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Nonmonotonic reasoning with multiple belief sets

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In complex reasoning tasks it is often the case that there is no single, correct set of conclusions given some initial information. Instead, there may be several such conclusion sets, which we will call *belief sets*. In the present paper we introduce nonmonotonic belief set operators and selection operators to formalize and to analyze structural aspects of reasoning with multiple belief sets. We define and investigate formal properties of belief set operators as absorption, congruence, supradeductivity and weak belief monotony. Furthermore, it is shown that for each belief set operator satisfying strong belief cumulativity there exists a largest monotonic logic underlying it, thus generalizing a result for nonmonotonic inference operations. Finally, we study abstract properties of selection operators connected to belief set operators, which are used to choose some of the possible belief sets.

1. Introduction

In a broad sense, reasoning can be viewed as an activity where an agent, given some initial information (or set of beliefs) X, performs some manipulation to this information and arrives at a new state with different information. So a (partial) view on a situation (in the domain the agent is reasoning about) is transformed to another partial view. In general the mechanism may be non-deterministic in the sense that multiple possible views on the world can result from the reasoning process. In the current paper we present an approach to formalize and to analyze structural aspects of reasoning of an agent with multiple belief sets.

If we want to formalize reasoning in this way, we must describe the input-output behavior of the agent's reasoning process. We propose to use belief set operators for this purpose. A belief set operator is a function B which assigns to a set of beliefs (information) X, given in some language L, a family of belief sets $B(X)$, described in the same language.

Different modes of reasoning give rise to different kinds of belief set operators. If we consider exhaustive classical propositional reasoning, a set of propositional beliefs X is mapped to the set $Cn(X)$ of propositional consequences of X, which is unique; so in this case there is only one belief set: $B(X) = \{Cn(X)\}$. However, if we look at nonmonotonic logics such as Autoepistemic Logic or Default Logic, an initial set of beliefs X may have none or more than one possible expansion (or extension).

In these two cases, the reasoning is conservative: the resulting belief sets extend the set of initial beliefs. But there are also modes of reasoning in which beliefs are retracted. This is the case in, for instance, contraction in belief revision, in which the contraction of a belief from a belief set is not uniquely determined. Also, when the set of initial beliefs is contradictory, and we want to remove the contradiction, one can select a consistent subset; this again can be done in more than one way.

Even though we have argued that in general a reasoning process may have multiple possible outcomes, an agent which has to act in a situation must commit itself somehow to one set of conclusions by using the information in the possible belief sets. In nonmonotonic logics, two different approaches to this problem are well-known: the credulous approach, where the agent believes anything from any possible extension (thus taking the union of the possible belief sets), and the sceptical approach, in which it only believes those facts which appear in all of the possible belief sets (taking their intersection).

A third approach is based on the situation where the agent has additional (control) knowledge allowing it to choose one of the possible belief sets as the "preferred" one. (Many nonmonotonic formalisms such as Autoepistemic Logic, Default Logic and Logic Programming have a prioritized or stratified variant.) As the different belief sets are usually based on different assumptions, and may even be mutually contradictory, we feel the credulous approach is not very realistic. Looking at belief revision in the AGM framework [1], when we retract a sentence φ from a belief set K, the maximally consistent subsets of K which do not contain φ (denoted $K \perp \varphi$), in a sense play the role of the possible belief sets. Contraction with φ is always the result of intersecting a number of these belief sets. Special cases of contraction are full meet contraction, in which all elements of $K \perp \varphi$ are intersected (analogously to sceptical inference), and maxi-choice contraction, in which just one element of $K \perp \varphi$ is selected (analogously to prioritized nonmonotonic logics).

In earlier work [12] we described the following hierarchy of five levels of abstraction for the specification of nonmonotonic reasoning.

1. *Specification of a set of intended models.*
   Specification of the global set of possible (intended) worlds and the beliefs that hold in them, abstracting from the specific underlying (multiple) belief states, the specific reasoning patterns that lead to them and the specific reasoning system generating these reasoning patterns.

2. *Specification of a set of intended multiple belief states.*
   Specification of the possible belief states for the agent abstracting from the specific reasoning patterns that lead to them and the specific reasoning system generating these reasoning patterns.

