Analysis of spectral points of the operators $T^*[\cdot]T$ and $TT^*\cdot$ in a Krein space.

André Ran and Michał Wojtylak
Vrije Universiteit Amsterdam
Let $A$ be a selfadjoint operator in $\mathcal{K}$. We call $A$ definitizable if $\rho(A) \neq \emptyset$ and there exists a (real or complex) polynomial $p$ such that $[p(A)f, f] \geq 0$ for $f \in \mathcal{D}(p(A))$.

Any polynomial $p$ satisfying the last inequality is called a definitizing polynomial for $A$. 

Let $A$ be a selfadjoint operator in $\mathcal{K}$. We call $A$ definitizable if $\rho(A) \neq \emptyset$ and there exists a (real or complex) polynomial $p$ such that $[p(A)f, f] \geq 0$ for $f \in \mathcal{D}(p(A))$.

Any polynomial $p$ satisfying the last inequality is called a definitizing polynomial for $A$.

We define the set of critical points of a definitizable operator $A$ as $c(A) := c_0(A) \cap \sigma(A) \cap \mathbb{R}$, where

$$c_0(A) := \bigcap_{p \text{ definitizing for } A} p^{-1}(0).$$

It is well known that $c(A) = c_0(A) \cap \mathbb{R}$. 
Regular and singular critical points

If $A$ is definitizable there is a spectral function $E(\Delta)$ for intervals with endpoints not critical points.
Regular and singular critical points

If $A$ is definitizable there is a spectral function $E(\Delta)$ for intervals with endpoints not critical points.

A critical point $\lambda_0$ of $A$ is called *regular* if

$$\lim_{x \downarrow \lambda_0} E(x, \lambda) \text{ exists for some } \lambda > \lambda_0,$$

$$\lim_{x \uparrow \lambda_0} E[\lambda, x) \text{ exists for some } \lambda < \lambda_0.$$

In $\Pi_\kappa$-space, equivalently the root subspace $\mathcal{R}(\lambda_0, A)$ is non-degenerate.

A critical point that is not regular is called *singular*. In $\Pi_\kappa$ space, in that case the root subspace is degenerate and the spectral function is unbounded at that point.
Regular and singular critical points

If $A$ is definitizable there is a spectral function $E(\Delta)$ for intervals with endpoints not critical points.

A critical point $\lambda_0$ of $A$ is called regular if

$$\lim_{x \downarrow \lambda_0} E(x, \lambda) \text{ exists for some } \lambda > \lambda_0,$$

$$\lim_{x \uparrow \lambda_0} E[\lambda, x) \text{ exists for some } \lambda < \lambda_0.$$

In $\Pi_\kappa$-space, equivalently the root subspace $\mathcal{R}(\lambda_0, A)$ is non-degenerate.

A critical point that is not regular is called singular. In $\Pi_\kappa$ space, in that case the root subspace is degenerate and the spectral function is unbounded at that point.

An isolated point of the real spectrum is never a singular critical point.
\( \sigma_+(A), \sigma_-(A) \)

A point \( \lambda_0 \) of the real spectrum which is not a critical point is in \( \sigma_+(A) \) (if \( E(\Delta) \) has positive definite range for a neighbourhood \( \Delta \) of \( \lambda_0 \)) or in \( \sigma_-(A) \) (if \( E(\Delta) \) has negative definite range for a neighbourhood \( \Delta \) of \( \lambda_0 \)).
Infinity as a critical point

Let $A$ be a definitizable operator. We write $\infty \in \rho(A)$ if and only if $A$ is bounded.
Infinity as a critical point

Let $A$ be a definitizable operator. We write $\infty \in \rho(A)$ if and only if $A$ is bounded.

We say that infinity is in the positive (negative) spectrum if there exists a real neighborhood of infinity $\tau$ such that $E(\tau)K$ is positive (negative).


Infinity as a critical point

Let $A$ be a definitizable operator. We write $\infty \in \rho(A)$ if and only if $A$ is bounded.

We say that infinity is in the positive (negative) spectrum if there exists a real neighborhood of infinity $\tau$ such that $E(\tau)K$ is positive (negative).

We call infinity a **critical point** of a definitizable operator $A$ if for each real neighborhood $\tau$ of infinity $E(\tau)K$ is indefinite.
Infinity as a critical point

Let $A$ be a definitizable operator. We write $\infty \in \rho(A)$ if and only if $A$ is bounded.

We say that infinity is in the positive (negative) spectrum if there exists a real neighborhood of infinity $\tau$ such that $E(\tau)\mathcal{K}$ is positive (negative).

We call infinity a critical point of a definitizable operator $A$ if for each real neighborhood $\tau$ of infinity $E(\tau)\mathcal{K}$ is indefinite.

