Stanley decomposition of the joint covariants of three quadratics

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This paper is dedicated to Richard Cushman on his 65th birthday

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Abstract

The Stanley decomposition of the joint covariants of three quadratics is computed using a new transvectant algorithm and computer algebra. This is sufficient to compute the general form of the normal form with respect to a nilpotent with three 3-dimensional irreducible blocks.

1 Introduction

The problem of describing the normal form of a differential equation at equilibrium with nilpotent linear part is known to be solvable once one has the Stanley decomposition of the covariants associated to the SL_2-action induced by the Jacobson-Morozov imbedding of the nilpotent into an $\mathfrak{sl}_2$, see [CS90], and, for the construction of the equivariants out of the covariants, [Mur02].

Remark 1 *The terminology Stanley decomposition was introduced in [CS90] and was later given a more technical meaning based on a conjecture of Stanley, see [Ape03a, Ape03b].*

In [MS07, SVM07] it has been shown how one can obtain the Stanley decomposition in the case of a reducible nilpotent, if the Stanley decompositions of the components are known. In terms of invariant theory, this opens the road to compute the Stanley decomposition for the space of joint covariants [Olv99, Olv01].

It also opens the way to investigate a more ambitious program suggested by Jim Murdock: find the Stanley decomposition of n quadratics. The Stanley decomposition for n lines is known, see [CSW88], and since the joint covariants for n quadratics can be described explicitly [GY03, Section 139], this is the next obvious question. To ask the same question for cubics seems to be too much.

Remark 2 *This type of question leads one into combinatorics. A few papers where normal form theory articles were cited outside this specific context are [Stu90, DS04] and [Zon05].*

In order to get some idea of the structure of the Stanley decomposition for n quadratics, the author has written a Form [Ver00] program to assist with the rather extensive computations. The results in this paper for three quadratics were derived using this program.

2 Classical invariant theory

Classical invariant theory considers so called *groundforms*

$$\mathfrak{g} = \sum_{i+j=n} \binom{n}{i} a_j X^i Y^j.$$

One then lets $A \in SL_2$ act on $\mathbb{R}^2$ (with the coordinates X and Y). This induces an action on the coefficients a_i as follows. If the new coordinates are called $\hat{X}, \hat{Y}$, then the new form looks like

$$\hat{\mathfrak{g}} = \sum_{i+j=n} \binom{n}{i} \hat{a}_i \hat{X}^i \hat{Y}^j.$$

Let $A \cdot (a_0, \ldots, a_n) = (\hat{a}_0, \ldots, \hat{a}_n)$. If we now let A act on $\mathbb{R}^2$ and A^{-1} on $\mathbb{R}^{n+1}$, the space of the coefficients, then the groundform will be invariant. Any invariant under this combined action is called a **covariant**. A covariant with degree zero in X, Y is called an **invariant**. Nowadays one thinks of a covariant as an irreducible representation space of SL_2 or its Lie algebra $\mathfrak{sl}_2$. The Lie algebra is given by

$$N = X\frac{\partial}{\partial Y}, \quad M = Y\frac{\partial}{\partial X}, \quad H = Y\frac{\partial}{\partial Y} - X\frac{\partial}{\partial X}$$

and in the same way induces a representation on $\mathbb{R}^{n+1}$ as before. We use the same notation N, M, H for the induced operators. The induced operators look very familiar to anyone with some basic knowledge of the finite dimensional representation theory of $\mathfrak{sl}_2$. In normal form theory one is interested in the **seminvariants**, that is the leading term of the covariant, say the coefficient of Y^m. One can now compute new seminvariants from old ones by a process called **transvection (Überschiebung)**, which is defined as follows.

Definition 1 *Suppose $f \in \mathcal{V}$ and $g \in \mathcal{W}$ are H-eigenvectors in* $\ker M$, *with weight w_f, w_g, respectively. Let for any $f \in \ker M$, $f_i = N^i f$ fori $= 0, \ldots, w_f$. For any $n \leq \min(w_f, w_g)$, define the mth transvectant of f and g, $\tau^m(f \otimes g)$, by*

$$\tau^m f \otimes g = m! \sum_{i+j=m} (-1)^i \binom{w_f - i}{m - i} f_i \otimes \binom{w_g - j}{m - j} g_j.$$

So $\tau^m : \mathcal{V} \otimes \mathcal{W} \to \mathcal{V} \otimes \mathcal{W}$.

