The pair of operators $T^*[T]$ and $TT^*$; J-dilations and canonical forms

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Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $H = H^*$ be bounded and invertible. The indefinite inner product on $\mathcal{H}$ given by $H$ is

$$[x, y] = \langle Hx, y \rangle.$$
Preliminaries

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $H = H^*$ be bounded and invertible. The indefinite inner product on $\mathcal{H}$ given by $H$ is

$$[x, y] = \langle Hx, y \rangle.$$ 

Likewise, let $\mathcal{K}$ be a Hilbert space, let $K = K^*$ be bounded and invertible. By abuse of notation we shall also denote the indefinite inner product on $\mathcal{K}$ given by $K$ by $[x, y]$. 

$T^{[*]}$ and $TT^{[*]}$
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For an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ denote by $T^{[*]}$ the adjoint with respect to the indefinite inner products, that is

$$[Tx, y] = [x, T^{[*]}y].$$
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For an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ denote by $T[*]$ the adjoint with respect to the indefinite inner products, that is

$[Tx, y] = [x, T[*]y]$.

In terms of the Hilbert space adjoint, we have

$T[*] = K^{-1}T^*H$.
Given $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, what can be said about the spectral properties of $TT^*\star$ in terms of the spectral properties of $T^*T$?

Observe: both these operators are selfadjoint in an indefinite inner product space: $TT^*\star$ is selfadjoint in the indefinite inner product given on $\mathcal{K}$ by $K$, $T^*T$ is selfadjoint in the indefinite inner product given on $\mathcal{H}$ by $H$. 

Problem statement

Given $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, what can be said about the spectral properties of $TT^*$ in terms of the spectral properties of $T^*T$?

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- Finite dimensional case
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- Finite dimensional case
- Pontryagin space case (i.e., $H$ has finitely many negative eigenvalues)
Problem statement

Given $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, what can be said about the spectral properties of $TT^{[*]}$ in terms of the spectral properties of $T^{[*]}T$?

Observe: both these operators are selfadjoint in an indefinite inner product space: $TT^{[*]}$ is selfadjoint in the indefinite inner product given on $\mathcal{K}$ by $K$, $T^{[*]}T$ is selfadjoint in the indefinite inner product given on $\mathcal{H}$ by $H$.

- Finite dimensional case
- Pontryagin space case (i.e., $H$ has finitely many negative eigenvalues)
- General Krein space

Note: the Hilbert space case is well-known.
Jordan chains

Focus on the finite dimensional case here.

Let \( \lambda \) be an eigenvalue of \( A \), and let \( n_0, \ldots, n_k \) be the lengths of Jordan chains of \( A \) corresponding to \( \lambda \) in decreasing order. Define the Segre characteristic: \( (n_j)_{j=0}^{\infty} \) for \( A \) at \( \lambda \) as

\[
    n_0, \ldots, n_k, 0, 0, \ldots
\]
Finite dimensional case: canonical form

In the finite dimensional case there is a canonical form for selfadjoint matrices in indefinite inner product spaces.

**Examples:**

Define \( P_k = \begin{pmatrix} \cdot & \cdot & \cdots & 1 \\ \cdot & \cdots & \cdots & \cdot \\ 1 \end{pmatrix} \) of size \( k \times k \) and \( J_k(\lambda) \) the \( k \times k \) Jordan block

\[
J_k(\lambda) = \begin{pmatrix} \lambda & 1 \\ & \ddots & \ddots & \ddots \\ & & \ddots & 1 \\ & & & \lambda \\
\end{pmatrix}
\]

- \( H = \varepsilon P_k, A = J_k(\lambda) \) with \( \lambda \in \mathbb{R}, \varepsilon = \pm 1 \)
- \( H = P_{2k}, A = J_k(\lambda) \oplus J_k(\bar{\lambda}) \) with \( \lambda \notin \mathbb{R} \)
Theorem For any pair of matrices \((H, A)\) with \(H = H^*\) invertible and \(HA = A^*H\) there exists an invertible matrix \(S\) such that \((S^*HS, S^{-1}AS)\) is a diagonal direct sum of blocks of the form as in the examples above.

The signs \(\varepsilon\), one for each block corresponding to a real eigenvalue of \(A\), are unique (up to trivialities).