3. *Specification of a set of intended reasoning patterns*.
   Specification of the reasoning patterns that lead to the intended possible belief states, abstracting from the specific reasoning system generating these reasoning patterns.

4. *Specification of a reasoning system*.
   Specification of an architecture for a reasoning system that when executed (by use of heuristic control knowledge) can generate the intended reasoning patterns.

5. *Implementation*.
   At this level an implemented reasoning system is described in any implementation environment (implementation code).

Of course, there exist connections between the levels in the sense that from a specification of a lower level of abstraction in an unambiguous manner a specification of each of the higher levels can be determined. One could say the specification at a lower level gives in some sense a refinement or specialization of the specification at the higher level (as in the case of conventional software specifications at different levels of abstraction). Given specifications of two different levels, relative verification is possible: to establish whether the lower level one indeed refines the higher level one. At a lower level different specifications can refine the same higher level specification. As a parallel one may think of development of programs using the method of (top down) stepwise refinement, e.g., according to Dijkstra's approach. Note however that other methods (other than top down stepwise refinement) are possible as well.

On the second level of abstraction, nonmonotonic reasoning is described by giving, for a set of initial facts, a set of belief states (the semantical counterpart of belief sets). The current paper, which extends the work reported in [13], can be viewed as an exploration of (the syntactical side of) the second level of this hierarchy.

In the current paper, in section 2 some basic background notions are introduced. In section 3 the notion of belief set operator is introduced, some illustrative examples are described (default logic, belief revision) and a number of properties of belief set operators are discussed. Section 4 links belief set operators to underlying monotonic logics and discusses semantical variants. In section 5 results are obtained on the semantics of a belief set operator in terms of the semantical notion of belief state operator. Moreover, results are obtained on the existence of a greatest underlying (monotonic) deductive system. In section 6 the notion of selection operator is introduced, formalizing an agent's commitment to some of its belief sets. Selection functions applied to the results of a belief set operator provide a set of (selective) inference operations. Such a set of inference operations can be viewed as an alternative formalization of multiple belief sets. Some formal relationships between sets of inference operations and belief set operators are established. Properties of selection operators are related to properties of the belief set operator and the inference operations resulting after selection. In section 7 conclusions are drawn and perspectives on further research are sketched.

2. Background and preliminaries

Let L be a nonempty language whose elements are denoted by ϕ, ψ, χ; $\mathcal{P}(X)$ denotes the power set of the set X. An operation $C: \mathcal{P}(L) \to \mathcal{P}(L)$ is called an *inference operation*, and the pair (L, C) is said to be an *inference system*. The operation C represents the notion of logical inference. An inference system (L, C_L) is a *closure system* and C_L a *closure operation* if it satisfies the following conditions: $X \subseteq C_L(X)$ (*inclusion*), $C_L(C_L(X)) = C_L(X)$ (*idempotence*), $X \subseteq Y \Rightarrow C_L(X) \subseteq C_L(Y)$ (*monotony*). An inference operation C_L satisfies *compactness* if $\phi \in C_L(X)$ implies the existence of a finite subset $Y \subseteq X$ such that $\phi \in C_L(Y)$. A closure system (L, C_L) is a *deductive system* if C_L satisfies compactness; then C_L is said to be a *deductive (inference) operation*. A set $X \subseteq L$ is closed under C_L if $C_L(X) = X$. The investigation of logics on the abstract level of inference operations was proposed and motivated by A. Tarski in [37]. The classical example of a deductive system is the inference system denoted by $\mathcal{L}_0 = (L_0, \mathit{Cn})$ which is based on classical propositional logic. Here L_0 is the set of propositional formulas based on a set of propositional variables *Var* and $\mathit{Cn}(X)$ can be defined as the smallest subset of L_0 containing the set $X \cup \mathit{Ax}$, where *Ax* is a suitable set of axioms, and which is closed with respect to the rule of modus ponens (see [3,23]).

