A Formalized Hierarchy of Probabilistic System Types

Proof Pearl

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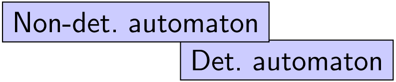
ITP 2015

Zoo of Probabilistic System Types

Det. automaton

Zoo of Probabilistic System Types



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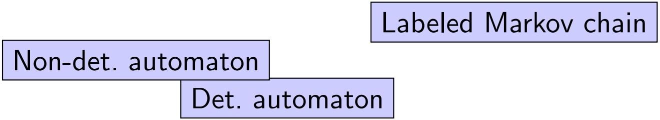

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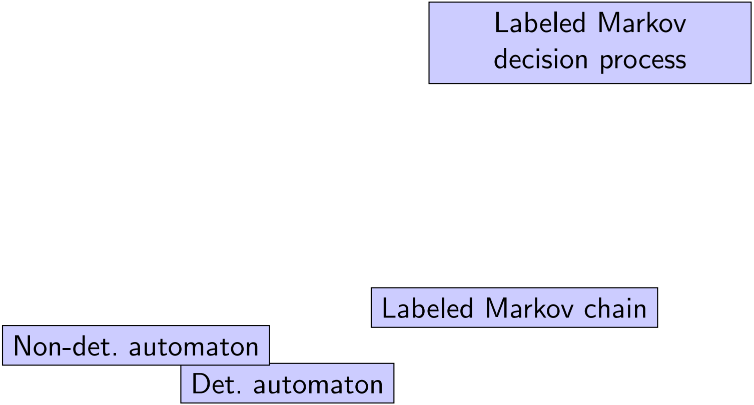

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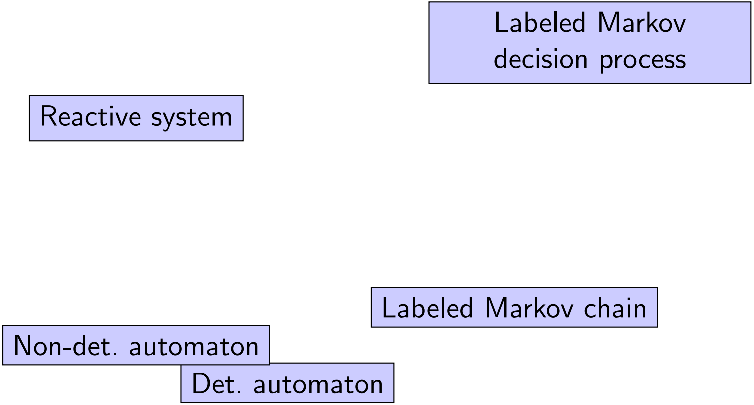

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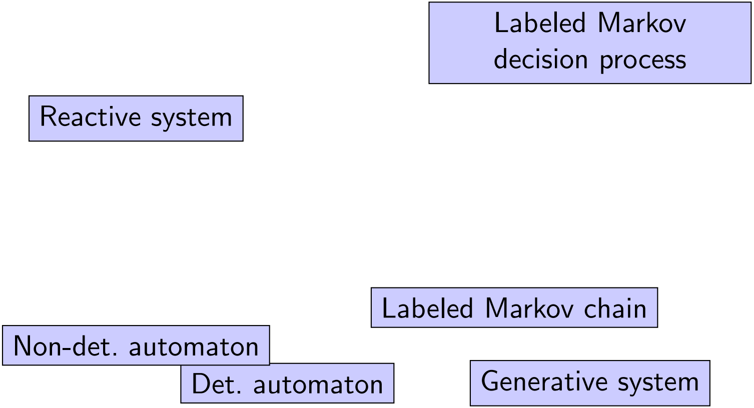

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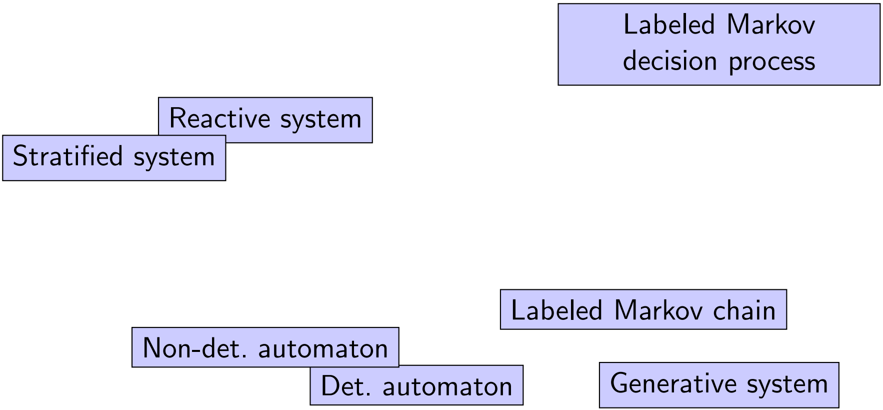