Lemma 1 *With $f, g \in \ker M$, $\tau^m f \otimes g \in \ker M$ and $w_{\tau^m f \otimes g} = w_f + w_g - 2m$.*

If one now simply contracts $f \otimes g$ to fg, then this is a seminvariant, which is the leading term of a covariant. Of course, after the contraction the result might be zero. We denote the contracted transvection by $(f, g)^{(m)}$.

3 Gordan and Hilbert

In 1868 Gordan proved that given a number of groundforms, any covariant could be computed using the transvection process in a finite number of steps. Emmy Noether worked as a student of Gordan on the ternary quartic, and this work must have been the motivation for the later search for abstraction. Hilbert gave a nonconstructive proof of the finite generation of covariants in 1890, and a constructive one in 1893. This almost killed invariant theory. Multilinear algebra was, under the influence of functional analysis, again with Hilbert in the foreground, almost completely replaced by linear algebra.

The computational problems, however, are still not solved. Ideally one would like to say that any arbitrary covariant can be written in a certain way depending on the basic covariants. Any such decomposition of the space of covariants will be called a **Stanley decomposition**. The technical meaning of this terminology is somewhat more complicated, cf. Remark 1. Examples of such Stanley decompositions can be found in [CS90] and in the following sections.

In [MS07, SVM07] a method is given to compute the Stanley decomposition of the joint of two sets of groundforms for which the Stanley decomposition is known. The idea behind this is to compute the tensor product of the two Stanley decompositions and use the Clebsch–Gordan formula to express this in terms of itself times an invariant. Since the tensor product can be taken over the ring of invariants (on which $\mathfrak{sl}_2$ acts trivially) this allows one to compute the product recursively.

4 Computational methods

The Stanley decomposition is not unique. How does one check that any given Stanley decomposition is correct? The first thing that needs to be verified is that the proposed sum is in fact direct. There are no shortcuts for this: it is usually boring but not extremely difficult work. The second thing is to check that the decomposition is complete. This can be done very effectively by a method introduced in [CS86]. The idea is to compute the generating function of the covariants from the Stanley decomposition, using the degree and the H-eigenvalue of the seminvariant as powers of t and u, respectively. This allows one to check that the seminvariants generate all polynomials when $\mathfrak{sl}_2$ is applied to them. When the generating function does this, it is called $(1, n)$-perfect in the n-dimensional situation, see [SVM07].

5 The Stanley decomposition of the covariants of one quadratic

Let k be a field with characteristic zero. Next to the groundform

$$\mathfrak{a} = a^0X^2 + 2a^1XY + a^2Y^2, \quad a^i \in k,$$

the only other covariant is the discriminant, which is the second transvectant of $\mathfrak{a}$ with itself:

$$\mathfrak{b} = (\mathfrak{a}, \mathfrak{a})^{(2)} \equiv a^0a^2 - (a^1)^2,$$

where the $\equiv$ symbol is used when we compute modulo multiplicative rational numbers. The ring of covariants is polynomial: $k[\mathfrak{a}, \mathfrak{b}]$. We are going to consider the covariants as a ring over the invariants, and since $\mathfrak{b}$ is an invariant, we define $\mathfrak{I} = k[\mathfrak{b}]$, and we see that the ring of covariants is $\mathfrak{I}[\mathfrak{a}]$.

6 The Stanley decomposition of the covariants of two quadratics

Let $\mathfrak{a}_1, \mathfrak{a}_2$ be the groundforms, and let

$$\mathfrak{b}_i = (\mathfrak{a}_i, \mathfrak{a}_i)^{(2)}, \quad i = 1, 2,$$

be their discriminants and

$$\mathfrak{c}_{12} = (\mathfrak{a}_1, \mathfrak{a}_2)^{(2)} \equiv a_1^0a_2^2 + a_1^2a_2^0 - 2a_1^1a_2^1$$

their joint invariant. Let $\mathfrak{I} = k[\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{c}_{12}]$. Then the Stanley decomposition is given by

$$\mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_2](\mathfrak{a}_1, \mathfrak{a}_2)^{(1)} \oplus \mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_2].$$

This calculation can be easily done by hand using the methods in [MS07] and is explicitly given in [SVM07, Section 12.7.2].

