{pmatrix} 0 & D_x\kappa \\ \kappa D_x & \tau D_x + D_x\tau \end{pmatrix} \tag{3.21}$$

$$\mathcal{E} = \begin{pmatrix} -\tau D_x - D_x\tau & D_x^2\frac{1}{\kappa}D_x - \frac{\tau^2}{\kappa}D_x \\ D_x\frac{1}{\kappa}D_x^2 - D_x\frac{\tau^2}{\kappa} & D_x(\frac{\tau}{\kappa^2}D_x + D_x\frac{\tau}{\kappa^2})D_x \end{pmatrix} \tag{3.22}$$

form a triplet of compatible Hamiltonian operators (where $P = \mathcal{E} + \mathcal{D} + \varkappa\mathcal{C}$, cf. (3.3)).

Proof. We prove this by checking the conditions of Theorem 7.8 and Corollary 7.21 in [Olv93]. Here we only give the details to show that $\mathcal{D}$ and $\mathcal{E}$ are compatible and leave the remaining, identical computations to the reader.

First we check the operators $\mathcal{D}$ and $\mathcal{E}$ are indeed Hamiltonian operators. The associated bi-vector of $\mathcal{D}$ is

$$\begin{aligned} \Theta_{\mathcal{D}} &= \frac{1}{2}\int (\theta \wedge \mathcal{D}\theta)\,dx \\ &= \frac{1}{2}\int (\kappa\vartheta \wedge \zeta_1 + \kappa_1\vartheta \wedge \zeta + \kappa\zeta \wedge \vartheta_1 + 2\tau\zeta \wedge \zeta_1)\,dx \\ &= \int (\kappa\zeta \wedge \vartheta_1 + \tau\zeta \wedge \zeta_1)\,dx, \end{aligned}$$

where $\theta = (\vartheta, \zeta)$. We need to check the vanishing of

$$\begin{aligned} \operatorname{Pr} V_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) &= \int ((\kappa_1\zeta + \kappa\zeta_1) \wedge \zeta \wedge \vartheta_1 + (\kappa\vartheta_1 + 2\tau\zeta_1 + \tau_1\zeta) \wedge \zeta \wedge \zeta_1)\,dx \\ &= \int (\kappa\zeta_1 \wedge \zeta \wedge \vartheta_1 + \kappa\vartheta_1 \wedge \zeta \wedge \zeta_1)\,dx \\ &= 0. \end{aligned}$$

The proof that $\mathcal{E}$ is Hamiltonian is a rather laborious calculation. Its associated bi-vector is

$$\begin{aligned} \Theta_{\mathcal{E}} &= \frac{1}{2}\int (\theta \wedge \mathcal{E}\theta)\,dx \\ &= \frac{1}{2}\int \Big(-2\tau\vartheta \wedge \vartheta_1 + \vartheta \wedge D_x^2\frac{\zeta_1}{\kappa} - \frac{\tau^2}{\kappa}\vartheta \wedge \zeta_1 \\ &\qquad + \zeta \wedge D_x\frac{\vartheta_2}{\kappa} - \zeta \wedge D_x\frac{\tau^2\vartheta}{\kappa} + \zeta \wedge D_x\frac{\tau\zeta_2}{\kappa^2} + \zeta \wedge D_x^2\frac{\tau\zeta_1}{\kappa^2}\Big)\,dx \\ &= \int \Big(\tau\vartheta_1 \wedge \vartheta + \frac{1}{\kappa}\vartheta_2 \wedge \zeta_1 + \frac{\tau^2}{\kappa}\zeta_1 \wedge \vartheta + \frac{\tau}{\kappa^2}\zeta_2 \wedge \zeta_1\Big)\,dx. \end{aligned}$$

The vanishing of $\Pr V_{\mathcal{E}\theta}(\Theta_{\mathcal{E}})$ can be proved by the fact that

$$\begin{aligned}
\Pr V_{\mathcal{E}\theta}(\kappa) &= -2\tau\vartheta_1 - \tau_1\vartheta + \frac{1}{\kappa}\zeta_3 - \frac{2\kappa_1}{\kappa^2}\zeta_2 + (\frac{2\kappa_1^2}{\kappa^3} - \frac{\kappa_2}{\kappa^2} - \frac{\tau^2}{\kappa})\zeta_1, \\
\Pr V_{\mathcal{E}\theta}(\tau) &= \frac{1}{\kappa}\vartheta_3 - \frac{\kappa_1}{\kappa^2}\vartheta_2 - \frac{\tau^2}{\kappa}\vartheta_1 + (\frac{\tau^2\kappa_1}{\kappa^2} - \frac{2\tau\tau_1}{\kappa})\vartheta \\
&\quad + \frac{2\tau}{\kappa^2}\zeta_3 + (\frac{3\tau_1}{\kappa^2} - \frac{6\tau\kappa_1}{\kappa^3})\zeta_2 + (\frac{\tau_2}{\kappa^2} - \frac{4\kappa_1\tau_1}{\kappa^3} - \frac{2\tau\kappa_2}{\kappa^3} + \frac{6\tau\kappa_1^2}{\kappa^4})\zeta_1
\end{aligned}$$

and integration by parts.