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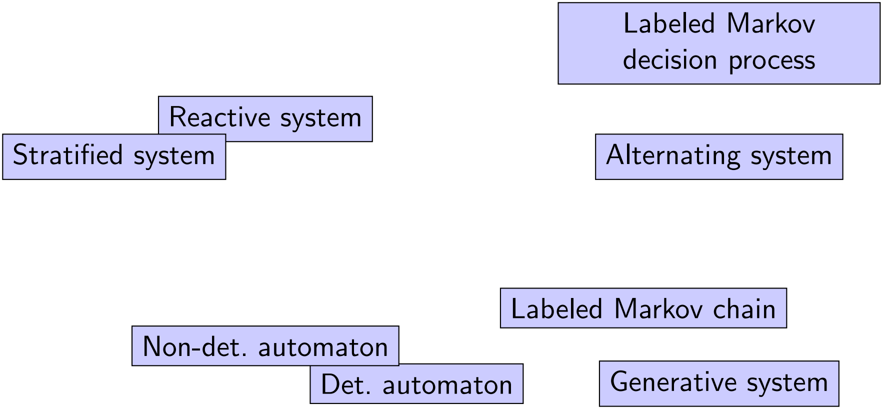

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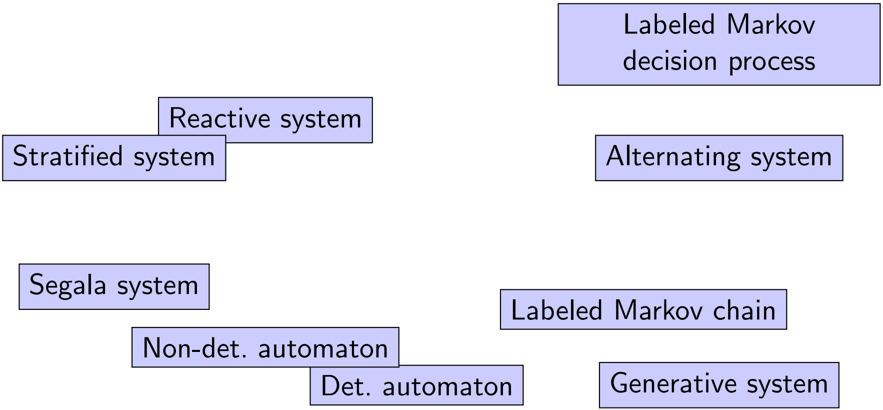