If infinity is a critical point we call it regular if the limits $\lim_{x \uparrow +\infty} E([\lambda, x])$ and $\lim_{x \downarrow -\infty} E([x, \lambda])$ exist in the strong operator topology for any (some) not critical $\lambda \in \mathbb{R}$, otherwise we call it singular.
Assume that $T^{[*]} T$ and $TT^{[*]}$ are (densely defined and) selfadjoint operators in $\mathcal{K}$; and that $T^{[*]} T$ and $TT^{[*]}$ have nonempty resolvent sets. Then $T^{[*]} T$ is definitizable if and only if $TT^{[*]}$ is.
Definitizability

Assume that $T^*[T]$ and $TT^*[T]$ are (densely defined and) selfadjoint operators in $\mathcal{K}$; and that $T^*[T]$ and $TT^*[T]$ have nonempty resolvent sets. Then $T^*[T]$ is definitizable if and only if $TT^*[T]$ is.

If $p(x)$ is a definitizing polynomial for $T^*[T]$, then $xp(x)$ is a definitizing polynomial for $TT^*[T]$. 
Definitizability

Assume that $T^*[T]$ and $TT^*[T]$ are (densely defined and) selfadjoint operators in $\mathcal{K}$; and that $T^*[T]$ and $TT^*[T]$ have nonempty resolvent sets. Then $T^*[T]$ is definitizable if and only if $TT^*[T]$ is.

If $p(x)$ is a definitizing polynomial for $T^*[T]$, then $xp(x)$ is a definitizing polynomial for $TT^*[T]$.

Indeed, for $f \in D((TT^*[T])p(TT^*[T]))$ we have

$$[(TT^*[T])p(TT^*[T])f, f] = [T^*[T]p(TT^*[T])f, T^*[T]f] = [p(T^*[T]T^*[T])f, T^*[T]f] \geq 0.$$
Standing assumption from now on:

\( T^{[*]} \) and \( TT^{[*]} \) are:

- densely defined, and selfadjoint,
- definitizable,
- have nonempty resolvent sets.
Nonzero spectral points

The nonzero finite spectra of $T^*[T]$ and $TT^*$ are the same, including all spectral properties.
Nonzero spectral points

The nonzero finite spectra of \( T^*T \) and \( TT^* \) are the same, including all spectral properties.

**Proposition** Infinity can be a critical point of at most one of the operators \( T^*T \) and \( TT^* \).

This uses the definitizability: if \( p(x) \) is definitizing for \( T^*T \) and \( p \) is of odd degree then \( \infty \) is critical point for \( T^*T \). Then \( xp(x) \) is of even degree, and is definitizing for \( TT^* \). So \( \infty \) is not a critical point for \( TT^* \).
Nonzero spectral points

The nonzero finite spectra of $T^* T$ and $TT^*$ are the same, including all spectral properties.

Proposition *Infinity can be a critical point of at most one of the operators $T^* T$ and $TT^*$.*

This uses the definitizability: if $p(x)$ is definitizing for $T^* T$ and $p$ is of odd degree then $\infty$ is critical point for $T^* T$. Then $xp(x)$ is of even degree, and is definitizing for $TT^*$. So $\infty$ is not a critical point for $TT^*$.

Moreover, if $T^* T x = \lambda_0 x$, $[x, x] > 0$, then $TT^*(Tx) = \lambda_0(Tx)$, $[Tx, Tx] = \lambda_0[x, x]$. 
Infinity as a spectral point

<table>
<thead>
<tr>
<th>$T^T^{[<em>]} \setminus T^{[</em>]} T$</th>
<th>$\infty \in \rho$</th>
<th>$\infty \in \sigma_+ \cup \sigma_-$</th>
<th>$\infty$ reg. crit.</th>
<th>$\infty$ sing. crit.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty \in \rho$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Infinity as a spectral point

<table>
<thead>
<tr>
<th>$TT^{[<em>]} \setminus T^{[</em>]}T$</th>
<th>$\infty \in \rho$</th>
<th>$\infty \in \sigma_+ \cup \sigma_-$</th>
<th>$\infty \text{ reg. crit.}$</th>
<th>$\infty \text{ sing. crit.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty \in \rho$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\infty \in \sigma_+ \cup \sigma_-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>
### Infinity as a spectral point

<table>
<thead>
<tr>
<th>$TT^{[<em>]} \setminus T^{[</em>]}T$</th>
<th>$\in \rho$</th>
<th>$\in \sigma_+ \cup \sigma_-$</th>
<th>$\text{reg. crit.}$</th>
<th>$\text{sing. crit.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\in \rho$</td>
<td>$+$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\in \sigma_+ \cup \sigma_-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>-</td>
</tr>
<tr>
<td>$\text{reg. crit.}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\text{sing. crit.}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Infinity as a spectral point
Zero as a spectral point