A semantics for a closure system (L, C_L) can be defined by a *model-theoretic system*. A model-theoretic system $(L, M, \models)$ is determined by a language L, a set (or class) M whose elements are called *worlds* and a *relation of satisfaction* $\models \subseteq M \times L$ between worlds and formulas. Given a model-theoretic system $(L, M, \models)$, we introduce the following notions. Let $X \subseteq L$, $\mathit{Mod}^{\models}(X) = \{m\colon m \in M$ and $m \models X\}$, where $m \models X$ if for every $\phi \in X$: $m \models \phi$. Let $K \subseteq M$, then $\mathit{Th}^{\models}(K) = \{\phi\colon \phi \in L$ and $K \models \phi\}$, where $K \models \phi$ if for all $m \in K$: $m \models \phi$. $C^{\models}(X) = \{\phi\colon \mathit{Mod}^{\models}(X) \subseteq \mathit{Mod}^{\models}(\phi)\}$, $X \models \phi$ if $\phi \in C^{\models}(X)$. Obviously, $(L, C^{\models})$ is a closure system and if $C^{\models}(X) = X$ then $\mathit{Th}^{\models}(\mathit{Mod}^{\models}(X)) = X$. $(L, M, \models)$ is said to be compact if the closure operation $C^{\models}$ is compact. The inference system (L, C_L) is *correct* (*complete*) with respect to the model-theoretic system $(L, M, \models)$ if $C_L(X) \subseteq C^{\models}(X)$ ($C_L(X) = C^{\models}(X)$). In case of completeness we say also that $(L, M, \models)$ *represents* (or *is adequate for*) (L, C_L). The model-theoretic system $(L_0, M, \models)$ of classical propositional logic is defined by the set $M = \{m \mid m\colon \mathit{Var} \to \{0, 1\}\}$ of all interpretations, Var being the set of propositional variables, and the relation $m \models F$ which means that the formula F is satisfied by the interpretation m.

The study of the general properties of inference operations $C: \mathcal{P}(L) \to \mathcal{P}(L)$ that do not satisfy monotony is well-established (see, e.g., [28]). A condition on inference operations is said to be *pure* if it concerns the operation alone without regard to its interrelations to a deductive system (L, C_L) representing a monotonic and compact logic. The most important pure conditions are the following:

- $X \subseteq Y \subseteq C(X) \Rightarrow C(Y) \subseteq C(X)$ (*cut*),
- $X \subseteq Y \subseteq C(X) \Rightarrow C(X) \subseteq C(Y)$ (*cautious monotony*),
- $X \subseteq Y \subseteq C(X) \Rightarrow C(X) = C(Y)$ (*cumulativity*).

Some impure conditions are: $C(X) \cap C(Y) \subseteq C(C_L(X) \cap C_L(Y))$ (*distributivity*), $C_L(X) \neq L \Rightarrow C(X) \neq L$ (*consistency preservation*).

An inference operation C is said to be *supraclassical* if it extends the consequence operation *Cn* of classical logic, i.e., $Cn(X) \subseteq C(X)$ for all $X \subseteq L$. If we assume an arbitrary deductive system (L, C_L) (where C_L is not necessarily *Cn*), then this condition can be generalized to the condition of *supradeductivity*: $C_L(X) \subseteq C(X)$. A system $\mathcal{IF} = (L, C_L, C)$ is said to be an *inference frame* if L is a language, C_L is a deductive inference operation on L, and $C_L(X) \subseteq C(X)$ (supradeductivity) is fulfilled. The operation C satisfies *left absorption* if $C_L(C(X)) = C(X)$; and C satisfies *congruence* or *right absorption* if $C_L(X) = C_L(Y) \Rightarrow C(X) = C(Y)$. C satisfies *full absorption* if C satisfies left absorption and congruence. An inference frame $\mathcal{DF} = (L, C_L, C)$ is said to be a *deductive inference frame* if it satisfies full absorption. In this case C is said to be *logical over* C_L, and (L, C_L) is a *deductive basis* for C.

The semantics of a deductive frame can be described by introducing a model operator based on a model-theoretic system [10,21]. $\mathcal{SF} = (L, M, \models, \Phi)$ is a *semantical frame* if $(L, M, \models)$ is a model-theoretic system and $\Phi : \mathcal{P}(L) \to \mathcal{P}(M)$ is a functor (called model operator) such that $\Phi(X) \subseteq Mod^{\models}(X)$. Let $C_\Phi(X) = Th^{\models}(\Phi(X))$. The operator Φ is said to be C_L-*invariant* if $(\forall X \subseteq L)(\Phi(X) = \Phi(C_L(X)))$. An important example of an invariant semantical frame is the frame $(L_0, M, \models, \Phi_{\min})$ of minimal reasoning in propositional logic. Here, $(L_0, M, \models)$ is the model-theoretic system of classical propositional logic, and $\Phi_{\min}(X)$ selects all minimal elements from $Mod^{\models}(X)$ with respect to the following partial ordering on interpretations: $m \leqslant n \Leftrightarrow \forall p \in Var$: $m(p) \leqslant n(p)$. The model operator $\Phi_{\min}$ represents the propositional version of circumscription introduced by McCarthy (see [29]).