Zoo of Probabilistic System Types

Labeled Markov decision process

Reactive system

Stratified system

Alternating system

Simple Segala system

Segala system

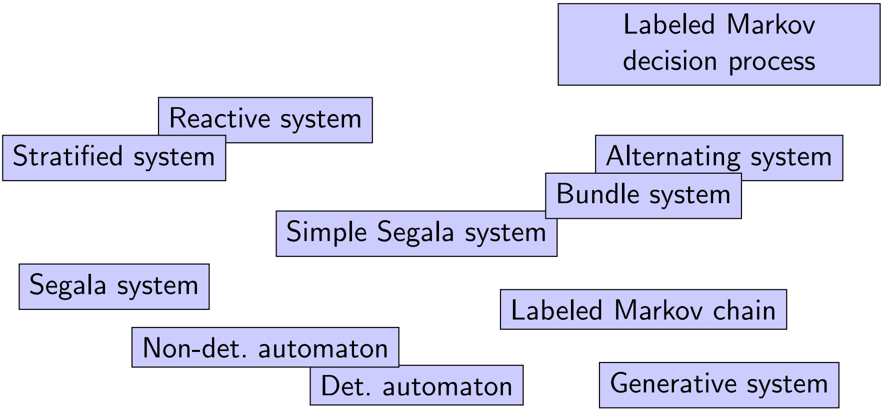
Labeled Markov chain

Non-det. automaton

Det. automaton

Generative system

Zoo of Probabilistic System Types



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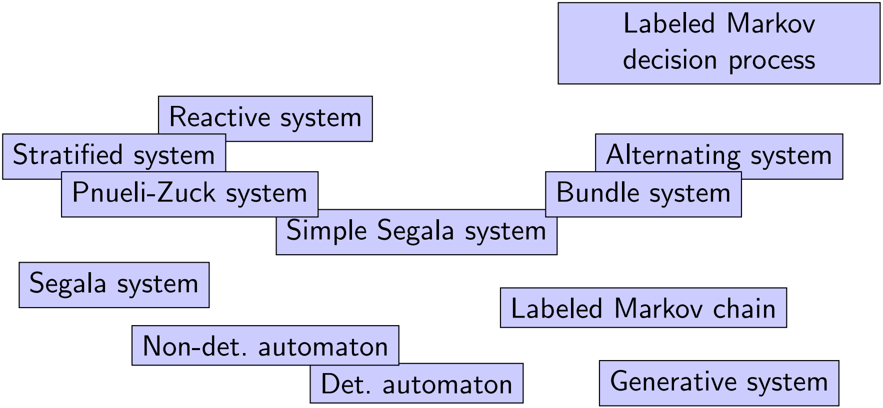

Zoo of Probabilistic System Types

Most general system

Labeled Markov decision process

Reactive system

Stratified system

Alternating system

Pnueli-Zuck system

Bundle system

Simple Segala system

Segala system

Labeled Markov chain

Non-det. automaton

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Hierarchy of Probabilistic System Types

Ana Sokolva – Coalgebraic Analysis of Probabilistic Systems (2005):

4.4 The hierarchy 107

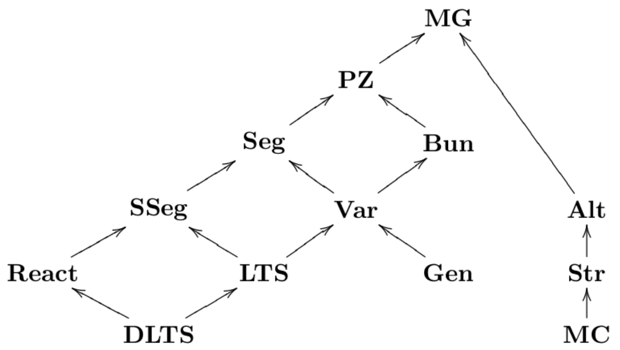

Figure 4.2: Hierarchy of probabilistic system types

Hierarchy of Probabilistic Systems Types

How to . . .

Hierarchy of Probabilistic Systems Types

How to . . .

. . . model system types?

Hierarchy of Probabilistic Systems Types

How to . . .

. . . model system types?

. . . compare systems of same type?

Hierarchy of Probabilistic Systems Types

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... model system types?

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... compare different system types?

Hierarchy of Probabilistic Systems Types

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Probability Mass Func. +
Eisbach

Coalgebras

- Functor F describes the system *type*

Coalgebras

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Deterministic System	$\alpha \times (\beta \Rightarrow \square)$
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- System (σ, s) of type F:

σ	type of states,
$s :: \sigma \Rightarrow \sigma\ F$	transition system

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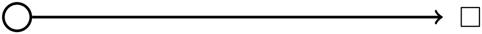
- (σ, s) is a F-coalgebra

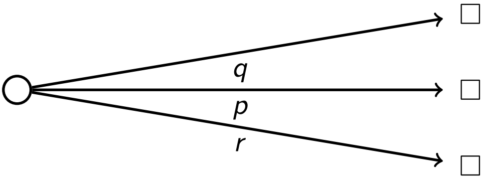
Types of Transition System



Property

Functor

Types of Transition System



Property

- Probability p

Functor

□ *pmf*

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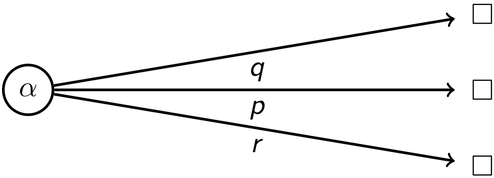

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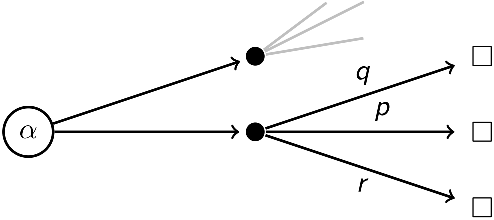

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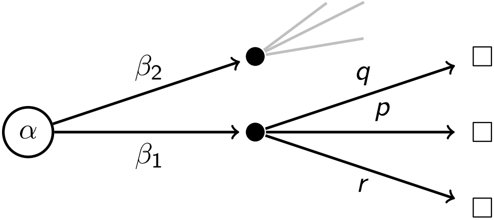