<table>
<thead>
<tr>
<th>$TT^{(<em>)} \setminus T^{(</em>)}T$</th>
<th>$0 \in \rho$</th>
<th>$0 \in \sigma_+ \cup \sigma_-$</th>
<th>$0$ reg. crit.</th>
<th>$0$ sing. crit.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \in \rho$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>
### Zero as a spectral point

<table>
<thead>
<tr>
<th>$TT^{(<em>)} \setminus T^{(</em>)}T$</th>
<th>$0 \in \rho$</th>
<th>$0 \in \sigma_+ \cup \sigma_-$</th>
<th>0 reg. crit.</th>
<th>0 sing. crit.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \in \rho$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$0 \in \sigma_+ \cup \sigma_-$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>0 reg. crit.</td>
<td></td>
<td></td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>0 sing. crit.</td>
<td></td>
<td></td>
<td></td>
<td>+</td>
</tr>
</tbody>
</table>

Table 2: Zero as a spectral point
Zero as a spectral point

<table>
<thead>
<tr>
<th>$TT^{[<em>]} \setminus T^{[</em>]} T$</th>
<th>$0 \in \rho$</th>
<th>$0 \in \sigma_{+} \cup \sigma_{-}$</th>
<th>$0$ reg. crit.</th>
<th>$0$ sing. crit.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \in \rho$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$0 \in \sigma_{+} \cup \sigma_{-}$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$0$ reg. crit.</td>
<td></td>
<td></td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$0$ sing. crit.</td>
<td></td>
<td></td>
<td></td>
<td>+</td>
</tr>
</tbody>
</table>

Table 2: Zero as a spectral point

All examples in $\Pi_1$. 
Example 1: $0 \in \rho(T^{[*]}T)$, regular c.p. for $TT^{[*]}$.

On $\ell^2(\mathbb{Z})$ define

$$J = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$  

$$T = \begin{pmatrix} \vdots \\ \vdots \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$
Example 1 continued

Computation:

\[ T^{[*]} T = I \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I, \]

\[ T T^{[*]} = I \oplus \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \oplus I. \]
Example 1 continued

Computation:

\[ T^{[*]}T = I \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I, \]

\[ TT^{[*]} = I \oplus \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \oplus I. \]

\( 0 \not\in \sigma(T^{[*]}T) \), and 0 is a regular critical point for \( TT^{[*]} \) as the algebraic root subspace corresponding to zero is nondegenerate.
Example 2: $0 \in \sigma_+(T^{[*]}T)$, and a singular c.p. for $TT^{[*]}$.

\[
\mathcal{K} = L^2[0, 1] \oplus \mathbb{C}^2 \oplus \ell^2.
\]

\[
J = I_{L^2[0,1]} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{\ell^2}.
\]

Obviously, $\mathcal{K}$ with the $J$-inner product is a $\Pi_1$-space.
Example 2: the operator

Consider the operator

\[ T := \begin{pmatrix} M\sqrt{t} & 0 & \pi(1) & 0 \\ \langle \cdot, 1 \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \pi(e_1) & 0 & S \end{pmatrix} \]

\( M_\phi \) denotes the multiplication operator by a bounded function \( \phi \),
\( S \) is the shift operator in \( \ell^2 \) (\( Se_j = e_{j+1} \)),
\( \pi(g) \) (where \( g \) is an element of some Hilbert space) maps \( x \in \mathbb{C} \) to \( xg \)
and \( 1 \in L^2[0, 1] \) is a function constantly equal one.
Example 2 continued

\[ T^{[*]} = \begin{pmatrix} M \sqrt{t} & 0 & \pi(1) & 0 \\ \langle \cdot, 1 \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & \langle \cdot, e_1 \rangle \\ 0 & 0 & 0 & S^* \end{pmatrix}, \]

\[ T^{[*]} T = \begin{pmatrix} M_t & 0 & \pi(\sqrt{t}) & 0 \\ \langle \cdot, \sqrt{t} \rangle & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{\ell^2} \end{pmatrix}, \]

\[ TT^{[*]} = \begin{pmatrix} M_t & 0 & \pi(\sqrt{t}) & \langle \cdot, e_1 \rangle 1 \\ \langle \cdot, \sqrt{t} \rangle & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \langle \cdot, 1 \rangle e_1 & 0 & 0 & \ell S S^* \end{pmatrix}. \]
Example 2 continued

