Hyponormal matrices and semidefinite invariant subspaces in indefinite inner product spaces

Joint work with:
Christian Mehl
Leiba Rodman
Indefinite inner product

On $\mathbb{C}^n$ define indefinite inner product given by an invertible $H = H^*$ as follows:

$$[x, y] = \langle Hx, y \rangle$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product.
The $H$-adjoint

$n \times n$ matrix $X$

$H$-adjoint:

$$[Xx, y] = [x, X[^*]y]$$

$$X[^*] = H^{-1}X^*H$$
Classes of matrices

- $H$-expansion if $X^*HX \geq H$

- $H$-selfadjoint if $X = X^*$

- $H$-unitary if $XX^* = I$

- $H$-dissipative if $iH(X - X^*) \leq 0$

- $H$-normal if $X^*X = XX^*$

- $H$-hyponormal if $H(X^*X - XX^*) \geq 0$
Invariant maximal $H$-nonnegative/nonpositive subspaces

Well known result:

**Theorem 1.** If $X$ is an $H$-expansion in a finite dimensional space, and $M_0$ is $X$-invariant and $H$-nonnegative then there exists a maximal $H$-nonnegative $X$-invariant subspace $M$ with $M_0 \subset M$.

Usual proof uses a fixed point argument.
Invariant maximal $H$-nonnegative/nonpositive subspaces II

Also well known: if $X$ is $H$-selfadjoint in a finite dimensional space then there is an explicit construction of an $X$-invariant maximal $H$-nonnegative subspace. Uses canonical form under the equivalence $(X, H) \mapsto (S^{-1}XS, S^*HS)$. Real cases done as well.

If $X$ is $H$-dissipative in finite dimensional space there is also an explicit construction. Uses a “simple” form under the equivalence above. Done for complex matrices in a complex space, currently under investigation are the real cases.
Recall: $X$ is $H$-normal if $X^*[X^*]X = XX^*[X^*].$

Canonical forms: when $H$ has only one negative eigenvalue there is a classification of the equivalence classes under the equivalence $(X, H) \mapsto (S^{-1}XS, S^*HS).$

Same when $H$ has two negative eigenvalues. Number of equivalence classes is (very) large.

Points to the fact that an explicit construction of invariant maximal nonnegative subspaces using a canonical form will be impossible.
Invariant maximal $H$-nonnegative/nonpositive subspaces for $H$-normal matrices

**Theorem 2.** Let $X$ be an $H$-normal matrix. Then there is an $X$-invariant maximal $H$-nonnegative subspace.

Proof in finite dimensional case: write $A = \frac{1}{2}(X + X^[*])$, $S = \frac{1}{2}(X - X^[*])$. Then $X = A + S$ and $AS = SA$. Prove by induction on the dimension of the space that $A$ and $S$ have a common invariant maximal $H$-nonnegative subspace.

Observation: Let $X$ be $H$-normal, $\mathcal{M}$ an $X$-invariant maximal $H$-nonnegative/nonpositive subspace. Then $\mathcal{M}$ is invariant also for $X^[*]$. 
Invariant maximal $H$-nonnegative/nonpositive subspaces
for $H$-normal matrices II: Extension results

Two extension results:

i. Let $X$ be $H$-normal, $\mathcal{M}_0$ an $H$-nonnegative $X$-invariant subspace which is also invariant for $X^\ast$. Then there exists an $X$-invariant maximal $H$-nonnegative subspace $\mathcal{M}$ containing $\mathcal{M}_0$.

ii. Let $X$ be $H$-normal, $\mathcal{M}_0$ an $H$-neutral $X$-invariant subspace. Then there exists an $X$-invariant maximal $H$-nonnegative subspace $\mathcal{M}$ containing $\mathcal{M}_0$. 
Caveat!! Not a simple result. Follows from the following:

iii. Let $X$ be a matrix, and suppose that
1. $\mathcal{M}_0$ is $H$-neutral and $X$-invariant,
2. $\mathcal{M}_0$ is also $XX[*] - X[*]X$-invariant,
3. $\mathcal{M}_0^{[\perp]} \cap (H^{-1}\mathcal{M}_0)^{[\perp]}$ is $XX[*] - X[*]X$-neutral.

Then there is a maximal $H$-nonnegative $X$-invariant subspace $\mathcal{M}$ such that $\mathcal{M}_0 \subset \mathcal{M} \subset \mathcal{M}_0^{[\perp]}$. 
Example

\[
X = \begin{pmatrix}
0 & 1 & -1 & -\frac{1}{2} \\
1 & 1 & -\frac{1}{2} & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}, \quad H = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Then \(X\) is \(H\)-normal, \(\mathcal{M}_0 = \text{span}\{e_1, e_2\}\) is \(X\)-invariant and \(H\)-nonnegative.