Now we prove that $\mathcal{D}$ and $\mathcal{E}$ form a Hamiltonian pair by checking

$$\Pr V_{\mathcal{D}\theta}(\Theta_{\mathcal{E}}) + \Pr V_{\mathcal{E}\theta}(\Theta_{\mathcal{D}}) = 0.$$

We compute

$$\begin{aligned}
&\Pr V_{\mathcal{D}\theta}(\Theta_{\mathcal{E}}) = \int \Big((2\tau\zeta_1 + \tau_1\zeta) \wedge \vartheta_1 \wedge \vartheta - \frac{\kappa_1}{\kappa^2}\zeta \wedge \vartheta_2 \wedge \zeta_1 \\
&+ \frac{2\tau}{\kappa}(\kappa\vartheta_1 + \tau_1\zeta) \wedge \zeta_1 \wedge \vartheta - \frac{\tau^2\kappa_1}{\kappa^2}\zeta \wedge \zeta_1 \wedge \vartheta \\
&+ \frac{1}{\kappa^2}(\kappa\vartheta_1 + \tau_1\zeta) \wedge \zeta_2 \wedge \zeta_1 - \frac{2\tau\kappa_1}{\kappa^3}\zeta \wedge \zeta_2 \wedge \zeta_1 \Big) dx \\
&= \int \Big(-\tau_1\vartheta \wedge \vartheta_1 \wedge \zeta - \frac{\kappa_1}{\kappa^2}\vartheta_1 \wedge \zeta \wedge \zeta_2 - (\frac{\kappa_2}{\kappa^2} - \frac{2\kappa_1^2}{\kappa^3})\vartheta_1 \wedge \zeta \wedge \zeta_1 \\
&+ (\frac{2\tau\tau_1}{\kappa} - \frac{\tau^2\kappa_1}{\kappa^2})\vartheta \wedge \zeta_1 \wedge \zeta - \frac{1}{\kappa}\vartheta_1 \wedge \zeta_1 \wedge \zeta_2 - (\frac{\tau_1}{\kappa^2} - \frac{2\tau\kappa_1}{\kappa^3})\zeta \wedge \zeta_1 \wedge \zeta_2 \Big) dx,
\end{aligned}$$

and

$$\begin{aligned}
&\Pr V_{\mathcal{E}\theta}(\Theta_{\mathcal{D}}) = \int \Big((-\tau_1\vartheta + \frac{1}{\kappa}\zeta_3 - \frac{2\kappa_1}{\kappa^2}\zeta_2 + (\frac{2\kappa_1^2}{\kappa^3} - \frac{\kappa_2}{\kappa^2} - \frac{\tau^2}{\kappa})\zeta_1) \wedge \zeta \wedge \vartheta_1 \\
&+ (\frac{1}{\kappa}\vartheta_3 - \frac{\kappa_1}{\kappa^2}\vartheta_2 - \frac{\tau^2}{\kappa}\vartheta_1 + (\frac{\tau^2\kappa_1}{\kappa^2} - \frac{2\tau\tau_1}{\kappa})\vartheta + \frac{2\tau}{\kappa^2}\zeta_3 + (\frac{3\tau_1}{\kappa^2} - \frac{6\tau\kappa_1}{\kappa^3})\zeta_2) \wedge \zeta \wedge \zeta_1 \Big) dx \\
&= \int \Big(\tau_1\vartheta \wedge \vartheta_1 \wedge \zeta + \frac{1}{\kappa}\vartheta_1 \wedge \zeta_1 \wedge \zeta_2 + \frac{\kappa_1}{\kappa^2}\vartheta_1 \wedge \zeta \wedge \zeta_2 - (\frac{2\kappa_1^2}{\kappa^3} - \frac{\kappa_2}{\kappa^2})\vartheta_1 \wedge \zeta \wedge \zeta_1 \\
&+ (\frac{\tau^2\kappa_1}{\kappa^2} - \frac{2\tau\tau_1}{\kappa})\vartheta \wedge \zeta \wedge \zeta_1 + (\frac{\tau_1}{\kappa^2} - \frac{2\tau\kappa_1}{\kappa^3})\zeta \wedge \zeta_1 \wedge \zeta_2 \Big) dx.
\end{aligned}$$

Thus the result follows. □

4. Integrable evolutions

4.1. **Two integrable canonical evolution equations.** Having a Hamiltonian triplet, with two of their members, $\mathcal{C}$ and $\mathcal{D}$, being nondegenerate, allows us to produce two hereditary operators. We insist again on the fact that these are tensors linked to the intrinsic geometry of Riemannian curves and not to any integrable system in particular. On the other hand, they are indeed recursion operators for two canonical integrable systems.

The vortex filament flow

The Hamiltonian pair $P = \mathcal{E} + \mathcal{D} + \varkappa\mathcal{C}$ and $\mathcal{C}$ gives us the hereditary (Nijenhuis) operator:

$$\begin{aligned}\mathfrak{R}_1 &= P\mathcal{C}^{-1}\\ &= \begin{pmatrix} D_x^2 - \tau^2 + \kappa^2 + \varkappa & -2\kappa\tau \\ 2D_x^2\frac{\tau}{\kappa} - D_x(\frac{\tau_1}{\kappa} - \frac{2\kappa_1\tau}{\kappa^2}) + 2\kappa\tau & D_x^2 + 2D_x\frac{\kappa_1}{\kappa} + \frac{\kappa_2}{\kappa} - \tau^2 + \kappa^2 + \varkappa \end{pmatrix}\\ &+ \begin{pmatrix} -2\tau\kappa_1 - \kappa\tau_1 \\ \frac{\kappa_3}{\kappa} - \frac{\kappa_1\kappa_2}{\kappa^2} - 2\tau\tau_1 + \kappa\kappa_1 \end{pmatrix} D_x^{-1}\,(0 \quad 1)\\ &+ \begin{pmatrix} \kappa_1 \\ \tau_1 \end{pmatrix} D_x^{-1}\,(\kappa \quad 0)\,.\end{aligned}$$