The inference operation C_Φ satisfies supradeductivity, and hence $(L, C^{\models}, C_\Phi)$ is an inference frame associated to $\mathcal{SF}$ and denoted by $IF(\mathcal{SF})$. An inference frame $\mathcal{I} = (L, C_L, C)$ is said to be *complete* for a semantical frame $(L, M, \models, \Phi)$ if (L, C_L) is complete with respect to $(L, M, \models)$ and $C = C_\Phi$. Representation theorems for classes of inference frames can be proved by using semantical frames based on the *Lindenbaum–Tarski* construction of maximal consistent sets. We recall the ingredients of this construction. Let (L, C_L) be a deductive system. A set $X \subseteq L$ is said to be relatively maximal consistent (*r-maximal*) iff there is a formula $\phi \in L$ such that $\phi \notin C_L(X)$ and for every proper super set $Y \supset X$ the condition $\phi \in C_L(Y)$ is satisfied. Let $\mathrm{rmax}(L)$ be the set of all r-maximal subsets of L. The Lindenbaum–Tarski semantics (abbreviated by LT-semantics) is defined by the model-theoretic system $(L, M, \models)$ where $M = \mathrm{rmax}(L)$ and $m \models \phi$ iff $\phi \in m$. Then $C^{\models} = C_L$. We collect some elementary results that can be formulated and proved within this framework [10].

Proposition 1. Let $\mathcal{F} = (L, C_L, C)$ be an inference frame satisfying left absorption. Then there exists a semantical frame $\mathcal{SF} = (L, M, \models, \Phi)$ such that $\mathcal{F}$ is complete with respect to $\mathcal{SF}$, i.e., $C_L = C^{\models}$ and $C = C_\Phi$.

Proof. Let $(L, M, \models)$ be the LT-semantics for (L, C_L) and $\Phi(X) = \{m\colon\ m \in M, C(X) \subseteq m\}$. It is easy to show that $C = C_\Phi$. □

Left absorption does not imply congruence. We get an adequateness result for deductive inference frames by using invariant semantical frames [10].

Proposition 2.

1. Let $\mathcal{F} = (L, C_L, C)$ be a deductive inference frame. Then there exists a semantical frame $\mathcal{S} = (L, M, \models, \Phi)$ such that Φ is an invariant model operator and $\mathcal{S}$ represents $\mathcal{F}$.
2. If Φ is an invariant model operator for the logical system $(L, M, \models)$ then $(L, C^{\models}, C_\Phi)$ is a deductive inference frame.

Proof. 1. Let $(L, M, \models)$ be the LT-semantics for (L, C_L) and $\Phi(X) = \mathit{Mod}^{\models}(C(X))$. Left absorption implies $C_\Phi = C$. Invariance of Φ follows from right absorption: since $C(C_L(X)) = C(X)$ we have

$$\Phi(X) = \mathit{Mod}^{\models}\bigl(C(X)\bigr) = \mathit{Mod}^{\models}\bigl(C\bigl(C_L(X)\bigr)\bigr) = \Phi\bigl(C_L(X)\bigr).$$

2. Let $(L, C^{\models}, C_\Phi)$ be a semantical frame and Φ an invariant model operator. By definition, $C_\Phi(X) = \mathit{Th}^{\models}(\Phi(X))$. Hence, $C_L(C_\Phi(X)) = C_\Phi(X)$. By invariance of Φ we have $\Phi(X) = \Phi(C_L(X))$, hence $C_\Phi(X) = C_\Phi(C_L(X))$, i.e., C_Φ satisfies right absorption. □

3. Belief set operators

Usually, there can be many different sets of beliefs that can be justified on the base of a set X of given knowledge. A set of such belief sets will be called a *belief set family*. In this section we adapt and generalize the framework of deductive and semantical frames to the case of belief set operators.

