Perturbations of bisemigroups

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Motivated in part by:


and in part by:


Motivation

LQ-optimal control.

System \( \dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0 \).

A generator of an exponentially decaying \( C_0 \)-semigroup acting on a separable Hilbert space \( \mathcal{H} \)

\( B \in \mathcal{L}(\mathcal{U}, \mathcal{H}) \), where also \( \mathcal{U} \) is a separable Hilbert space.

Cost function

\[
J(u, x_0) = \int_0^\infty \langle Qx(t), x(t) \rangle + \langle Ru(t), u(t) \rangle \, dt
\]

\( Q = Q^* \geq 0, Q \in \mathcal{L}(\mathcal{H}), R = R^* > 0, R \in \mathcal{L}(\mathcal{U}) \).

Known result:

\[
\min_{u \text{ admissible}} J(u, x_0) = \langle X_+ x_0, x_0 \rangle
\]

where \( X_+ \) satisfies

1. for \( x \in \mathcal{D}(A) \) we have

\[
(X_+ B R^{-1} B^* X_+ - A^* X_+ - X_+ A - Q)x = 0,
\]

2. \( A - B R^{-1} B^* X_+ \) generates an exponentially decaying \( C_0 \)-semigroup.

See, e.g.,


Finite dimensional approach

\[
H = \begin{pmatrix} A & -B R^{-1} B^* \\ -Q & -A^* \end{pmatrix}
\]

Let \( \mathcal{M}_+ \) be the invariant subspace of \( H \) corresponding to the open left half plane

\[
\mathcal{M}_+ = \text{Im} \left( \begin{pmatrix} I \\ X \end{pmatrix} \right) \text{ for some } X = X^*
\]

and \( X \) satisfies the algebraic Riccati equation

\[
X B R^{-1} B^* X - A^* X - X A - Q = 0
\]

and

\[
\sigma(A - B R^{-1} B^* X) = \sigma(H|_{\mathcal{M}_+})
\]
Infinite dimensional analogue

**Problem** We would like to see the infinite dimensional result in the same way.

Idea: set $H_0 = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$

This generates an exponentially decaying bisemigroup.

View $H$ as a bounded perturbation of this.

Under what conditions does $H$ generate an exponentially decaying bisemigroup?

If so, what can we say about $\mathcal{M}_+^+$?

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Bisemigroups

A closed and densely defined linear operator $S$ on a Banach space $\mathcal{X}$ is called **exponentially dichotomous** if there is a projection $P$ commuting with $S$ such that

$-S|\text{Im } P$ and $S|\text{Ker } P$

generate exponentially decaying $C_0$-semigroups.

The **bisemigroup** generated by $S$

$$E(t; S) = \begin{cases} 
 e^{tS}(I - P), & t > 0 \\
 -e^{tS}P, & t < 0.
\end{cases}$$

Its **separating projection** $P$ is given by

$$P = -E(0^-; S) = I_{\mathcal{X}} - E(0^+; S).$$

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First perturbation result: BGK

**Theorem 1.** (Bart, Gohberg, Kaashoek)

$S$ exponentially dichotomous.

$\tilde{S}$ closed, densely defined with $\mathcal{D}(\tilde{S}^2) \subset \mathcal{D}(S^2)$, and such that there is $\epsilon' > 0$ with

$$\{ \lambda \in \mathbb{C} : |\text{Re } \lambda| \leq \epsilon' \} \subset \rho(S) \cap \rho(\tilde{S})$$

and

$$\sup_{|\text{Re } \lambda| \leq \epsilon'} |\lambda^2|| (\lambda - \tilde{S})^{-1} - (\lambda - S)^{-1} | < \infty.$$  

Then $\tilde{S}$ is exponentially dichotomous.

Formula for separating projection: for $x \in \mathcal{D}(\tilde{S}^2)$

$$\tilde{P}x = -\frac{1}{2\pi i} \int_{\epsilon' - i\infty}^{\epsilon' + i\infty} \frac{1}{\lambda^2}(\lambda - \tilde{S})^{-1} \tilde{S}^2x d\lambda$$
Try to apply to Hamiltonian operator

\[ S = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \]

\[ \tilde{S} = H = \begin{pmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{pmatrix} \]

The domain condition

\[ D(S^2) \subset D(S^2) \]

is complicated.

Can we do better?

Two possibilities: strenghen the assumptions on \( S \), or strenghen the assumptions on the perturbation.

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**Second perturbation result: LRR**

Assume \( S \) generates an analytic bisemigroup (i.e., the constituent semigroups are analytic).

Then the separating projection

\[ P_x = \frac{1}{2\pi i} \int_{e^{-i\infty}}^{e^{+i\infty}} \frac{1}{\lambda^2} S(\lambda - S)^{-1} Sx \, d\lambda \]

for \( x \in D(S) \).