From its expression one can identify equations

$$\left\{ \begin{array}{l} \kappa_{t_1} = -2\tau\kappa_1 - \kappa\tau_1, \\ \tau_{t_1} = \frac{\kappa_3}{\kappa} - \frac{\kappa_1\kappa_2}{\kappa^2} - 2\tau\tau_1 + \kappa\kappa_1 \end{array}\right. \tag{4.1}$$

and

$$\left\{ \begin{array}{l} \kappa_{t_1'} = \kappa_3 + \frac{3}{2}\kappa^2\kappa_1 - 3\kappa_1\tau^2 - 3\kappa\tau\tau_1 + \varkappa\kappa_1, \\ \tau_{t_1'} = D_x(\tau_2 + \frac{3}{2}\kappa^2\tau - \tau^3 + 3\frac{\kappa_2\tau}{\kappa} + 3\frac{\kappa_1\tau_1}{\kappa} + \varkappa\tau), \end{array}\right. \tag{4.2}$$

that is, $\mathfrak{R}_1\begin{pmatrix}\kappa_1\\ \tau_1\end{pmatrix}$, as symmetries of the operator $\mathfrak{R}_1$. One can also identify their cosymmetries to be $(\kappa, 0)$ and $(0, 1)$, both of them deriving from conservation laws with densities $\frac{\kappa^2}{2}$ and τ, respectively. These (commuting) equations have the operator $\mathfrak{R}_1$ as recursion operator ([SW00]), which generates a hierarchy of symmetries, cosymmetries and conservation laws for the equations. One can think of it as a flow in two simultaneous directions: we start with an arbitrary (closed) curve, we can now flow out this curve along both equations and use t_1 and t_1' as canonical local coordinates besides x to describe the 3–dimensional Riemannian manifold.

Equations (4.1) and (4.2) are the evolutions of curvature and torsion associated to the best known versions of the vortex filament equations. Vortex filament equations are the nonlinear evolution of curves describing the time development of a very thin vortex tube. The ones associated to our integrable systems are

$$\gamma_t = \kappa\mathbf{e}_3; \tag{4.3}$$

$$\gamma_t = \frac{1}{2}\kappa^2\mathbf{e}_1 + \kappa'\mathbf{e}_2 + \kappa\tau\mathbf{e}_3. \tag{4.4}$$

In the Euclidean case, the geometry and integrability of (4.3) and some generalizations have been extensively studied in [LP91, DS94] and [YS98]. The curve evolution (4.4) and its relation to (4.3) has also been studied in [LP91].

Evolution equation (4.1) is easily seen to be a bi-Hamiltonian system since

$$u_{t_1} = \mathcal{C}\delta\mathcal{H}_1 = (\mathcal{E} + \mathcal{D} + \varkappa\mathcal{C})\delta\mathcal{H}_0,$$

where $u = (\kappa, \tau)$, $\mathcal{H}_0 = \int \frac{\kappa^2}{2}dx$ and $\mathcal{H}_1 = \int \left(\frac{1}{8}\kappa^4 - \frac{1}{2}\kappa_1^2 - \frac{1}{2}\tau^2\kappa^2\right)dx$. In fact, notice that $\mathcal{H}_0$ is in the kernel of $\mathcal{C}$. This fact allows us to drop $\varkappa\mathcal{C}$ from the Hamiltonian pair of this particular equation and to consider the simpler recursion operator $(\mathcal{E} + \mathcal{D})\mathcal{C}^{-1}$, the corresponding to the flat case, as valid for the general case. This simplification does not hold for the second evolution

Equation (4.2) is also bi-Hamiltonian since

$$u_{t'_1} = \mathcal{C}\delta\mathcal{H}_3 = (\mathcal{E} + \mathcal{D} + \varkappa\mathcal{C})\delta\mathcal{H}_2,$$

with $\mathcal{H}_2 = \int \frac{1}{2}\kappa^2\tau dx$ and $\mathcal{H}_3 = \int \left(\kappa\kappa_2\tau - \frac{1}{2}\kappa_1^2\tau - \frac{1}{2}\tau^3\kappa^2 + \frac{3}{8}\kappa^4\tau\right) dx + \varkappa\mathcal{H}_2$.

Therefore, evolutions (4.1) and (4.2) are completely integrable systems. Geometric evolutions (4.3) and (4.4) would also be integrable in the sense that their associated κ, τ evolutions are, and given that $\kappa(t,x), \tau(t,x)$ determines $u(t,x)$ up to the action of the group.

After minor calculations one can easily see that the operator $\mathcal{D}$ is not a Hamiltonian operator for either of these two equations. This implies that $\mathcal{D}$ *is not in the hierarchy generated by* $\mathcal{C}$ *and* $\mathcal{E}$.

A second integrable system

Since the operator $\mathcal{D}$ is also nondegenerate, the different choice of Hamiltonian pair $\mathcal{J} = \mathcal{E} + \varkappa\mathcal{C}$ and $\mathcal{D}$ (we will see shortly the reason why we drop $\mathcal{D}$ in this pair) gives us the hereditary operator:

$$\begin{aligned}
\mathfrak{R}_2 &= \mathcal{J}\mathcal{D}^{-1} \\
&= \begin{pmatrix} \frac{1}{\kappa^2}D_x^2 - \frac{5\kappa_1}{\kappa^3}D_x - \frac{4\kappa_2}{\kappa^3} + \frac{12\kappa_1^2}{\kappa^4} + \frac{3\tau^2}{\kappa^2} & -\frac{2\tau}{\kappa} \\ -\frac{2\tau_1}{\kappa^3}D_x + \frac{2\tau^3}{\kappa^3} - \frac{3\tau_2}{\kappa^3} + \frac{9\kappa_1\tau_1}{\kappa^4} & \frac{1}{\kappa^2}D_x^2 - \frac{3\kappa_1}{\kappa^3}D_x - \frac{\kappa_2}{\kappa^3} + \frac{3\kappa_1^2}{\kappa^4} - \frac{\tau^2}{\kappa^2} \end{pmatrix} \\
&\quad + \varkappa\begin{pmatrix} \frac{1}{\kappa^2} & 0 \\ -2\frac{\tau}{\kappa^3} & \frac{1}{\kappa^2} \end{pmatrix} \\
&\quad - \begin{pmatrix} \frac{\kappa_3}{\kappa^3} - 9\frac{\kappa_1\kappa_2}{\kappa^4} + 12\frac{\kappa_1^3}{\kappa^5} - 3\frac{\tau\tau_1}{\kappa^2} + 3\frac{\tau^2\kappa_1}{\kappa^3} + \varkappa\frac{\kappa_1}{\kappa^3} \\ \frac{\tau_3}{\kappa^3} - 6\frac{\kappa_1\tau_2}{\kappa^4} - 3\frac{\kappa_2\tau_1}{\kappa^4} - 3\frac{\tau^2\tau_1}{\kappa^3} + 3\frac{\kappa_1\tau^3}{\kappa^4} + 12\frac{\kappa_1^2\tau_1}{\kappa^5} + \varkappa\left(\frac{\tau}{\kappa^3}\right)_1 \end{pmatrix} D_x^{-1} \begin{pmatrix} 1 & 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} \tau_1 \\ \frac{2\tau\tau_1}{\kappa} - \frac{\tau^2\kappa_1}{\kappa^2} + \varkappa\frac{\kappa_1}{\kappa^2} \end{pmatrix} D_x^{-1} \begin{pmatrix} -\frac{\tau}{\kappa^2} & \frac{1}{\kappa} \end{pmatrix}.
\end{aligned}$$