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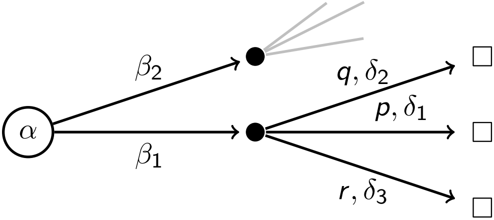

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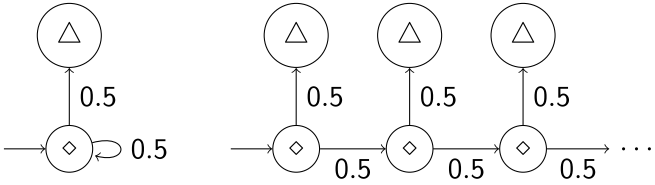
▶ Reactive β	$\alpha \times (\beta \Rightarrow \square\ \mathit{pmf})$
▶ Generative δ	$\alpha \times (\beta \Rightarrow (\delta \times \square)\ \mathit{pmf})$

Bisimulation

$$F = \alpha \times (\Box\ pmf)$$



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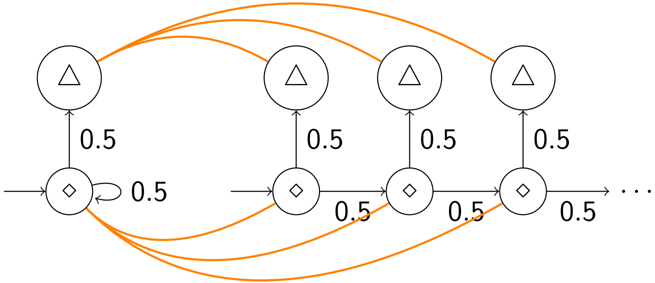

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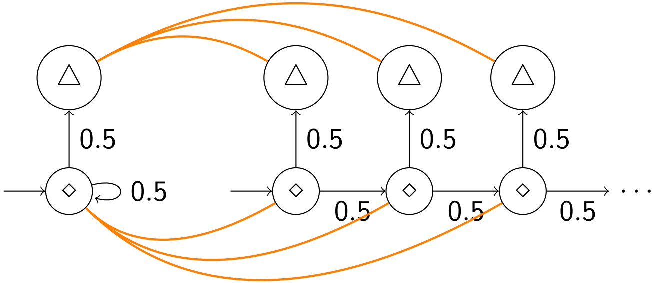

- Expresses *observational* equality
- Bisimulation relation $R :: (\sigma \times \tau)$ *set* for a system type F:

$$\forall (x, y) \in R.\ (s\,x, t\,y) \in rel_F\ R$$

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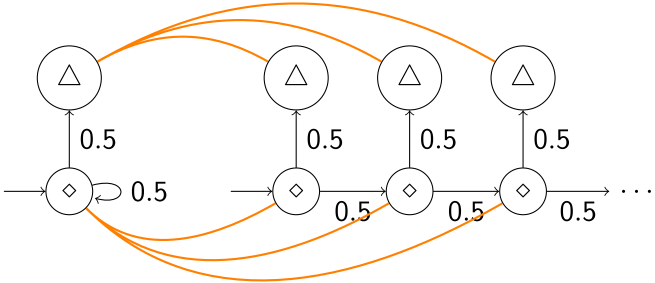

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- State x in system s is *bisimilar* to state y in system t iff $\exists$ bisimulation relation R with $(x, y) \in R$

Coalgebras in Isabelle/HOL

Idea: Analyse transition systems modulo *bisimulation*!

$$\text{Equality} :\Longleftrightarrow \text{Bisimulation}$$

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Example (Labeled Markov Chains where $F = \alpha \times \Box\ pmf$):

$$\textbf{codatatype } \alpha\ mc = MC\ (\alpha \times \alpha\ mc\ pmf)$$

Bounded Natural Functors (BNFs)

Traytel, Popescu & Blanchette: Foundational, compositional (co)datatypes for HOL

Codatatype only allows nesting through BNFs:

- Examples: products, sums, functions, lists

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- Has map and set functions:

$$\begin{array}{lll} \mathit{map}_F & :: & (\sigma \Rightarrow \tau) \Rightarrow \sigma\ F \Rightarrow \tau\ F \\ \mathit{set}_F & :: & \sigma\ F \Rightarrow \sigma\ \mathit{set} \end{array}$$

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- Has relator lifting relations R component wise:

$$rel_F :: (\sigma \times \tau)\ set \Rightarrow (\sigma\ F \times \tau\ F)\ set$$
$$rel_F\ R := \{(map_F\ \pi_1\ z, map_F\ \pi_2\ z) \mid z :: (\sigma \times \tau)\ F.\ set_F\ z \subseteq R\}$$