These signs are called the sign characteristic of the pair.
Canonical form II

Each selfadjoint matrix in a finite dimensional Krein space is determined (up to similarity) by its sign characteristic: Segre characteristic with signs.
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\[ \begin{array}{cccccc}
+ & - & - & 0 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & 0 \\
\end{array} \ldots \]

denotes
Canonical form II

Each selfadjoint matrix in a finite dimensional Krein space is determined (up to similarity) by its sign characteristic: Segre characteristic with signs. For nilpotent matrices:

\[
\begin{array}{cccccc}
\uparrow & - & - \\
3 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots
\end{array}
\]

denotes \( S^{-1}AS = J_3(0) \oplus J_2(0) \oplus J_1(0) \)
Canonical form II

Each selfadjoint matrix in a finite dimensional Krein space is determined (up to similarity) by its sign characteristic: Segre characteristic with signs. For nilpotent matrices:

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3 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots
\end{pmatrix}
\]

denotes \( S^{-1} AS = J_3(0) \oplus J_2(0) \oplus J_1(0) \)

and the indefinite inner product is given by

\[
S^* HS = (+1) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus (-1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (-1)(1)
\]
Problem statement

Compare the sign characteristics for $T^*[T]$ and $TT^*[T]$. 
Problem statement

Compare the sign characteristics for $T^{[*]}T$ and $TT^{[*]}$.

Only the zero eigenvalue is problematic $\leadsto$ nilpotent case (explanation in the next few slides)

$$\begin{pmatrix} T^{[*]}T \\ TT^{[*]} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \dagger & 4 & \dagger \\ 4 & -4 & -4 \end{pmatrix} & \begin{pmatrix} \dagger & 3 & 2 \\ \dagger & -1 & 1 \end{pmatrix} \\ \begin{pmatrix} \dagger & 4 & 4 \\ 4 & -4 & 4 \end{pmatrix} & \begin{pmatrix} \dagger & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

What is possible, what is not?
Problem statement

Compare the sign characteristics for $T^* T$ and $TT^*$. 

Only the zero eigenvalue is problematic $\rightsquigarrow$ nilpotent case (explanation in the next few slides)

\[
\begin{pmatrix}
T^* T \\
TT^*
\end{pmatrix}
= 
\begin{pmatrix}
\begin{array}{cccc}
\dagger & - & \dagger & 2 \\
\dagger & - & \dagger & 1 \\
\dagger & - & - & -
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{cccc}
4 & 4 & 3 & 2 \\
4 & 4 & 1 & 0 \\
4 & 4 & 1 & 0
\end{array}
\end{pmatrix}
\]

What is possible, what is not?

Solved by Mehl, Mehrmann, Xu [2009]

Bolshakov, Reichstein [1995], more or less hidden in the paper.
Theorem [Flanders] If $A$ is an $n \times p$ matrix and $B$ is a $p \times n$ matrix, then:

- the nonzero eigenvalues of $AB$ and $BA$ coincide, partial multiplicities included,
Flanders’ Theorem

Theorem [Flanders] If $A$ is an $n \times p$ matrix and $B$ is a $p \times n$ matrix, then:

- the nonzero eigenvalues of $AB$ and $BA$ coincide, partial multiplicities included,

- let $n_1 \geq n_2 \geq \cdots$ denote the sizes of the Jordan blocks of $AB$ corresponding to the zero eigenvalue, and let $m_1 \geq m_2 \geq \cdots$ denote the sizes of the Jordan blocks of $BA$ corresponding to the zero eigenvalue, then $|n_j - m_j| \leq 1$ for all $j$. 
Flanders’ Theorem

Theorem [Flanders] If $A$ is an $n \times p$ matrix and $B$ is a $p \times n$ matrix, then:

- the nonzero eigenvalues of $AB$ and $BA$ coincide, partial multiplicities included,

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\[
\begin{pmatrix}
T^{[*]}T \\
T'T^{[*]} \\
\end{pmatrix}
\begin{pmatrix}
\begin{array}{cccc}
4 & -4 & 3 & -2 \\
4 & 4 & -2 & -1 \\
4 & 4 & 4 & -1 \\
4 & 4 & 4 & 0 \\
\end{array}
\end{pmatrix}
\]
Nonzero real eigenvalues: sign characteristic