Definition 3. A belief set operator B is a function that assigns a belief set family to each set of initial facts: $B : \mathcal{P}(L) \to \mathcal{P}(\mathcal{P}(L))$.

1. B satisfies inclusion if $(\forall X)$ $(\forall T \in B(X))$ $(X \subseteq T)$.
2. B satisfies non-inclusiveness if $(\forall X)$ $(\forall UV \in B(X))$ $(U \subseteq V \Rightarrow U = V)$.
3. The *kernel* $K_B : \mathcal{P}(L) \to \mathcal{P}(L)$ of B is defined by $K_B(X) = \bigcap B(X)$.

We collect several examples of belief set operators.

Example 4 (Default logic). Let D be a set of defaults. For $X \subseteq L$, and $\Delta = (X, D)$ let $\mathcal{E}(\Delta)$ denote the set of (Reiter) extensions of the default theory Δ. The belief set operator B_D can be defined as follows: $B_D(X) = \mathcal{E}(\Delta)$. The kernel of B_D gives the sceptical conclusions of a default theory.

Example 5 (Belief revision). Let φ be a sentence. For a deductively closed belief set K, define $K \perp \varphi = \{T \mid T \subseteq K \backslash \{\varphi\}, T = Cn(T)$ and T is maximal with respect to these properties$\}$. These maximal subsets can play the role of possible belief sets resulting from the contraction of φ from K. Define a belief set operator $B_{-\varphi}$ by $B_{-\varphi}(X) = Cn(X) \perp \varphi$ if $Cn(X) \perp \varphi$ is not empty, and $B_{-\varphi}(X) = Cn(X)$ otherwise (this only occurs if φ is a tautology). The kernel of this operator yields a special contraction function, called a *full meet contraction function*.

Example 6 (Poole systems). Let $\Sigma = (D, E)$, $D \cup E \subseteq L$; the elements of D are called defaults, the elements of E are said to be constraints. A set $\delta \subseteq D$ is a basis for $X \subseteq L$ if the set $X \cup \delta \cup E$ is consistent and δ is maximal with this property. Let $Cons_\Sigma(X) = \{\delta\colon \delta \subseteq D$ and δ is a basis for $X\}$. Then define $B_\Sigma(X) = \{Cn(X \cup \delta)\colon \delta \in Cons_\Sigma(X)\}$. Obviously, B_Σ is a belief set operator.

Structural properties of inference operations (like monotony, cut or cautious monotony) can be generalized to properties of belief set operators, usually in more than one way. The simplest way is to relate everything to the kernel. For instance, we could say that B is monotonic if and only if its kernel K_B is. But this definition does not at all consider the structure of the belief sets, and we can define more refined versions of such properties that *do* take into account the structure of the belief sets.

In order to define these properties, it will be convenient to introduce an information ordering on belief set families. For belief sets there is already a natural notion of degree of information (a belief set T contains more information than a belief set S if $S \subseteq T$). Using this new ordering of information, the properties of belief set operators resemble their counterparts for inference operations.

Definition 7. Let $\mathcal{A}$, $\mathcal{B}$ be belief set families. We say $\mathcal{B}$ contains more information than $\mathcal{A}$, denoted $\mathcal{A} \preceq \mathcal{B}$, if $(\forall T \in \mathcal{B})$ $(\exists S \in \mathcal{A})$ $(S \subseteq T)$. We write $\mathcal{A} \equiv \mathcal{B}$ if $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \preceq \mathcal{A}$.

If one of the arguments in the above definition is a singleton belief set family, we will often omit the parentheses and write $X \preceq \mathcal{A}$ instead of $\{X\} \preceq \mathcal{A}$. Thus, we can also write $X \preceq Y$ instead of $X \subseteq Y$. So in words this definition says that a belief set family $\mathcal{B}$ is considered to have more information than $\mathcal{A}$ if any of the sets of $\mathcal{B}$ extends some of the sets of $\mathcal{A}$. This also means that it may happen that a belief set in $\mathcal{A}$ has no extending belief set in $\mathcal{B}$. One can think of the belief sets as (partial) possible worlds: the less possible worlds the agent considers, the more sure she is of the state of affairs of the outside world. So the more possibilities, the less knowledge an agent has. On the other hand, the possible states in $\mathcal{B}$ must contain more information than their counterparts in $\mathcal{A}$. Note that this condition implies that

$\bigcap \mathcal{A} \subseteq \bigcap \mathcal{B}$. We introduce the following formal properties of belief set operators capturing essential features of a rational agent.