Again, from its expression we can identify the cosymmetries in $\mathfrak{R}_2$, that is $(1, \quad 0)$ and $\left(-\frac{\tau}{\kappa^2}, \quad \frac{1}{\kappa}\right)$, deriving from conservation laws, with densities κ and $\frac{\tau}{\kappa}$, respectively.

The equations

$$\text{(4.5)} \qquad \begin{cases} \kappa_{t_2} = \tau_1, \\ \tau_{t_2} = \frac{2\tau\tau_1}{\kappa} - \frac{\tau^2\kappa_1}{\kappa^2} + \varkappa\frac{\kappa_1}{\kappa^2} \end{cases}$$

and

$$\text{(4.6)} \qquad \begin{cases} \kappa_{t'_2} = \frac{\kappa_3}{\kappa^3} - 9\frac{\kappa_1\kappa_2}{\kappa^4} + 12\frac{\kappa_1^3}{\kappa^5} - 3\frac{\tau\tau_1}{\kappa^2} + 3\frac{\tau^2\kappa_1}{\kappa^3} + \varkappa\frac{\kappa_1}{\kappa^3}, \\ \tau_{t'_2} = \frac{\tau_3}{\kappa^3} - 6\frac{\kappa_1\tau_2}{\kappa^4} - 3\frac{\kappa_2\tau_1}{\kappa^4} - 3\frac{\tau^2\tau_1}{\kappa^3} + 3\frac{\kappa_1\tau^3}{\kappa^4} + 12\frac{\kappa_1^2\tau_1}{\kappa^5} + \varkappa\left(\frac{\tau}{\kappa^3}\right)_1 \end{cases}$$

are symmetries of the operator, and so they have $\mathfrak{R}_2$ as recursion operator.

Equation (4.6) is a generalization of the equations studied by Ivey to the case of manifolds with constant curvature,

Again, evolution equation (4.5) is easily seen to be a bi-Hamiltonian system. This is true since

$$u_{t_2} = \mathcal{D}\delta\mathcal{H}_1 = \mathcal{J}\delta\mathcal{H}_0,$$

where $u = (\kappa, \tau)$, $\mathcal{H}_0 = -\int \kappa dx$ and $\mathcal{H}_1 = \int \left(\frac{\tau^2}{2\kappa} + \varkappa\frac{1}{2\kappa}\right) dx$. We also have that equation (4.6) is bi-Hamiltonian:

$$u_{t_2'} = \mathcal{D}\delta\mathcal{H}_3 = \mathcal{J}\delta\mathcal{H}_2,$$

with $\mathcal{H}_2 = -\int \frac{\tau}{\kappa} dx$ and $\mathcal{H}_3 = \int \left(\frac{\tau\kappa_1^2}{\kappa^5} - \frac{\kappa_1\tau_1}{\kappa^4} - \frac{\tau^3}{2\kappa^3} - \varkappa\frac{\tau}{2\kappa^3}\right) dx$. However, operator $\mathcal{C}$ is not a Hamiltonian operator for either of these two equations. This implies that *$\mathcal{C}$ is not in the hierarchy generated by $\mathcal{D}$ and $\mathcal{E}$*. Therefore, *the triplet $\mathcal{E}, \mathcal{C}, \mathcal{D}$ is a true Hamiltonian triplet.*

Notice that both $\mathcal{H}_0$ and $\mathcal{H}_2$ lie on the kernel of the operator $\mathcal{D}$. This explains why we dropped from the beginning $\mathcal{D}$ from the Hamiltonian pair, considering $(\mathcal{E}+\varkappa\mathcal{C}, \mathcal{C})$ rather than the expected, but more complicated, $(\mathcal{E}+\mathcal{D}+\varkappa\mathcal{C}, \mathcal{C})$. Thus, (4.5) and (4.6) still have geometric evolutions of curves associated to them. Namely,

$$\gamma_t = -\mathbf{e}_3$$

is associated to (4.5) and the one associated to (4.6) is

$$\gamma_t = -\frac{1}{\kappa}\mathbf{e}_1 + \frac{\kappa_1}{\kappa^3}\mathbf{e}_2 + \frac{\tau}{\kappa^2}\mathbf{e}_3.$$

They would also be integrable in the same sense that was mentioned above for the previous set of integrable equations.