If $\lambda$ is a real nonzero eigenvalue of $T^{[*]} T$ with partial multiplicities $n_1, n_2, \cdots$ and corresponding signs $\varepsilon_1, \varepsilon_2, \cdots$, then $\lambda$ is a real nonzero eigenvalue of $TT^{[*]}$ with partial multiplicities $n_1, n_2, \cdots$ and corresponding signs $\text{sign}(\lambda)\varepsilon_1, \text{sign}(\lambda)\varepsilon_2, \cdots$. 
**Nonzero real eigenvalues: sign characteristic**

If $\lambda$ is a real nonzero eigenvalue of $T^{[*]}T$ with partial multiplicities $n_1, n_2, \cdots$ and corresponding signs $\varepsilon_1, \varepsilon_2, \cdots$, then $\lambda$ is a real nonzero eigenvalue of $TT^{[*]}$ with partial multiplicities $n_1, n_2, \cdots$ and corresponding signs $\text{sign}(\lambda)\varepsilon_1, \text{sign}(\lambda)\varepsilon_2, \cdots$.

Thus, the signs corresponding to negative eigenvalues change.
Nonzero real eigenvalues: sign characteristic

If $\lambda$ is a real nonzero eigenvalue of $T^{[*]} T$ with partial multiplicities $n_1, n_2, \cdots$ and corresponding signs $\varepsilon_1, \varepsilon_2, \cdots$, then $\lambda$ is a real nonzero eigenvalue of $T T^{[*]}$ with partial multiplicities $n_1, n_2, \cdots$ and corresponding signs $\text{sign}(\lambda) \varepsilon_1, \text{sign}(\lambda) \varepsilon_2, \cdots$.

Thus, the signs corresponding to negative eigenvalues change.

So, it remains to consider the case where $\sigma(T^{[*]} T) = \{0\}$.
Given $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ suppose there exist decompositions

$$
\mathcal{H} = \mathcal{H}_0 \bigdot{\bigdot{\bigdot}} \mathcal{H}_1 \bigdot{\bigdot{\bigdot}} (\mathcal{H}_2 \bowtie \mathcal{H}_3), \quad \mathcal{K} = \mathcal{K}_0 \bigdot{\bigdot{\bigdot}} \mathcal{K}_1 \bigdot{\bigdot{\bigdot}} (\mathcal{K}_2 \bowtie \mathcal{K}_3),
$$

where $\mathcal{H}_0, \mathcal{K}_0, \mathcal{H}_1$ and $\mathcal{K}_1$ are Krein spaces, $\mathcal{H}_2$ and $\mathcal{H}_3$ ($\mathcal{K}_2$ and $\mathcal{K}_3$) are skewly linked neutral spaces such that $\mathcal{H}_2 \bowtie \mathcal{H}_3$ ($\mathcal{K}_2 \bowtie \mathcal{K}_3$) is a Krein space.

The operator $T$ has the following representation with respect to the above decomposition

$$
T = \begin{pmatrix}
T_0 & 0 & T_{02} & 0 \\
0 & 0 & 0 & 0 \\
T_{20} & 0 & T_2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$
J-dilation and J-restriction continued

If

$$\hat{T} = \begin{pmatrix} T_0 & T_{02} \\ T_{20} & T_2 \end{pmatrix}$$

is 1-1 and onto (infinite dimensional case: has dense range), then

$T_0$ is called a rigid J-restriction of $T$

$T$ is called a rigid J-dilation of $T_0$.

**Theorem** If $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional spaces then there exists a rigid J-restriction $T_0$ of $T$. 
J-dilation and J-restriction continued

If

\[ \hat{T} = \begin{pmatrix} T_0 & T_{02} \\ T_{20} & T_2 \end{pmatrix} \]

is 1-1 and onto (infinite dimensional case: has dense range), then

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Theorem If \( \mathcal{H} \) and \( \mathcal{K} \) are finite dimensional spaces then there exists a rigid J-restriction \( T_0 \) of \( T \).