The only maximal \(H\)-nonnegative subspace containing \(\mathcal{M}_0\) is \(\text{span}\{e_1, e_2, e_4\}\) and that is not \(X\)-invariant.
Another extension result

**Theorem 3.** Let $X$ be $H$-normal and let $\mathcal{M}_0$ be an $H$-positive $X$-invariant subspace. Decompose $\mathbb{C}^n = \mathcal{M}_0 + \mathcal{M}_0^{[\perp]}$. Denote by $P$ the projection onto $\mathcal{M}_0^{[\perp]}$ along $\mathcal{M}_0$, and define

$$X_{22} = PX|_{\mathcal{M}_0^{[\perp]}} : \mathcal{M}_0^{[\perp]} \to \mathcal{M}_0^{[\perp]}.$$

On $\mathcal{M}_0^{[\perp]}$ define an indefinite inner product inherited from $H$. Assume that either

$$\sigma(X_{22} + X_{22}^*) \subset \mathbb{R} \text{ or } \sigma(X_{22} - X_{22}^*) \subset i\mathbb{R}.$$ 

Then there exists an $X$-invariant maximal $H$-nonnegative subspace $\mathcal{M}$ containing $\mathcal{M}_0$. 
$H$-hyponormal matrices

Recall definition: $X$ is $H$-hyponormal if $X^{[*]}X - XX^{[*]}$ is $H$-nonnegative, that is

$$H(X^{[*]}X - XX^{[*]}) \geq 0.$$ 

Problem How far can we extend results on invariant maximal nonnegative/nonpositive subspaces to $H$-hyponormal matrices?

Important role played by $H$-selfadjoint part $A$ and $H$-skew-adjoint part $S$:

$$A = \frac{1}{2}(X + X^{[*]}), \quad S = \frac{1}{2}(X - X^{[*]}).$$
Invariant maximal $H$-nonnegative/nonpositive subspaces for $H$-hyponormal matrices

Main existence result:

**Theorem 4.** Let $X$ be $H$-hyponormal. If either the eigenvalues of $X + X^*$ are all real, or the eigenvalues of $X - X^*$ are all purely imaginary (including zero), then there exist an $X$-invariant maximal $H$-nonnegative subspace $M_+$ and an $X$-invariant maximal $H$-nonpositive subspace $M_-$ that are both also invariant for $X^*$.

Very much a finite-dimensional result. Based on the fact that under the conditions of the theorem $A$ and $S$ will have a common eigenvector, and induction on the size of the matrix.
Extension of invariant $H$-nonpositive subspace

Let $X$ be $H$-hyponormal, and let $\mathcal{M}_-$ be $X$-invariant and $H$-nonpositive. Let $\mathcal{M}_0 = \mathcal{M}_- \cap \mathcal{M}_-^{[\perp]}$. Decompose

$$\mathcal{M}_-^{[\perp]} = \mathcal{M}_0 \dot{+} \mathcal{M}_{nd},$$

where $\mathcal{M}_{nd}$ is $H$-nondegenerate.

Assumption: $\mathcal{M}_0$ is also $X$-invariant.
Denote by $X_{nd}$ and $H_{nd}$ the compressions of $X$ and $H$ onto $M_{nd}$.

Assumptions on $X$: either one of the following three conditions hold:

a. $\sigma(X_{nd} + X_{nd}^{[*]}) \subset \mathbb{R}$
b. $\sigma(X_{nd} - X_{nd}^{[*]}) \subset i\mathbb{R}$
c. $X_{nd}$ is $H_{nd}$-normal.

Then there is an $X$-invariant maximal $H$-nonpositive subspaces $\mathcal{M}$ containing $\mathcal{M}_-$. The conditions on $X$ are independent of the choice of $\mathcal{M}_{nd}$.

A Pontryagin space version holds as well (finite number of positive squares).
References


Chr. Mehl, A.C.M. Ran en L. Rodman: Extension to maximal semidefinite invariant subspaces for hyponormal matrices in indefinite inner products. Submitted for publication.
ILAS 2006: Welcome to Amsterdam!!!

http://staff.science.uva.nl/~brandts/ILAS06

Amsterdam canal scene in September

Photo: A.S. Tanenbaum
IWOTA July 2007: North-West University, Potchefstroom, South Africa

On behalf of the organizers: you are cordially invited!

Potchefstroom Campus Main Building

More information: Manfred Möller and André Ran are on the organizing committee. We want to tell you all about it!