7 The Stanley decomposition of the covariants of three quadratics

Let $\mathfrak{a}_i, i = 1, 2, 3$ be the groundforms, and let

$$\mathfrak{b}_i = (\mathfrak{a}_i, \mathfrak{a}_i)^{(2)}, \quad i = 1, 2, 3,$$

be their discriminants and

$$\mathfrak{c}_{ij} = (\mathfrak{a}_i, \mathfrak{a}_j)^{(2)}, \quad 1 \leq i < j \leq 3,$$

their joint invariant. Let $\mathfrak{J} = k[\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3, \mathfrak{c}_{12}, \mathfrak{c}_{23}]$ and $\mathfrak{I} = \mathfrak{J}[\mathfrak{c}_{13}]$. Then the Stanley decomposition is given by

$$\begin{aligned} & \mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3](\mathfrak{a}_1, \mathfrak{a}_2)^{(1)}\mathfrak{a}_2 \oplus \mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3](\mathfrak{a}_2, \mathfrak{a}_3)^{(1)} \oplus \mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3](\mathfrak{a}_1, \mathfrak{a}_2)^{(1)}(\mathfrak{a}_2, \mathfrak{a}_3)^{(1)} \oplus \mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3]\mathfrak{a}_2 \\ \oplus\ & \mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_3] \oplus \mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_3](\mathfrak{a}_1, \mathfrak{a}_2)^{(1)} \oplus \mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_3](\mathfrak{a}_1, \mathfrak{a}_3)^{(1)} \oplus \mathfrak{I}[\mathfrak{a}_1, \mathfrak{a}_3](\mathfrak{a}_1, \mathfrak{a}_2)^{(1)}(\mathfrak{a}_1, \mathfrak{a}_3)^{(1)} \\ \oplus\ & \mathfrak{I}[\mathfrak{a}_3]((\mathfrak{a}_1, \mathfrak{a}_2)^{(1)}, \mathfrak{a}_3)^{(1)} \oplus \mathfrak{I}[\mathfrak{a}_3]((\mathfrak{a}_1, \mathfrak{a}_2)^{(1)}, \mathfrak{a}_3)^{(2)}. \end{aligned}$$

In Section 4 the question has already been raised how one can verify that this is true. Consider the generating function (see [CS90]), which in this case is given by

$$\begin{aligned} & \frac{u^2 t}{(1-u^2t)^3(1-t^2)^5} + \frac{u^2t^2}{(1-u^2t)^3(1-t^2)^5} + \frac{u^4t^3}{(1-u^2t)^3(1-t^2)^5} + \frac{u^4t^4}{(1-u^2t)^3(1-t^2)^5} \\ + \quad & \frac{1}{(1-u^2t)^2(1-t^2)^6} + 2\,\frac{u^2t^2}{(1-u^2t)^2(1-t^2)^6} + \frac{u^4t^4}{(1-u^2t)^2(1-t^2)^6} + \frac{u^2t^3}{(1-u^2t)(1-t^2)^6} \\ + \quad & \frac{t^3}{(1-u^2t)(1-t^2)^6}. \end{aligned}$$

It can be easily verified that this generating function is $(1, 9)$-perfect, by multiplying with u, differentiating with respect to u, and putting $u = 1$ and checking that the result is

$$\frac{1}{(1-t)^9}.$$

This means that if we identify the covariants with their leading terms and then apply one of the nilpotent elements to these, all polynomials in 9 variables are generated. This shows that we have enough elements in our Stanley decomposition, but one still needs to show that there can be no relations, that is to say, the direct sums in the expression are really direct sums. Since the chance that the generating function is perfect after a long and winded calculation, but the answer is wrong, seems rather remote, this last test is (as usual) left to the doubting reader.

If we now want to go to the equivariant vector field, that is, the vector field in normal form with respect to a nilpotent with three 3-dimensional irreducible blocks, we notice that theoretically there is no problem. The number of terms in the normal form, based on the calculation above, will be 81.

This is done by introducing dual groundforms $\mathfrak{u}_i$ (in terms of the $\partial/\partial a_i^j, j = 0, 1, 2$) representing vectors in the copies of $\mathbb{R}^3$, and computing the tensorproduct of the Stanley decomposition of the covariants with these $\mathbb{R}^3$'s. For instance, $(\mathfrak{a}_i, \mathfrak{u}_i)^{(2)} \equiv \sum_{j=0}^{2} a_i^j \partial/\partial a_i^j$. This computation is easier than the one we did before, since one of the components is a finite-dimensional space. Full details of this calculation can be found in [MS07, SVM07].

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