Actually also true in Pontryagin spaces.
With respect to a rigid J-restriction we have

\[
T = \begin{pmatrix}
T_0 & 0 & T_{02} & 0 \\
0 & 0 & 0 & 0 \\
T_{20} & 0 & T_2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad T^{[*]} = \begin{pmatrix}
T_0^{[*]} & 0 & 0 & T_{20}^+ \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
T_{02}^+ & 0 & 0 & T_2^+
\end{pmatrix}
\]

\[
T^{[*]}T = \begin{pmatrix}
T_0^{[*]}T_0 & 0 & T_0^{[*]}T_{02} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
T_{02}^+T_0 & 0 & T_{02}^+T_{02} & 0
\end{pmatrix}, \quad TT^{[*]} = \begin{pmatrix}
T_0T_0^{[*]} & 0 & 0 & T_0T_{20}^+ \\
0 & 0 & 0 & 0 \\
T_{20}T_0^{[*]} & 0 & 0 & T_{20}T_{20}^+
\end{pmatrix}.
\]
Define the operators

\[ T_{0\bullet} = \begin{pmatrix} T_0 & 0 & T_{02} & 0 \end{pmatrix}, T_{\bullet 0} = \begin{pmatrix} T_0 \\ 0 \\ T_{20} \\ 0 \end{pmatrix}. \]

Then

\[ T^{[*]}T = T_{0\bullet}^{[*]}T_{0\bullet}, \quad TT^{[*]} = T_{\bullet 0}T_{\bullet 0}. \]
Reduction of the problem

\[(T^{[*]} T)^j = T_0^{[*]} (T_0 T_0^{[*]})^{j-1} T_0^{[*]}, \quad j \geq 1,\]
\[(T T^{[*]})^j = T_0 (T_0^{[*]} T_0)^{j-1} T_0^{[*]}, \quad j \geq 1.\]

**Theorem** Let \( \mathcal{H} \) and \( \mathcal{K} \) be finite dimensional. If \( T_0 \) is a rigid restriction of \( T \), and \( T^{[*]} T \) is nilpotent with Segre characteristic

\[n_1 \geq n_2 \geq n_3 \geq \cdots,\]

then the Segre characteristic of \( T_0 T_0^{[*]} \) is

\[m_1 \geq m_2 \geq m_3 \geq \cdots,\]

where

\[m_j = \max\{n_j - 1, 0\}.\]
Decomposing pairs

A pair of spaces $\mathcal{E} \subset \mathcal{H}$ and $\mathcal{F} \subset \mathcal{K}$ is said to decompose $T$ if they are nondegenerate, $T\mathcal{E} \subset \mathcal{F}$ and $T^{[*]}\mathcal{F} \subset \mathcal{E}$. Note: in that case $\mathcal{E}^{[\perp]}$ and $\mathcal{F}^{[\perp]}$ decompose $T$ as well.

$$\mathcal{H} = \mathcal{E}_1 \bigoplus \mathcal{E}_2 \bigoplus \cdots \bigoplus \mathcal{E}_n$$

$$\mathcal{K} = \mathcal{F}_1 \bigoplus \mathcal{F}_2 \bigoplus \cdots \bigoplus \mathcal{F}_n$$

$\mathcal{E}_i, \mathcal{F}_i \ (i = 1, \ldots, n)$— non degenerate (or trivial) subspaces
Decomposing pairs

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$$\mathcal{H} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_n$$

$$\mathcal{K} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_n$$

$\mathcal{E}_i, \mathcal{F}_i (i = 1, \ldots, n)$— non degenerate (or trivial) subspaces
Decomposing pairs

A pair of spaces $\mathcal{E} \subset \mathcal{H}$ and $\mathcal{F} \subset \mathcal{K}$ is said to decompose $T$ if they are nondegenerate, $T\mathcal{E} \subset \mathcal{F}$ and $T^\star \mathcal{F} \subset \mathcal{E}$. Note: in that case $\mathcal{E}^\perp$ and $\mathcal{F}^\perp$ decompose $T$ as well.

\[
\mathcal{H} = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_n
\]

\[
T \downarrow \uparrow T^\star \quad T \downarrow \uparrow T^\star \quad T \downarrow \uparrow T^\star
\]

\[
\mathcal{K} = \mathcal{F}_1 \mathcal{F}_2 \cdots \mathcal{F}_n
\]

$\mathcal{E}_i, \mathcal{F}_i \ (i = 1, \ldots, n)$— non degenerate (or trivial) subspaces

$(\mathcal{E}_i, \mathcal{F}_i)$ – decomposing pair.
Types of decomposing pairs

We say that a decomposing pair \((\mathcal{E}_i, \mathcal{F}_i)\) is of
Types of decomposing pairs

We say that a decomposing pair \((E_i, F_i)\) is of type (i) if the sign char. for \(\begin{pmatrix} T^{[*]} T & E_i \\ TT^{[*]} & F_i \end{pmatrix} \) are \(\begin{pmatrix} + & - \\ k & k \\ + & - \end{pmatrix} \) \((k \geq 1)\)
Types of decomposing pairs