4.2. **Some transformations.** In this section we describe the transformations that simplify our integrable hierarchies, the well–known Hasimoto transformation, which takes the vortex filament flow to the nonlinear Schrödinger equation, and the hodographic transformation taking the second hierarchy in the nonflat case to a decoupled potential KdV system. This last transformation "straightens" the hierarchy when the curvature of the manifold is zero. We first describe briefly how a change of variable will affect the Poisson brackets. Suppose we have coordinates (x, u) and a Poisson bracket in these coordinates given by

$$\{f, g\} = \int \frac{\delta f}{\delta u} \cdot \mathcal{D}\frac{\delta g}{\delta u} dx.$$

We map (y, v) by an invertible map Φ to (x, u) coordinates. The effect of this change of variables on the variational derivatives of f and g is known to be given by $\left(\frac{\delta v}{\delta u}\right)^* \frac{\delta\Phi^* f}{\delta v}$ and $\left(\frac{\delta v}{\delta u}\right)^* \frac{\delta\Phi^* g}{\delta v}$, respectively, where $\left(\frac{\delta v}{\delta u}\right)^*$ denotes the operator conjugate to $\frac{\delta v}{\delta u}$ and $\Phi^* f = f(\Phi(y, v))$. From here we obtain the transformation of our Poisson bracket under the map Φ

$$\Phi^*\{f, g\} = \int \left[\left(\frac{\delta v}{\delta u}\right)^* \frac{\delta\Phi^* f}{\delta v}\right] \cdot \left[\mathcal{D}\left(\frac{\delta v}{\delta u}\right)^* \frac{\delta\Phi^* g}{\delta v}\right] \left|\frac{\delta x}{\delta y}\right| dy.$$

Let $\Phi^*\mathcal{D} = \frac{\delta v}{\delta u}\mathcal{D}(\frac{\delta v}{\delta u})^* \left|\frac{\delta x}{\delta y}\right|$. Then we define

$$\{\hat{f}, \hat{g}\}' = \int \frac{\delta\hat{f}}{\delta v} \cdot \Phi^*\mathcal{D}\frac{\delta\hat{g}}{\delta v} dy \tag{4.7}$$

to be the new bracket, with associated tensor $\Phi^*\mathcal{D}$. Since we have

$$\Phi^*\{f, g\} = \{\Phi^* f, \Phi^* g\}',$$

the skew-symmetry and Jacobi identity follow automatically.

The Hasimoto transformation

We describe how the first integrable group of equations changes under the Hasimoto transformation:

$$\phi = \kappa e^{i\int \tau dx}.$$

Notice that

$$\begin{aligned}
\phi_t &= (\kappa_t + i\kappa \int \tau_t dx) e^{i\int \tau dx} \\
\phi_1 &= (\kappa_1 + i\kappa\tau) e^{i\int \tau dx} \\
\phi_2 &= (\kappa_2 + 2i\kappa_1\tau + i\kappa\tau_1 - \kappa\tau^2) e^{i\int \tau dx} \\
\phi_3 &= (\kappa_3 + 3i\kappa_2\tau + 3i\kappa_1\tau_1 + i\kappa\tau_2 - 3\kappa\tau\tau_1 - 3\kappa_1\tau^2 - i\kappa\tau^3) e^{i\int \tau dx}.
\end{aligned}$$

We can now rewrite (4.1) as

$$\phi_{t_1} = i\phi_2 + \frac{i}{2}|\phi|^2\phi, \tag{4.8}$$

which is nonlinear Schrödinger equation, and (4.2) as

$$\phi_{t_1'} = \phi_3 + \frac{3}{2}|\phi|^2\phi_1,$$

which is modified KdV equation.

To show how the transformation affects the Hamiltonian pair, we write it as

$$u + iv = \kappa e^{i\int \tau dx},$$

which gives the matrix

$$Q = \begin{pmatrix} \cos(\int \tau dx) & -\kappa \sin(\int \tau dx) D_x^{-1} \\ \sin(\int \tau dx) & \kappa \cos(\int \tau dx) D_x^{-1} \end{pmatrix}$$

describing the action of the transformations on the vectorfields, the Hamiltonian operator, say $\mathcal{C}$, transforms as $\mathcal{C} \mapsto Q\mathcal{C}Q^*$ according to (4.7), since $x = y$ so that $\frac{\delta x}{\delta y} = 1$. We find that

$$Q\mathcal{C}Q^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$Q\mathcal{D}Q^* = \begin{pmatrix} -vD_x^{-1}uD_x + D_xuD_x^{-1}v & -D_xuD_x^{-1}u - vD_x^{-1}vD_x \\ uD_x^{-1}uD_x + D_xvD_x^{-1}v & uD_x^{-1}vD_x - D_xvD_x^{-1}u \end{pmatrix}$$

$$Q\mathcal{E}Q^* = \begin{pmatrix} 0 & -D_x^2 \\ D_x^2 & 0 \end{pmatrix}$$

The new recursion operator, the transformed of $\mathfrak{R}_1$ is

$$\begin{aligned}
\bar{\mathfrak{R}}_1 &= \varkappa \mathrm{I} \\
&+ \begin{pmatrix} D_x^2 + u^2 + v^2 + u_1 D_x^{-1} u - v D_x^{-1} v_1 & v D_x^{-1} u_1 + u_1 D_x^{-1} v \\ u D_x^{-1} v_1 + v_1 D_x^{-1} u & D_x^2 + u^2 + v^2 - u D_x^{-1} u_1 + v_1 D_x^{-1} v \end{pmatrix} \\
&= \varkappa \mathrm{I} - \begin{pmatrix} -vD_x^{-1}u & -D_x - vD_x^{-1}v \\ D_x + uD_x^{-1}u & uD_x^{-1}v \end{pmatrix}^2 = \varkappa \mathrm{I} - \mathfrak{R}_{nls}^2,
\end{aligned}$$

where I denotes the identity matrix. It is interesting to notice that $\mathfrak{R}_{nls}$ is a recursion operator of (4.8). In fact, if we look at the recursion operator prior to the Hasimoto transformation we now realize that

$$\mathfrak{R}_1 = \varkappa \mathrm{I} - \begin{pmatrix} -\tau & -D_x\kappa D_x^{-1} \\ D_x\frac{1}{\kappa}D_x + \kappa & -D_x\tau D_x^{-1} \end{pmatrix}^2$$

