Integral operators with semi-separable kernel and with symmetries

collaboration with G.J. Groenewald and M.A. Petersen

(Potchefstroom Campus, North-West University, SA)

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Preliminaries and introduction

Integral operator with semi-separable kernel

\[ T : L_2([a, b], \mathcal{U}) \to L_2([a, b], \mathcal{Y}) \]

\[ y(t) = (Tu)(t) = D(t)u(t) + \int_a^b k(t, s)u(s) \, ds \]

where

\[ k(t, s) = \begin{cases} 
F_1(t)G_1(s), & a \leq s < t \leq b, \\
-F_2(t)G_2(s), & a \leq t < s \leq b,
\end{cases} \]

Here \( F_i(t) : \mathcal{X}_i \to \mathcal{Y} \) and \( G_i(t) : \mathcal{U} \to \mathcal{X}_i, \ i = 1, 2 \) are square integrable.

All spaces \( \mathcal{U}, \mathcal{Y}, \mathcal{X}_i \) are finite dimensional.
Realization

Gohberg-Kaashoek, mid eighties Systems with boundary values

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t), & a \leq t \leq b, \\
y(t) &= C(t)x(t) + D(t)u(t), & a \leq t \leq b, \\
(I - P)x(a) + PU(b)^{-1}x(b) &= 0.
\end{align*}
\]

where \( A(t) \) is integrable, \( B(t) \) and \( C(t) \) are square integrable, \( P \) is a projection and \( U(t) \) is the fundamental operator corresponding to \( A(t) \), i.e., \( \dot{U}(t) = A(t)U(t), \ U(a) = I. \)

The system is well-posed, that is \( u(t) \equiv 0 \) implies the existence of a unique solution (which is the zero solution).
In that case

\[ y(t) = D(t)u(t) + \int_a^b k(t, s)u(s) \, ds \]

where

\[ k(t, s) = \begin{cases} 
C(t)U(t)(I - P)U(s)^{-1}B(s), & a \leq s < t \leq b, \\
-C(t)U(t)PU(s)^{-1}B(s), & a \leq t < s \leq b,
\end{cases} \]

So, the input-output operator of the system with boundary values is of the semi-separable type.
Conversely, any such operator admits a realization of this type:

\[ A(t) = 0, B(t) = \begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix}, C(t) = \begin{pmatrix} F_1(t) & F_2(t) \end{pmatrix}, P = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{X}_2} \end{pmatrix}, \]

on the state space \( \mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \).

A realization of \( T \) of the form

\[ \Theta = (A(t), B(t), C(t), D(t); I - P, PU(b)^{-1}) \]

with \( P \) a projection is called an **SB-realization** of the integral operator \( T \).
Problem

Characterize in terms of the SB-realization when such integral operators will be

- selfadjoint,
- $J$-unitary for some invertible $J = J^*$,
- positive definite,
- contractive,
- positive real or dissipative.
Intermezzo: time invariant case

Let \( b = +\infty \), and \( k(t, s) = k(t - s) \). Let

\[
k(t) = Ce^{tA}B
\]

be a minimal realization.

Then the corresponding integral operator is selfadjoint if and only if \( k(t) = k(-t)^* \) for real \( t \).

In terms of the minimal realization, the integral operator is selfadjoint if and only if there exists an invertible skew-hermitian matrix \( S = -S^* \) such that

\[
SA = -A^*S, \quad SB = -C^*.
\]
Elements in the proof

$(A, B, C)$ is a minimal realization if and only if $(-A^*, C^*, -B^*)$ is a minimal realization.

The proof is now based on Kalman’s state space isomorphism theorem: two minimal realizations are similar and the similarity is unique. That similarity is the matrix $S$.