We say that a decomposing pair \((E_i, F_i)\) is of

- **type (i)** if the sign char. for \(\begin{pmatrix} T^{[*]}T | E_i \end{pmatrix} \) are
  \[
  \begin{pmatrix}
    + & - \\
    k & k \\
    + & - \\
    k & k \\
    \pm & - \\
    k \\
    \pm & - \\
    k + 1
  \end{pmatrix}
  \]
  \((k \geq 1)\)

- **type (ii)** if the sign char. for \(\begin{pmatrix} TT^{[*]} | F_i \end{pmatrix} \) are
  \[
  \begin{pmatrix}
    + & - \\
    k & k \\
    \pm & - \\
    k \\
    \pm & - \\
    k + 1
  \end{pmatrix}
  \]
  \((k \geq 1)\)
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We say that a decomposing pair \((E_i, F_i)\) is of

- **type (i)** if the sign char. for \(\begin{pmatrix} T^*[T|E_i] \\ TT^*[T|F_i] \end{pmatrix}\) are
  \[
  \begin{pmatrix}
  k & k \\
  k & k \\
  \pm k & \pm k \\
  k & k \\
  \end{pmatrix}
  \]
  \((k \geq 1)\)

- **type (ii)**

- **type (iii)**

\(T^*[T\ and\ TT^*\]

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Types of decomposing pairs

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  \[
  \begin{pmatrix}
  + & - \\
  k & k \\
  \pm & k \\
  k & k \\
  \pm & \pm \\
  k + 1 \\
  k + 1 \\
  k \\
  0 \\
  \pm \\
  1
  \end{pmatrix}
  \]

  \((k \geq 1)\)

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- **type (iii)**

- **type (iv)**
Types of decomposing pairs

We say that a decomposing pair \((\mathcal{E}_i, \mathcal{F}_i)\) is of

- **type (i)** if the sign char. for \(\begin{pmatrix} T^{[*]} T \mid \mathcal{E}_i \end{pmatrix} \) are
  \[
  \begin{pmatrix}
  + & - \\
  k & -k \\
  \pm & k \\
  k & k \\
  0 & 1
  \end{pmatrix}
  \] (\(k \geq 1\))

- **type (ii)**
  \[
  \begin{pmatrix}
  \pm & k \\
  k & k \\
  k + 1 & k \\
  k & k \\
  0 & 1
  \end{pmatrix}
  \] (\(k \geq 1\))

- **type (iii)**
  \[
  \begin{pmatrix}
  \pm & k \\
  k & k \\
  k + 1 & k \\
  k & k \\
  0 & 1
  \end{pmatrix}
  \] (\(k \geq 1\))

- **type (iv)**
  \[
  \begin{pmatrix}
  0 & 1 \\
  -1 & 0 \\
  -1 & 0 \\
  0 & 1
  \end{pmatrix}
  \]

- **type (v)**
  \[
  \begin{pmatrix}
  0 & 1 \\
  0 & 1
  \end{pmatrix}
  \]
Final result

**Theorem** Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional and let the matrices $TT^\ast$ and $T^\ast T$ be nilpotent. Then there exist subspaces $E_i$ of $\mathcal{H}$ and $F_i$ of $\mathcal{K}$ $(i = 1, \ldots, p)$ such that

$$
\begin{align*}
\mathcal{H} &= E_1 \updownarrow T^\ast E_2 \updownarrow \cdots \updownarrow T^\ast E_n \\
\mathcal{K} &= F_1 \updownarrow T^\ast F_2 \updownarrow \cdots \updownarrow T^\ast F_n
\end{align*}
$$

and each pair $E_i, F_i$ is of the type (i)-(v).
Theorem Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional and let the matrices $TT^{[*]}$ and $T^{[*]}T$ be nilpotent. Then there exist subspaces $\mathcal{E}_i$ of $\mathcal{H}$ and $\mathcal{F}_i$ of $\mathcal{K}$ $(i = 1, \ldots, p)$ such that

\[
\mathcal{H} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_n
\]

\[
T \downarrow \uparrow T^{[*]} \quad T \downarrow \uparrow T^{[*]} \quad T \downarrow \uparrow T^{[*]}
\]

\[
\mathcal{K} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_n
\]

and each pair $\mathcal{E}_i$, $\mathcal{F}_i$ is of the type (i)-(v).