Furthermore, if we let

$$\mathcal{F} = \begin{pmatrix} D_x & \frac{\tau}{\kappa}D_x \\ D_x\frac{\tau}{\kappa} & -D_x \end{pmatrix}$$

and

$$\mathcal{G} = \begin{pmatrix} 0 & 0 \\ 0 & D_x\frac{1}{\kappa}D_x\frac{1}{\kappa}D_x \end{pmatrix}$$

then $\mathcal{F}$ and $\mathcal{G}$ are compatible Hamiltonian operators, such that

$$\left((\mathcal{F}-\mathcal{G})\mathcal{C}^{-1}\right)^2 = \varkappa I - \mathfrak{R}_1.$$

Indeed, we can now see that $\mathcal{E}+\mathcal{D} = -(\mathcal{F}-\mathcal{G})\mathcal{C}^{-1}(\mathcal{F}-\mathcal{G})$ and hence, evolution (4.1) is biHamiltonian with respect to the simpler Hamiltonian pair $\{\mathcal{G}-\mathcal{F}, \mathcal{C}\}$. Its new associated Hamiltonian is given by $\hat{\mathcal{H}}_0 = \frac{1}{2}\int \kappa^2\tau dx$. This is exactly the Hamiltonian functional corresponding to (4.2). Thus *in the flat case* the second evolution (4.2) is now in the hierarchy of (4.1) with respect to the new pair, as it is already known (see [LP98]).

Proposition 2. *$\{\mathcal{G}, \mathcal{F}, \mathcal{C}\}$ form a Hamiltonian triplet.*

Proof. The proof of this proposition is a straight forward calculation of the kind performed in Theorem 3. □

Notice that, since the classical filament flow equation (4.1) is identically induced for both flat and nonflat cases ($\mathcal{H}_0$ is in the kernel of $\mathcal{C}$), we could consider the Hamiltonian pair $(\mathcal{E}+\mathcal{D}, \mathcal{C})$, and indeed $(\mathcal{G}-\mathcal{F}, \mathcal{C})$, to integrate it. The second evolution in the hierarchy would *not* be (4.2) for the nonflat case, but rather its flat analogous.

The hodographic transformation

We describe first the transformation that affects the second integrable group of equations. We will see later the effect on the associated Hamiltonian pair.

The first step in this transformation is to define $\kappa = p_1$ and $\tau = q_1$, so that both systems (4.5) and (4.6) become

$$\left\{ \begin{array}{l} p_{t_2} = q_1, \\ q_{t_2} = \frac{q_1^2}{p_1} - \varkappa\frac{1}{p_1}, \end{array}\right. \tag{4.9}$$

and

$$\left\{ \begin{array}{l} p_{t_2'} = \frac{p_3}{p_1^3} - 3\frac{p_2^2}{p_1^4} - \frac{3}{2}\frac{q_1^2}{p_1^2} - \varkappa\frac{1}{2p_1^2}, \\ q_{t_2'} = \frac{q_3}{p_1^3} - 3\frac{p_2q_2}{p_1^4} - \frac{q_1^3}{p_1^3} + \varkappa\frac{q_1}{p_1^3}. \end{array}\right. \tag{4.10}$$

We now define the *hodograph* transformation (cf. [Olv93]) by

$$u = x \quad v = q, \quad y = p.$$

If the old independent variables are (x,t), we call the new (y,τ) and we find that

$$\begin{aligned}
1 &= \frac{\partial x}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial \tau}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial p}{\partial x} = u_1 p_1,\\
0 &= \frac{\partial x}{\partial t} = \frac{\partial u}{\partial y}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial t}{\partial t} = \frac{\partial u}{\partial y}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial \tau} = u_1 p_t + u_\tau,\\
q_1 &= \frac{\partial q}{\partial x} = \frac{\partial v}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial \tau}\frac{\partial t}{\partial x} = \frac{\partial v}{\partial y}\frac{\partial p}{\partial x} = v_1 p_1,\\
q_t &= \frac{\partial q}{\partial t} = \frac{\partial v}{\partial y}\frac{\partial p}{\partial t} + \frac{\partial v}{\partial \tau}\frac{\partial t}{\partial t} = v_1 p_t + v_\tau.
\end{aligned}$$

This implies

$$\begin{aligned}
u_1 &= \frac{1}{p_1}, \quad u_2 = \frac{\partial u_1}{\partial x}\frac{\partial x}{\partial y} = -\frac{p_2}{p_1^2}u_1 = -\frac{p_2}{p_1^3},\\
u_3 &= \frac{\partial u_{yy}}{\partial x}\frac{\partial x}{\partial y} = -u_1\frac{\partial}{\partial x}\frac{p_2}{p_1^3} = -\frac{p_3}{p_1^4} + 3\frac{p_2^2}{p_1^5},\\
u_\tau &= -u_1 p_t = -\frac{p_t}{p_1}, \quad v_1 = \frac{q_1}{p_1}\\
v_\tau &= q_t - v_1 p_t = q_t - \frac{q_1}{p_1}p_t
\end{aligned}$$

We can now rewrite the equations as

$$\left\{ \begin{array}{rcl} u_{\tau_2} &=& -v_1, \\ v_{\tau_2} &=& -\varkappa u_1, \end{array} \right. \tag{4.11}$$

and

$$\left\{ \begin{array}{rcl} u_{\tau_2'} &=& u_3 + \frac{3}{2}u_1 v_1^2 + \frac{1}{2}\varkappa u_1^3, \\ v_{\tau_2'} &=& v_3 + \frac{1}{2}v_1^3 + \frac{3}{2}\varkappa v_1 u_1^2. \end{array} \right. \tag{4.12}$$

We show later that the second equation is in the hierarchy of the first, also in the nonflat case!