Definition 8. Let B be a belief set operator.

1. B satisfies belief monotony if $(\forall X \forall Y)$ $(X \preceq Y \Rightarrow B(X) \preceq B(Y))$.
2. B satisfies weak belief monotony if $(\forall XY)$ $(X \preceq Y \preceq B(X) \Rightarrow B(X) \preceq B(Y))$.
3. B satisfies belief transitivity if
$$(\forall XYS) \quad \bigl(S \in B(X) \text{ and } X \subseteq Y \subseteq S \Rightarrow K_B(Y) \subseteq S\bigr).$$[1]
4. B satisfies belief cut if $(\forall XY)$ $(X \preceq Y \preceq B(X) \Rightarrow B(Y) \preceq B(X))$.
5. B satisfies belief cumulativity if it satisfies weak belief monotony and belief cut.
6. B satisfies strong belief cumulativity if it satisfies belief cumulativity and belief transitivity.
7. B satisfies strong belief cut if
$$(\forall XYS) \quad \bigl(S \in B(X) \text{ and } X \subseteq Y \subseteq S \Rightarrow \bigl(\exists T \in B(Y)\bigr) \ (T \subseteq S)\bigr).$$

It is easy to check that strong belief cut implies belief cut and belief transitivity.

In [38] a belief set operator B satisfying inclusion is said to be cumulative if it satisfies belief transitivity and the following condition that we call in the present paper *local belief monotony*: $(\forall XYS)$ $(S \in B(X)$ and $X \subseteq Y \subseteq S \Rightarrow B(Y) \subseteq B(X))$. A weaker form of this notion is defined by the following condition: $(\forall XYS)$ $(S \in B(X)$ and $X \subseteq Y \subseteq S \Rightarrow B(X) \preceq B(Y))$. All these properties are generalizations of the notion of cautious monotony for inference operations to the case of belief set operators. Similarly, there are alternative versions of the generalization of cut and cumulativity to belief set operators. There is not yet a complete analysis of these properties and their interrelations. The following holds:

Proposition 9. Let B be a belief set operator satisfying inclusion.

1. If B is belief monotonic then K_B is monotonic.
2. If B satisfies belief transitivity or belief cut then K_B satisfies cut.
3. If B satisfies weak belief monotony then K_B satisfies cautious monotony.

Proof. 1. $X \subseteq Y \Rightarrow B(X) \preceq B(Y) \Rightarrow \bigcap B(X) \subseteq \bigcap B(Y)$.

2. Suppose B satisfies belief cut, and suppose $X \subseteq Y \subseteq K_B(X)$, then certainly $X \preceq Y \preceq B(X)$, so $B(Y) \preceq B(Y)$ whence $\bigcap B(Y) \preceq \bigcap B(Y)$. Now suppose B satisfies belief transitivity, and suppose $X \subseteq Y \subseteq K_B(X)$. Let $T \in B(X)$, then $X \subseteq Y \subseteq T$ so $\bigcap B(Y) \subseteq T$. It follows that $\bigcap B(Y) \subseteq \bigcap B(X)$.

[1] This property is called *cumulative transitivity* in [38].

3. If $X \subseteq Y \subseteq \bigcap B(X)$ then $X \preceq Y \preceq B(X)$ so $B(X) \preceq B(Y)$. It follows that $\bigcap B(X) \subseteq \bigcap B(Y)$. □

So all of the properties of definition 8 are generalizations of the corresponding properties of inference operations. Given a belief set operator B with desirable properties, the associated inference operation K_B has analogous properties.