Theorem is due to Mehl, Mehrmann, Xu in a different form.
**Theorem** Let \( \mathcal{H} \) and \( \mathcal{K} \) be finite dimensional and let the matrices \( TT[^{*}] \) and \( T[^{*}]T \) be nilpotent. Then there exist subspaces \( \mathcal{E}_i \) of \( \mathcal{H} \) and \( \mathcal{F}_i \) of \( \mathcal{K} \) \((i = 1, \ldots, p)\) such that

\[
\begin{align*}
\mathcal{H} &= \mathcal{E}_1 \uparrow \mathcal{E}_2 \uparrow \cdots \uparrow \mathcal{E}_n \\
\mathcal{K} &= \mathcal{F}_1 \uparrow \mathcal{F}_2 \uparrow \cdots \uparrow \mathcal{F}_n
\end{align*}
\]

and each pair \( \mathcal{E}_i, \mathcal{F}_i \) is of the type (i)-(v).

Theorem is due to Mehl, Mehrmann, Xu in a different form.
Example

\[
\begin{pmatrix}
T^*[T] \\
T[T]^*[T]
\end{pmatrix}
\begin{pmatrix}
* \\
4
\end{pmatrix} - \text{impossible}
\]
Example

\[
\begin{pmatrix}
T^{[*]}T \\
TT^{[*]}
\end{pmatrix}
\begin{pmatrix}
\ast \\
4
\end{pmatrix} - \text{impossible}
\]

\[
\begin{pmatrix}
\pm & -k \\
+k & k
\end{pmatrix}
\begin{pmatrix}
\pm & k \\
\pm & k
\end{pmatrix}
\begin{pmatrix}
\pm & k + 1 \\
\pm & k
\end{pmatrix}
\begin{pmatrix}
0 \\
\pm & 1
\end{pmatrix}
\begin{pmatrix}
\pm & 1 \\
0
\end{pmatrix}
\]
Example

\[
\begin{pmatrix}
T^*[T] \\
TT^*[T]
\end{pmatrix}
\begin{pmatrix}
4 \\
4
\end{pmatrix}
- \text{impossible}
\]

\[
\begin{pmatrix}
\pm k \\
\pm k
\end{pmatrix}
\begin{pmatrix}
\pm k \\
\pm k
\end{pmatrix}
\begin{pmatrix}
k+1 \\
k+1
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\begin{pmatrix}
\pm 1 \\
\pm 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

Although there exists matrices \( A, B \) such that the Segre characteristics of \( AB \) and \( BA \) is (4) [Flanders 1951]
Method of proof

Proof (R.-Wojtylak): use the reduction procedure
Method of proof

Proof (R.-Wojtylak): use the reduction procedure and induction w.r.t. the length of the longest Jordan chain of $T^{[*]}T$ and $TT^{[*]}$.

Case $T^{[*]}T = 0$, $TT^{[*]} = 0$ simple.
Method of proof

Proof (R.-Wojtylak): use the reduction procedure and induction w.r.t. the length of the longest Jordan chain of $T^*[T]$ and $TT^*$. 

Case $T^*[T] = 0$, $TT^*[T] = 0$ simple.

$T^*[T] \rightsquigarrow$ J-restriction $T_0T_0^*$
Method of proof

Proof (R.-Wojtylak): use the reduction procedure and induction w.r.t. the length of the longest Jordan chain of $T[^*]T$ and $TT[^*]$.

Case $T[^*]T = 0$, $TT[^*] = 0$ simple.

$T[^*]T \rightsquigarrow$ J-restriction $T_0T_0[^*]$

5 4 4 3 2 1 1 1 0 0 0 0 ... - Segre characteristic for $T[^*]T$

4 3 3 2 1 0 0 0 0 0 0 ... - Segre characteristic for $T_0T_0[^*]$

For $T_0T_0[^*]$ we apply the induction hypothesis.
Method of proof

Proof (R.-Wojtylak): use the reduction procedure and induction w.r.t. the length of the longest Jordan chain of $T^*[T]$ and $TT^*$. 

Case $T^*[T] = 0$, $TT^* = 0$ simple.