Let us look at equation (4.12) a little bit closer. Assume $\varkappa \neq 0$ and let $\bar{u} = u$ and $\bar{v} = \frac{1}{\sqrt{\varkappa}}v$. Then

$$\left\{ \begin{array}{l} \bar{u}_{\tau_2'} = \bar{u}_3 + \frac{3}{2}\varkappa\bar{u}_1\bar{v}_1^2 + \frac{1}{2}\varkappa\bar{u}_1^3 \\ \bar{v}_{\tau_2'} = \bar{v}_3 + \frac{1}{2}\varkappa\bar{v}_1^3 + \frac{3}{2}\varkappa\bar{u}_1^2\bar{v}_1 \end{array} \right. .$$

Let $X = \bar{u} + \bar{v}$ and $Y = \bar{u} - \bar{v}$. Then

$$\left\{ \begin{array}{l} X_{\tau_2'} = X_3 + \frac{1}{2}\varkappa X_1^3 \\ Y_{\tau_2'} = Y_3 + \frac{1}{2}\varkappa Y_1^3 \end{array} \right. , \tag{4.13}$$

which is a decoupled set of potential modified KdV equations. This simplification only occurs on the nonflat case, while the flat case cannot be decoupled. In this sense, the flat case is a singular case in the family.

Notice that if we take $\tau = 0$, the 2-dimensional case, system (4.6) reduces to

$$\kappa_{t_2'} = \frac{\kappa_3}{\kappa^3} - 9\frac{\kappa_1\kappa_2}{\kappa^4} + 12\frac{\kappa_1^3}{\kappa^5} + \varkappa\frac{\kappa_1}{\kappa^3}, \tag{4.14}$$

the generalization of the evolution found in [Ive01] to nonflat Riemannian manifolds. Using the hodographic transformation, this equation would become

$$u_{\tau_3} = u_3 + \frac{1}{2}\varkappa u_1^3. \tag{4.15}$$

Next we describe the transformation as applied to the Hamiltonian pairs. The matrices

$$Q = \begin{pmatrix} D_x^{-1} & 0 \\ 0 & D_x^{-1} \end{pmatrix}$$

and

$$R = \begin{pmatrix} -\frac{1}{p_1} & 0 \\ -\frac{q_1}{p_1} & 1 \end{pmatrix}$$

describe the action of the transformations on the vectorfields

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix}, \quad \begin{pmatrix} p_t \\ q_t \end{pmatrix},$$

respectively, the Hamiltonian operator, say $\mathcal{D}$, transforms as

$$\mathcal{D} \mapsto Q\mathcal{D}Q^* \mapsto RQ\mathcal{D}Q^*R^*.$$

We remark that $D_x = \frac{1}{u_1} D_y$, as can be easily checked from the definitions of the second transformation. We find that

$$\bar{\mathcal{C}} = RQ\mathcal{C}Q^*R^* \left|\frac{\delta x}{\delta y}\right| = \begin{pmatrix} 0 & u_1 D_y^{-1} u_1 \\ u_1 D_y^{-1} u_1 & v_1 D_y^{-1} u_1 + u_1 D_y^{-1} v_1 \end{pmatrix},$$

$$\bar{\mathcal{D}} = RQ\mathcal{D}Q^*R^* \left|\frac{\delta x}{\delta y}\right| = \begin{pmatrix} 0 & D_y^{-1} \\ D_y^{-1} & 0 \end{pmatrix}$$

and

$$\bar{\mathcal{E}} = RQ\mathcal{E}Q^*R^* \left|\frac{\delta x}{\delta y}\right| = \begin{pmatrix} u_1 D_y^{-1} v_1 + v_1 D_y^{-1} u_1 & v_1 D_y^{-1} v_1 + D_y \\ v_1 D_y^{-1} v_1 + D_y & 0 \end{pmatrix}$$

The hodographic transform is a Poisson transformation between the triplet $\{\mathcal{E}, \mathcal{D}, \mathcal{C}\}$ and the new triplet $\{\bar{\mathcal{E}}, \bar{\mathcal{D}}, \bar{\mathcal{C}}\}$. The following corollary is thus obvious.

Corollary 2. *The operators $\bar{\mathcal{C}}, \bar{\mathcal{D}}, \bar{\mathcal{E}}$ form a Hamiltonian triplet. Equation (4.12) is biHamiltonian with respect to the Hamiltonian pair $\left(\bar{\mathcal{E}} + \varkappa\bar{\mathcal{C}}, \bar{\mathcal{D}}\right)$ associated Hamiltonian functionals $\bar{\mathcal{H}}_2 = -\int u_1 v_1 dy$ and $\bar{\mathcal{H}}_3 = \int \left(u_2 v_2 - \frac{1}{2} v_1^3 u_1 - \frac{1}{2}\varkappa v_1 u_1^3\right) dy$, respectively.*

Note: strictly speaking the definition of Poisson brackets given in section 2.2 does not include these tensors, since their entries are not defined in terms of differential operators. Nevertheless, they are if we allow nonlocal terms and the formalization of this kind of Poisson geometry.

Our final theorem states that the hodographic transform, followed by the decoupling map, is in fact a Poisson map between the space endowed with the Poisson bracket defined by $\mathcal{E} + \varkappa\mathcal{C}$ and the space endowed with a decoupled pair of the known potential modified KdV Hamiltonian structure (we are not entering here in the details of how these spaces should actually be defined, to avoid further complications).

Theorem 4. *Consider the decoupling transformation $\sigma(y, u, v) = (y, X, Y)$, where X and Y are defined as in (4.13). Denote by H the hodographic transformation. Then, if $\varkappa \neq 0$, H followed by σ is a Poisson map taking $\mathcal{E} + \varkappa\mathcal{C}$ to a decoupled pair of modified potential KdV Hamiltonian structures*

$$\hat{\mathcal{E}} + \varkappa\hat{\mathcal{C}} = \frac{2}{\sqrt{\varkappa}} \begin{pmatrix} X_1 D_y^{-1} X_1 + \varkappa D_y & 0 \\ 0 & -Y_1 D_y^{-1} Y_1 - \varkappa D_y \end{pmatrix} \tag{4.16}$$