Bounded Natural Functors (BNFs)

Example:

$R = \{(a, i), (b, j), (c, k), (d, l)\}$

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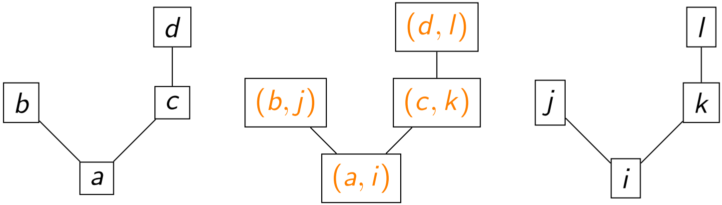

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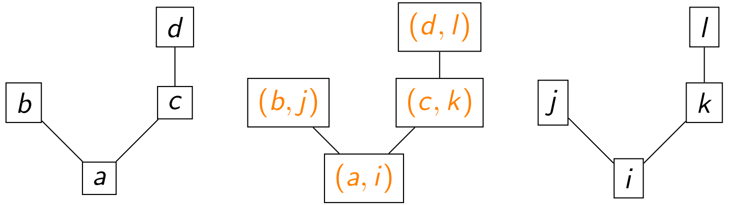

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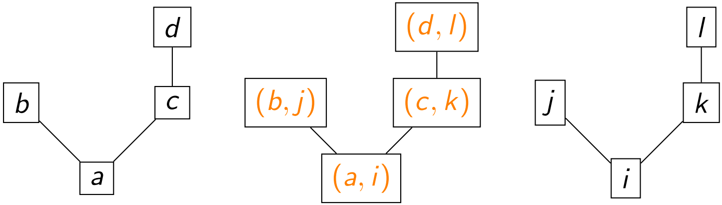

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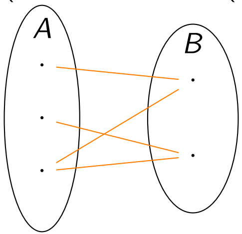
  Relator $(\forall a \in A.\ \exists b \in B.\ (a, b) \in R) \wedge (\forall b \in B.\ \exists a \in A.\ (a, b) \in R)$



BNF: Probability Mass Function

Model probabilistic transitions!

$$\begin{aligned} \mu :: \sigma\ \textit{pmf} &\approx \mu :: \sigma \Rightarrow [0,1],\quad \textstyle\sum_x \mu\ x = 1 \\ &\approx \mu :: \sigma\ \textit{measure},\quad \mu\,\mathcal{U} = 1,\quad \textit{discrete} \end{aligned}$$

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Map is $\lambda x. \displaystyle\sum_{f \ y = x} y$

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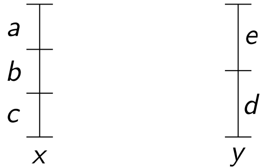

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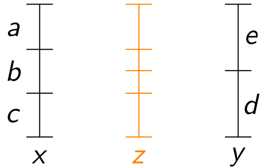

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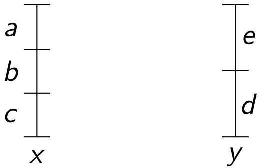

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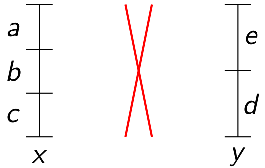

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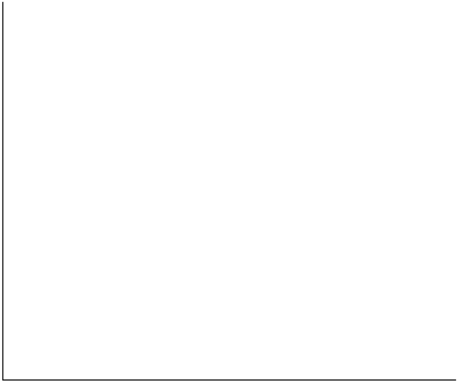
$\neg rel_{pmf}\ R'\ x\ y$