$T^*[T] \sim J\text{-restriction } T_0T_0^*$

5 4 4 3 2 1 1 1 0 0 0 0 ... - Segre characteristic for $T^*[T]
4 3 3 2 1 0 0 0 0 0 0 0 ... - Segre characteristic for $T_0T_0^*$

For $T_0T_0^*$ we apply the induction hypothesis.
The pair $T_0T_0^*$ and $T_0^*T_0$ can be decomposed in
The pair $T_0T_0^*$ and $T_0^*T_0$ can be decomposed in

$$
\mathcal{H}_0 = \mathcal{E}_1 \quad \cdots \quad \mathcal{E}_n
$$

$$
\mathcal{K}_0 = \mathcal{F}_1 \quad \cdots \quad \mathcal{F}_n
$$

$T_0 \downarrow \uparrow T_0^*$

$T_0 \downarrow \uparrow T_0^*$

$T_0 \downarrow \uparrow T_0^*$

$T_0 \downarrow \uparrow T_0^*$
Long story short

The pair $T^*T_0$ and $T_0^*T_0$ can be decomposed in

$$
\mathcal{H}_0 = \mathcal{E}_1 \downarrow \uparrow T_0^* \quad \mathcal{E}_n
$$

$$
\mathcal{K}_0 = \mathcal{F}_1 \downarrow \uparrow T_0 \quad \mathcal{F}_n
$$

Lift this to

$$
\mathcal{H} = \hat{\mathcal{E}}_1 \downarrow \uparrow T^* \quad \hat{\mathcal{E}}_n
$$

$$
\mathcal{K} = \hat{\mathcal{F}}_1 \downarrow \uparrow T \quad \hat{\mathcal{F}}_n
$$
Long story short

The pair $T_0 T_0^{[*]}$ and $T_0^{[*]} T_0$ can be decomposed in

$$
\mathcal{H}_0 = \mathcal{E}_1 \uparrow \cdots \uparrow \mathcal{E}_n
$$
$$
\mathcal{K}_0 = \mathcal{F}_1 \uparrow \cdots \uparrow \mathcal{F}_n
$$

Lift this to

$$
\mathcal{H} = \mathcal{\hat{E}}_1 \uparrow \cdots \uparrow \mathcal{\hat{E}}_n \uparrow \mathcal{\hat{E}}_0
$$
$$
\mathcal{K} = \mathcal{\hat{F}}_1 \uparrow \cdots \uparrow \mathcal{\hat{F}}_n \uparrow \mathcal{\hat{F}}_0
$$

such that $(\mathcal{E}_i, \mathcal{F}_i)$ has the same type as $(\mathcal{\hat{F}}_i, \mathcal{\hat{E}}_i)$ and $T^{[*]} T|_{\mathcal{\hat{E}}_0} = 0$, $TT^{[*]}|_{\mathcal{\hat{F}}_0} = 0$. 

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The pair $T_0 T_0^*$ and $T_0^* T_0$ can be decomposed in

$$
\begin{align*}
\mathcal{H}_0 &= \mathcal{E}_1 \uparrow \cdots \uparrow \mathcal{E}_n \\
T_0 &\downarrow \uparrow T_0^* \\
\mathcal{K}_0 &= \mathcal{F}_1 \uparrow \cdots \uparrow \mathcal{F}_n \\
\end{align*}
$$

Lift this to

$$
\begin{align*}
\mathcal{H} &= \hat{\mathcal{E}}_1 \uparrow \cdots \uparrow \hat{\mathcal{E}}_n \uparrow \hat{\mathcal{E}}_0 \\
T &\downarrow \uparrow T^{[*]} \\
\mathcal{K} &= \hat{\mathcal{F}}_1 \uparrow \cdots \uparrow \hat{\mathcal{F}}_n \uparrow \hat{\mathcal{F}}_0 \\
\end{align*}
$$

such that $(\mathcal{E}_i, \mathcal{F}_i)$ has the same type as $(\hat{\mathcal{F}}_i, \hat{\mathcal{E}}_i)$ and $T^{[*]} T|_{\hat{\mathcal{E}}_0} = 0$, $TT^{[*]}|_{\hat{\mathcal{F}}_0} = 0$. Method: $\begin{pmatrix} T_0 & 0 & T_0^2 & 0 \end{pmatrix} \hat{\mathcal{E}}_i = \mathcal{F}_i$. 
References


A.C.M. Ran and M. Wojtylak: The pair of operators $T^*T$ and $TT^*$; J–dilations and canonical forms. Accepted for publication Integral Equations and Operator Theory.