**Remark** $iA$ is selfadjoint in the indefinite inner product given by $iS$. That fact, and a lot of Krein space theory was used in the study of the time invariant case.
Two SB-realizations

\[ \Theta_i = \left( A_i(t), B_i(t), C_i(t), D(t); (I - P_i), P_i U_i(b)^{-1} \right) \]

are called similar if there exists an invertible operator \( E : \mathcal{X}_1 \to \mathcal{X}_2 \) and an absolutely continuous function \( S(t) : \mathcal{X}_1 \to \mathcal{X}_2, a \leq t \leq b \), with invertible operator values, such that a.e. on \([a, b]\)

\[
\begin{align*}
A_2(t) &= S(t) A_1(t) S(t)^{-1} + \dot{S}(t) S(t)^{-1}, \\
B_2(t) &= S(t) B_1(t), \\
C_2(t) &= C_1(t) S(t)^{-1}, \\
(I - P_2) S(a) &= E (I - P_1), \\
P_2 U_2(b)^{-1} S(b) &= E P_1 U_1(b)^{-1}.
\end{align*}
\]

Then \( U_2(t) = S(t) U_1(t) S(a)^{-1} \).
Minimality

An SB-realization is called *SB-minimal* if among all SB-realizations with the same input-output operator the dimension of the state space is as small as possible.

Big problem: this is not unique up to similarity.
USB-class

**Def.** 1. We say that an integral operator with semi-separable kernel is in the USB-class if up to similarity it has a unique SB-minimal realization.

**Example:** if \( F_1, F_2, G_1 \) and \( G_2 \) are analytic then \( T \) is in the USB-class.

So far all Gohberg-Kaashoek
Let $T$ be in the USB-class and let for $i = 1, 2$

$$\Theta_i = \left( A_i(t), B_i(t), C_i(t), D(t); (I - P_i), P_iU_i(b)^{-1} \right)$$

be two SB-minimal realizations of $T$. Then the two are similar.

**Theorem 1.** (GPR, 2002) *The similarity between two SB-minimal SB-realizations of an integral operator in the USB-class is unique.*
The adjoint

If $\Theta = (A(t), B(t), C(t), D(t); I - P, PU(b)^{-1})$ is an SB-realization of $T$, then

$$\Theta^* = (-A(t)^*, C(t)^*, -B(t)^*, D(t)^*; P^*, (I - P^*)U(b)^*)$$

is an SB-realization of $T^*$. 

The fundamental operator for $\Theta^*$ is $U(t)^{-*}$. 
Selfadjointness in the USB-class

**Theorem 2.** Let $T$ be in the USB-class and let

$$\Theta = \left( A(t), B(t), C(t), D(t); I - P, P U(b)^{-1} \right)$$

be an SB-minimal SB-realization of $T$. Then $T = T^*$ if and only if $D(t) = D(t)^*$ and there is a unique invertible $S(t)$ such that

$$\dot{S}(t) = -A(t)^*S(t) - S(t)A(t),$$

$$P^*S(a) = S(a)(I - P),$$

$$C(t)^* = S(t)B(t),$$

$$S(t) = -S(t)^*. $$
Unitarity

Let \( J = J^* = J^{-1} \) be a signature matrix.

**Theorem 3.** Under the same assumptions on \( T \) and its \( \Theta \): \( T \) is \( J \)-unitary, i.e., \( T^*JT = J \) if and only if there is an invertible \( S(t) \) such that

\[
\dot{S}(t) = -A(t)^*S(t) - S(t)A(t) + C(t)^*JC(t),
\]

\[
P^*S(a) = S(a)(I - P),
\]

\[
C(t)^*J = S(t)B(t),
\]

\[
S(t) = S(t)^*.
\]
Further

We continu the paper with

- positivity: $I + T > 0$, with $D(t) \equiv 0$, this involves factorization,

- positive real: $T + T^* > 0$, with $D(t)$ constant,

- dissipative: $-i(T - T^*) > 0$, also with $D(t)$ constant,

- strictly contractive: $\|T\| < 1$, with $D(t) \equiv 0$.

All characterizations involve Riccati differential equations.
Let \( \Theta = (A(t), B(t), C(t), 0; I - P, P U(b)^{-1}) \) be an SB-minimal SB-realization of \( T \) in the USB class, assume that \( \Theta \) is similar to \( \Theta^* \), and that the similarity is given by \( S(t) \).

**Theorem 4.** The following statements are equivalent:

a. \( I + T \) is positive definite,

b. the Riccati differential equation

\[
\dot{Y}(t) = (Y(t)PS(a)^{-*} - P^*)U(t)^*C(t)^*C(t)U(t)(S(a)^{-1}P^*Y(t) - P),
\]

\[
Y(a) = 0,
\]

has a solution on \([a, b]\). This solution is positive semidefinite, increasing and satisfies \( Y(t)(I - P) = 0 \) for all \( a \leq t \leq b \).