Proving the Relator Property

$$rel_{pmf}\ R \circ rel_{pmf}\ Q \subseteq rel_{pmf}\ (R \circ Q)$$

lines



Proving the Relator Property

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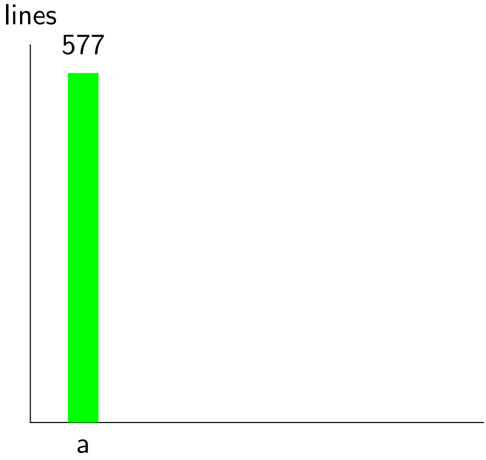

a Elementary, based on A. Sokolova's thesis

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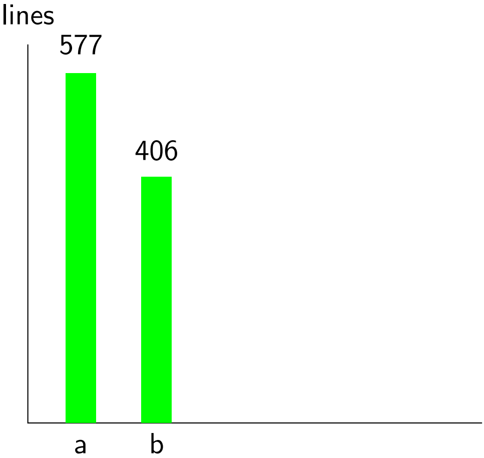

- a Elementary, based on A. Sokolova's thesis
- b Using map_{pmf}

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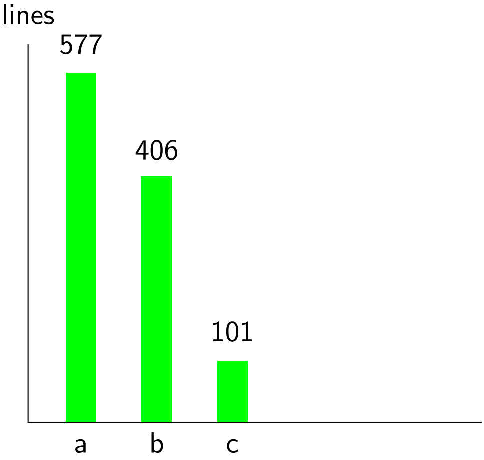

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- c Based on Jonsson et al.

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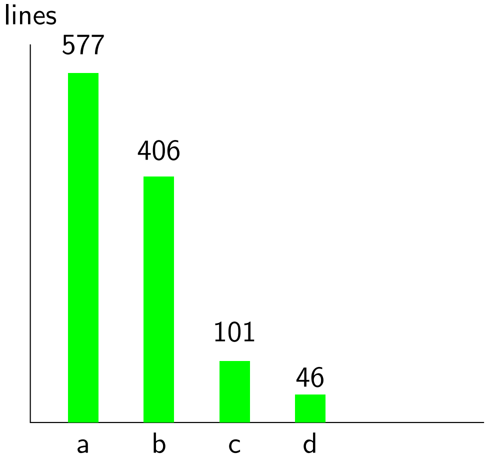

- a Elementary, based on A. Sokolova's thesis
- b Using map_{pmf}
- c Based on Jonsson et al.
- d Using map_{pmf}, $bind_{pmf}$, and $cond_{pmf}$

Proving the Relator Property

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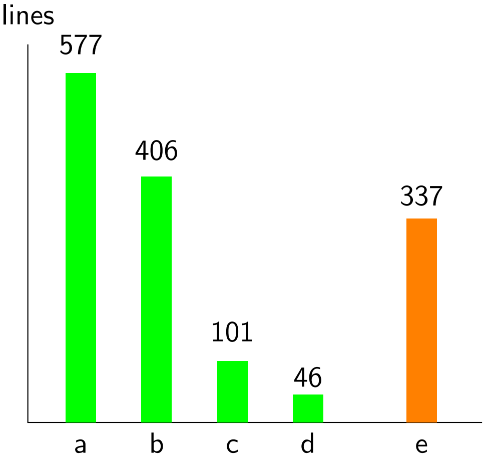

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- d Using map_{pmf}, $bind_{pmf}$, and $cond_{pmf}$
- e Zanella et al. in Coq