\[ T^{[*]} = \begin{pmatrix} M_{\sqrt{t}} & 0 & \pi(1) & 0 \\ \langle \cdot, 1 \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & \langle \cdot, e_1 \rangle \\ 0 & 0 & 0 & S^{*} \end{pmatrix}, \]

\[ T^{[*]} T = \begin{pmatrix} M_t & 0 & \pi(\sqrt{t}) & 0 \\ \langle \cdot, \sqrt{t} \rangle & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{\ell^2} \end{pmatrix}, \]

\[ T T^{[*]} = \begin{pmatrix} M_t & 0 & \pi(\sqrt{t}) & \langle \cdot, e_1 \rangle \cdot 1 \\ \langle \cdot, \sqrt{t} \rangle & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \langle \cdot, 1 \rangle e_1 & 0 & 0 & S S^{*} \end{pmatrix}. \]

Zero is not in the point spectrum of \( T^{[*]} T \), and as the root subspace of \( T T^{[*]} \) corresponding to zero is degenerate, zero is a singular critical point for \( T T^{[*]} \).
Jordan chains

Focus on the $\Pi_\kappa$ case here.

Recall: Jordan chains of a selfadjoint operator $A$ in $\Pi_\kappa$ can not be longer than $2\kappa + 1$.

For each eigenvalue $\lambda$ the root subspace $\mathcal{R}(\lambda, A)$ is well-defined.

$$\mathcal{R}(\lambda, A) = \mathcal{K}_0 \dot{+} \mathcal{K}_1,$$

where $\mathcal{K}_0$ is finite dimensional, $A \mid_{\mathcal{K}_1} = \lambda \cdot I$. 
Jordan chains

Focus on the $\Pi_\kappa$ case here.

Recall: Jordan chains of a selfadjoint operator $A$ in $\Pi_\kappa$ can not be longer than $2\kappa + 1$.

For each eigenvalue $\lambda$ the root subspace $\mathcal{R}(\lambda, A)$ is well-defined.

$$\mathcal{R}(\lambda, A) = \mathcal{K}_0 + \mathcal{K}_1,$$

where $\mathcal{K}_0$ is finite dimensional, $A |_{\mathcal{K}_1} = \lambda \cdot I$.

Let $n_0, \ldots, n_k$ be the lengths of Jordan chains of $A |_{\mathcal{K}_0}$ in decreasing order.

Define the Segre characteristic: $(n_j)_{j=0}^\infty$ for $A$ at $\lambda$ as

$$n_0, \ldots, n_k, 1, \ldots, 1, 0, 0, \ldots$$

with $\dim \mathcal{K}_1$.
Jordan chains for $T^*T$ and $TT^*$

Known already (van der Mee, R., Rodman):
the negative part of the spectrum of $T^*T$ is finite,
thus there are no singular critical points on the negative part of the real axes.
Moreover, all the algebraic root spaces corresponding to negative eigenvalues are finite dimensional.
Jordan chains for $T^{[*]} T$ and $TT^{[*]}$

Known already (van der Mee, R., Rodman):
the negative part of the spectrum of $T^{[*]} T$ is finite,
thus there are no singular critical points on the negative part of the real axes.
Moreover, all the algebraic root spaces corresponding to negative eigenvalues are finite dimensional.

**Theorem** Let $\lambda$ be a complex number, and denote by $(n_j)_{j=1}^{\infty}$ and $(m_j)_{j=1}^{\infty}$ the Segre characteristics for $T^{[*]} T$ and $TT^{[*]}$ respectively, corresponding to $\lambda$. If $\lambda \neq 0$ then $n_j = m_j$ for all $j \in \mathbb{N}$.
If $\lambda = 0$ then $|n_j - m_j| \leq 1$ for $j \in \mathbb{N}$. 
Jordan chains for $T^*[\dagger] T$ and $TT^*[\dagger]$

Known already (van der Mee, R., Rodman):
the negative part of the spectrum of $T^*[\dagger] T$ is finite,
thus there are no singular critical points on the negative part of the real axes.
Moreover, all the algebraic root spaces corresponding to negative eigenvalues are finite dimensional.

**Theorem** Let $\lambda$ be a complex number, and denote by $(n_j)_{j=1}^\infty$ and $(m_j)_{j=1}^\infty$ the Segre characteristics for $T^*[\dagger] T$ and $TT^*[\dagger]$ respectively, corresponding to $\lambda$. If $\lambda \neq 0$ then $n_j = m_j$ for all $j \in \mathbb{N}$.
If $\lambda = 0$ then $|n_j - m_j| \leq 1$ for $j \in \mathbb{N}$.

Follows from a result of Flanders.
Finite dimensional case

In principle completely solved in several ways: results of Mehl, Mehrmann, Xu give a complete solution.
Results of Ran, Wojtylak give a reduction procedure.
References