**Theorem 2.** (Langer, Ran, van de Rotten) \( S \) dichotomous, and for some \( \varepsilon \)

\[ \{ \lambda \in \mathbb{C} : |\text{Re} \lambda| \leq \varepsilon \} \subset \rho(S) \]

and for some \( \gamma > 0 \) and \( \beta > \frac{1}{2} \)

\[ \| (\lambda - S)^{-1} \| < \frac{\gamma}{1 + |\lambda|^{\beta}}, \quad |\text{Re} \lambda| \leq \varepsilon \]

For \( B \) bounded let \( \tilde{S} = S + B \). Assume that

\[ \{ \lambda \in \mathbb{C} : |\text{Re} \lambda| \leq \varepsilon \} \subset \rho(\tilde{S}) \]

Then \( \tilde{S} \) is dichotomous as well.

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**Third perturbation result: MR**

**Theorem 3.** (van der Mee, Ran)

\( S \) exponentially dichotomous

\( B \) compact,

\[ \tilde{S} = S + B \text{ with } D(\tilde{S}) = D(S). \]

Suppose that \( i\mathbb{R} \subset \rho(\tilde{S}) \).

Then \( \tilde{S} \) is exponentially dichotomous.

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**Fourth perturbation result: MR**

Our next result uses the concept of immediately norm continuous semigroups.

A semigroup \( T(t) \) is called immediately norm continuous if \( T(t) \) is norm continuous for \( t > 0 \).

**Examples**

- Analytic semigroups,

- immediately compact semigroups (i.e., \( T(t) \) is a compact operator for \( t > 0 \))

- immediately differentiable semigroups

**Theorem 4.** Let $\mathcal{H}$ be a complex Hilbert space and let $(T(t))_{t \geq 0}$ be an uniform exponentially stable $C_0$-semigroup on $\mathcal{H}$. Then $(T(t))_{t \geq 0}$ is immediately norm continuous if and only if the resolvent $(\lambda - A)^{-1}$ of its infinitesimal generator $A$ vanishes in the norm as $\lambda \to \infty$ along the imaginary line.

K.-J. Engel and R. Nagel ibid. Section II 4.20.

**Theorem 5.** (van der Mee, Ran)

$S$ exponentially dichotomous with immediately norm continuous constituent semigroups

$B$ bounded,

$\tilde{S} = S + B$ with $\mathcal{D}(\tilde{S}) = \mathcal{D}(S)$.

Suppose that

$$\{\lambda \in \mathbb{C} : |\text{Re}\lambda| < \varepsilon\} \subset \rho(\tilde{S})$$

Then $\tilde{S}$ is exponentially dichotomous with immediately norm continuous constituent semigroups.

**Sketch of proof**

If $\tilde{S}$ generates a bisemigroup then the resolvent identity gives that it satisfies

$$E(t; \tilde{S})x - \int_{-\infty}^{\infty} E(t-\tau; S)BE(\tau; \tilde{S})x\,d\tau = E(t; S)x,$$

where $x \in \mathcal{H}$ and $0 \neq t \in \mathbb{R}$.

Conversely, consider the convolution equation

$$u(t, x) - \int_{-\infty}^{\infty} E(t-\tau; S)Bu(\tau, x)\,d\tau = E(t; S)x.$$

The symbol of the convolution integral equation

$$I + (\lambda - S)^{-1}B = (\lambda - S)^{-1}(\lambda - \tilde{S})$$

tends to $I$ in the norm as $\lambda \to \infty$ in the strip $|\text{Re}\lambda| \leq \varepsilon$.

Moreover, it is a compact perturbation of the identity which only takes invertible values on the imaginary axis. Thus there exists $\varepsilon_0 \in (0, \varepsilon]$ such that the symbol only takes invertible values on the strip $|\text{Re}\lambda| \leq \varepsilon_0$.

By the Bochner-Phillips theorem the convolution equation has a unique solution

$$u(\cdot; x) = E(\cdot; \tilde{S})x.$$

Using results from Bart-Gohberg-Kaashoek we can conclude that this is a bisemigroup.
Application to the ARE

If $A$ generates an immediately norm continuous exponentially stable semigroup, and $D$ and $Q$ are bounded, such that the resolvent of $H = \begin{pmatrix} A & -D \\ -Q & -A^* \end{pmatrix}$ contains a strip around $i\mathbb{R}$, then $H$ generates a bisemigroup.

Moreover, these assumptions imply the existence of a bounded solution $X$ of the ARE

$$(XD - XA - A^*X - Q)x = 0, \quad x \in D(A)$$

such that $A - DX$ is exponentially stable and $X$ maps $D(A)$ into $D(A^*)$.

As another application: if only one of $D$ or $Q$ is compact and $A$ generates an exponentially stable semigroup then $H$ generates a bisemigroup and we again get the existence of a bounded solution of the ARE.