Proof. It is a straightforward calculation to show that

$$\sigma^* H^*(\mathcal{E} + \varkappa \mathcal{C}) = \frac{2}{\sqrt{\varkappa}} \begin{pmatrix} X_1 D_y^{-1} X_1 + \varkappa D_y & 0 \\ 0 & -Y_1 D_y^{-1} Y_1 - \varkappa D_y \end{pmatrix},$$

by conjugation, as usual (see (4.7)). The tensor $X_1 D_y^{-1} X_1 + \varkappa D_y$ is well-known to be a Hamiltonian structure for the potential KdV. Potential modified KdV equation is obtained with the particular choice of Hamiltonian $\mathcal{H} = -\frac{\sqrt{\varkappa}}{4} \int X_1^2 dy$. □

The new recursion operator, the transformed of $\mathfrak{R}_2$ is

$$\begin{aligned}
\bar{\mathfrak{R}}_2 = & \begin{pmatrix} u_1 D_y^{-1} v_1 + v_1 D_y^{-1} u_1 & v_1 D_y^{-1} v_1 + D_y \\ v_1 D_y^{-1} v_1 + D_y & 0 \end{pmatrix} \begin{pmatrix} 0 & D_y \\ D_y & 0 \end{pmatrix} \\
& + \varkappa \begin{pmatrix} 0 & u_1 D_y^{-1} u_1 \\ u_1 D_y^{-1} u_1 & v_1 D_y^{-1} u_1 + u_1 D_y^{-1} v_1 \end{pmatrix} \begin{pmatrix} 0 & D_y \\ D_y & 0 \end{pmatrix} \\
= & \begin{pmatrix} {D_y}^2 + v_1^2 + \varkappa u_1^2 & 2u_1 v_1 \\ 2\varkappa u_1 v_1 & {D_y}^2 + v_1^2 + \varkappa u_1^2 \end{pmatrix} \\
& - \mathfrak{N} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_y^{-1} \begin{pmatrix} v_2 & u_2 \end{pmatrix} - \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_y^{-1} \begin{pmatrix} v_2 & u_2 \end{pmatrix} \mathfrak{N}
\end{aligned}$$

where

$$\mathfrak{N} = \begin{pmatrix} 0 & 1 \\ \varkappa & 0 \end{pmatrix}.$$

Applying $\bar{\mathfrak{R}}_2$ to

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} u_3 + \frac{3}{2} u_1 v_1^2 + \frac{1}{2} \varkappa u_1^3 \\ v_3 + \frac{1}{2} v_1^3 + \frac{3}{2} \varkappa u_1^2 v_1 \end{pmatrix},$$

so now the third order equation is in the image of $\bar{\mathfrak{R}}_2$. This was not the case in the κ, τ coordinates, so the present coordinates give us a clearer idea of the interrelation of the two integrable systems as finally belonging to the same hierarchy.

We notice that apparently we can multiply a symmetry with $\mathfrak{N}$ and obtain a new symmetry. This is because $\mathfrak{N}$ and $\bar{\mathfrak{R}}_2$ commute. Thus we can consider $\mathfrak{N}$ as another recursion operator. It is obvious that it is hereditary, since $\mathfrak{N}^2 = \varkappa I$ and so it is defining a complex structure when $\varkappa \neq 0$. The complete hierarchy is generated by applying powers of $\bar{\mathfrak{R}}_2$ to the trivial symmetry

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

and then add the image of these under $\mathfrak{N}$ to this collection.

5. Conclusions

In this paper we associate a triplet of compatible Hamiltonian structures to the intrinsic geometry of curves in 3–dimensional Riemannian manifolds. From Theorem 2 and 3 we can conclude the following interesting fact: if a flow of curves evolves following an arc–length preserving evolution of the form (3.1) with (h_3, h_1) being the gradient of a certain functional $\mathcal{H}$ with respect to κ and τ, then the associated flow of the curvatures is Hamiltonian with respect to the Hamiltonian structure (3.3) presented here, with Hamiltonian functional $\mathcal{H}$. The condition of being a gradient amounts to the Fréchet derivative of (h_3, h_1) being self-adjoint and it is a rather natural condition one needs to impose to have a Hamiltonian. Furthermore, even if no Hamiltonian functional is associated to (h_3, h_1), the flow of the curvature evolution (3.2) will still lie on the Poisson leaves corresponding to the Poisson structure defined by (3.3), so that any possible Casimir element would be constant along the flow. This would be true for *any* arc-length preserving evolution of curvatures induced by geometric evolutions of Riemannian curves.

If in addition to (h_3, h_1) representing the gradient of a Hamiltonian functional, the associated (κ, τ) evolution is also Hamiltonian with respect to either (3.21) or (3.22), then the PDE evolution would be completely integrable. We present several examples of such evolutions, namely the well–known vortex filament equations and the system of equations announced in [Ive01]). We unify the study of the vortex filament flow equations on Riemannian manifolds with constant curvature and we study the geometry of the second group of integrable systems in detail, showing that, in the nonflat case, it is Poisson equivalent to a system of decoupled potential KdV equations. To our knowledge, a complete classification of integrable systems associated to this Hamiltonian triplet has not yet being found and it is an interesting problem in itself. A similar classification for the planar Euclidean case can be found in [Ive01].

The Poisson geometry of infinite dimensional Poisson brackets is largely unexplored, except for some special examples. The implications of the geometrical relationship presented here for the Poisson geometry of the associated brackets is a problem which has not been studied and that could have very important consequences. The presence of the projective group is essential in the understanding of the Poisson leaves of the Adler–Gel'fand–Dikii bracket and so one could expect a similar role in the case of the Euclidean group as related to these Riemannian manifolds. The geometrical reasons that allow the appearance of these tensors are not at all understood, not only which geometrical facts produce an skew–symmetric tensor but, more interestingly, what are the geometrical properties that allow one to obtain the Jacobi identity. The resolution of this question would be of the highest interest and it might link these Hamiltonian evolutions to evolutions of Drinfel'd and Sokolov type (cf. [DS84]). The study of other homogeneous spaces is still open. Some cases are currently under consideration by the authors.

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