System Types

Name	**Functor**	**Codatatype**
Markov chain	$\sigma\ \mathit{pmf}$	MC
Labeled MC	$\alpha \times \sigma\ \mathit{pmf}$	$\alpha\ \mathit{LMC}$
Labeled MDP	$\alpha \times \sigma\ \mathit{pmf}\ \mathit{set}_1^{\kappa}$	$\alpha\ \mathit{LMDP}^{\kappa}$
Det. automaton	$\alpha \Rightarrow \sigma\ \mathit{option}$	$\alpha\ \mathit{DLTS}$
Non-det. automaton	$(\alpha \times \sigma)\ \mathit{set}^{\kappa}$	$\alpha\ \mathit{LTS}^{\kappa}$
Reactive system	$\alpha \Rightarrow \sigma\ \mathit{pmf}\ \mathit{option}$	$\alpha\ \mathit{React}$
Generative system	$(\alpha \times \sigma)\ \mathit{pmf}\ \mathit{option}$	$\alpha\ \mathit{Gen}$
Stratified system	$\sigma\ \mathit{pmf} + (\alpha \times \sigma)\ \mathit{option}$	$\alpha\ \mathit{Str}$
Alternating system	$\sigma\ \mathit{pmf} + (\alpha \times \sigma)\ \mathit{set}^{\kappa}$	$\alpha\ \mathit{Alt}^{\kappa}$
Simple Segala system	$(\alpha \times \sigma\ \mathit{pmf})\ \mathit{set}^{\kappa}$	$\alpha\ \mathit{SSeg}^{\kappa}$
Segala system	$(\alpha \times \sigma)\ \mathit{pmf}\ \mathit{set}^{\kappa}$	$\alpha\ \mathit{Seg}^{\kappa}$
Bundle system	$(\alpha \times \sigma)\ \mathit{set}^{\kappa}\ \mathit{pmf}$	$\alpha\ \mathit{Bun}^{\kappa}$
Pnueli-Zuck system	$(\alpha \times \sigma)\ \mathit{set}^{\kappa_1}\ \mathit{pmf}\ \mathit{set}^{\kappa_2}$	$\alpha\ \mathit{PZ}^{\kappa_1,\kappa_2}$
Most general system	$(\alpha \times \sigma + \sigma)\ \mathit{set}^{\kappa_1}\ \mathit{pmf}\ \mathit{set}^{\kappa_2}$	$\alpha\ \mathit{MG}^{\kappa_1,\kappa_2}$

Hierarchy of Probabilistic System Types

Ana Sokolva – Coalgebraic Analysis of Probabilistic Systems (2005):

4.4 The hierarchy 107

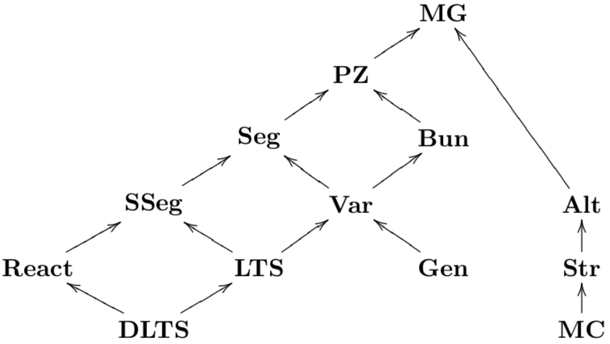


Figure 4.2: Hierarchy of probabilistic system types

Proving the Hierarchy in Isabelle/HOL

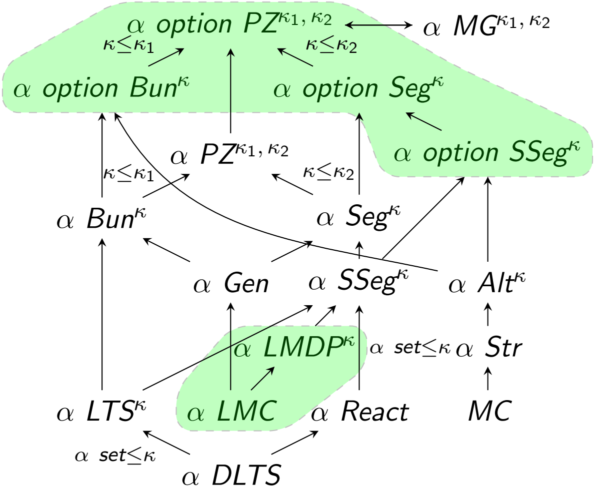
G is at least as expressive as F, iff
$\exists G\_of\_F :: \sigma\ F \Rightarrow \sigma\ G$
preserving and reflecting bisimilarity

Lift to $\overline{G\_of\_F} :: \tau_F \Rightarrow \tau_G$

Theorem: $\overline{G\_of\_F}$ is injective

Proof: by coinduction
(proof method in Eisbach)

Hierarchy


α option PZ^{κ_1,κ_2}
α MG^{κ_1,κ_2}
κ≤κ1
κ≤κ2
α option Bun^{κ}
α option Seg^{κ}
α option $SSeg^{\kappa}$
α PZ^{κ_1,κ_2}
κ≤κ1
κ≤κ2
α Bun^{κ}
α Seg^{κ}
α Gen
α $SSeg^{\kappa}$
α Alt^{κ}
α $LMDP^{\kappa}$
α set≤κ
α Str
α LTS^{κ}
α LMC
α React
MC
α set≤κ
α DLTS

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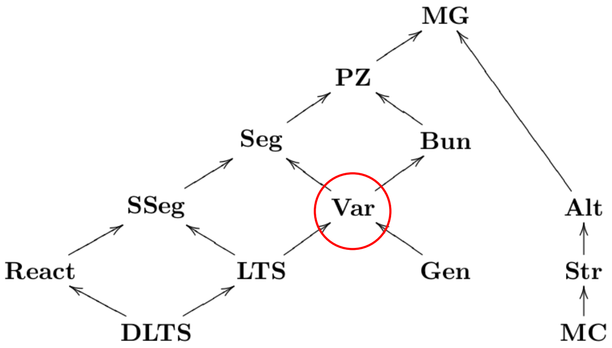

Figure 4.2: Hierarchy of probabilistic system types

Problem with Vardi Systems

- $\mathit{Var}^{\kappa} = (\alpha \times \Box)\ \mathit{pmf} + (\alpha \times \Box)\ \mathit{set}^{\kappa}$
  with: $\mathit{return}_{\mathit{pmf}}\ (a, s) = \{(a, s)\}$

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Problem with Vardi Systems

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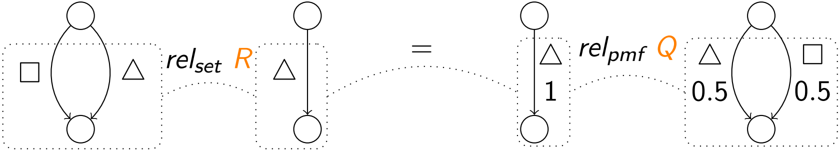

$(x, y) \in R \leftrightarrow y = \triangle$ $(x, y) \in Q \leftrightarrow x = \triangle$

Problem with Vardi Systems

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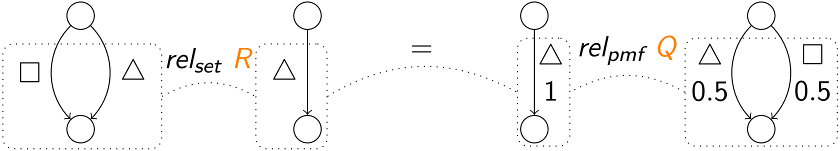
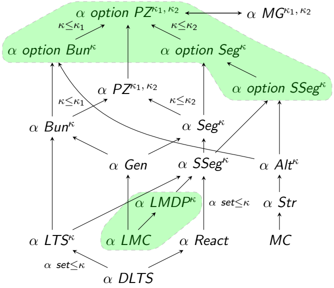


$(x, y) \in R \leftrightarrow y = \triangle$ $\quad$ $(x, y) \in Q \leftrightarrow x = \triangle$

- Approach not possible
  That is even a flaw in the original proof!

Conclusion

- Formalized hierarchy of probabilistic systems types



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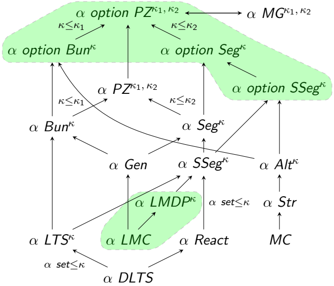

- Found *two* flaws:

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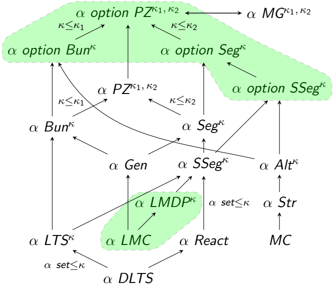

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  - Bisimulation on Vardi systems is no equiv. relation

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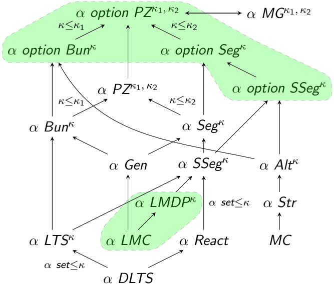

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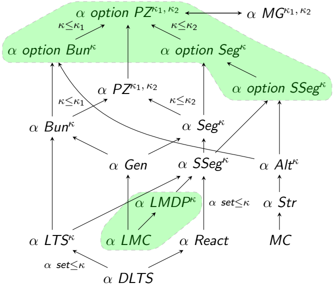

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$$\texttt{codatatype} + \text{PMF} + \text{Eisbach} = \text{Hierarchy}$$