Given an inference operation C, there are of course in general many belief set operators B such that $K_B = C$, the most trivial being $B(X) = \{C(X)\}$. One could ask whether there are *non-trivial* belief set operators B with $K_B = C$ which have interesting structural properties, and if there is a general way of obtaining them. The results in [28], building on results in [25], show that this can be done using preferential models. We will briefly sketch this. A preferential model is a triple $\langle M, \models, < \rangle$ where M is a set of states, $\models$ is any relation between states and formulas and $<$ is a relation between models. A state $m \in M$ *preferentially satisfies* a set of formulas A, denoted $m \models_< A$, if $m \models A$ and there is no $n \in M$ such that $n < m$ and $n \models A$. An inference operation $C_<$ can then be defined by $C_<(X) = \{\varphi \in L \mid \forall m \in M, m \models_< X \Rightarrow m \models \varphi\}$. A preferential model is called *smooth*, if for any $X \subseteq L$ and $m \in M$ such that $m \models X$, there exists a state $n \in M$ such that $n \leqslant m$ and $n \models_< X$. The basic result of [25], proved independently by [27], is that for any cumulative inference operation C, there is a smooth preferential model $\langle M, \models, < \rangle$ such that $C = C_<$. But this also gives rise to a belief set operator, in the sense that the theory of each state preferentially satisfying X can be seen as a belief set. If we set (in the notation of [28]) $E_m = \{\varphi \in L \mid m \models \varphi\}$ for each $m \in M$, then a belief set operator B can be defined by $B(X) = \{E_m \mid m \models_< X\}$. It is easy to see that $K_B = C_<$. Moreover, this belief set operator satisfies the properties defined in definition 8.

Proposition 10. Given a cumulative inference operation C, there exists a non-trivial belief set operator B satisfying all the properties in definition 8 such that $K_B = C$.

Proof. Given C, let B be defined as above. Then B satisfies weak belief monotony: suppose $X \preceq Y \preceq B(X)$. Let $E_m \in B(Y)$, then $m \models_< Y$ so $m \models X$, and by smoothness there exists $n \leqslant m$ such that $n \models_< X$. As $Y \preceq B(X)$ we have $n \models Y$, so $n = m$. We have found $E_n \subseteq E_m$ and $E_n \in B(X)$ so $B(X) \preceq B(Y)$. Furthermore, B satisfies strong belief cut. Suppose $E_m \in B(X)$ and $X \subseteq Y \subseteq E_m$, then $m \models Y$ so there exists $n \leqslant m$ such that $n \models_< Y$. Since $X \subseteq Y$ we have $n \models X$ so $n = m$. We have found $E_n \in B(Y)$ such that $E_n \subseteq E_m$. These two properties imply all the other ones. □

4. Belief frames

We now connect a belief state system with a compact monotonic logic which can be considered as a deductive basis. Many non-classical forms of reasoning are built ‘on top of’ a monotonic logic (L, C_L).

Definition 11.

1. A system $\mathcal{B}F = (L, C_L, B)$ is said to be a *belief set frame* if the following conditions are satisfied:
   (a) L is a language and C_L is a *deductive inference operation* on L.
   (b) B is a belief set operator on L satisfying non-inclusiveness and inclusion.
2. B satisfies belief left absorption iff $C_L(T) = T$ for every $T \in B(X)$, and B satisfies belief congruence or C_L-invariance iff $C_L(X) = C_L(Y)$ implies $B(X) = B(Y)$. B satisfies full absorption iff B satisfies belief left absorption and congruence.
3. A belief set frame $\mathcal{DF} = (L, C_L, B)$ is said to be a *deductive belief set frame* if it satisfies full absorption. In this case the system (L, C_L) is called a *deductive basis* for B.

Proposition 12. Let $\mathcal{B}F = (L, C_L, B)$ be a belief set frame satisfying strong belief cumulativity. Then $\mathcal{BF}$ satisfies belief left absorption and belief congruence, i.e., $\mathcal{BF}$ is a deductive belief set frame.

Proof. From belief transitivity it follows that for every $T \in B(X)$ the condition $K_B(T) \subseteq T$ is satisfied, hence $K_B(T) = T$. By supradeductivity we get $C_L(T) \subseteq K_B(T)$, thus $C_L(T) = T$.

Assume $C_L(X) = C_L(Y)$. Since $K_B : \mathcal{P}(L) \to \mathcal{P}(L)$ is cumulative it follows that (L, C_L, K_B) is a deductive frame, hence $K_B(X) = K_B(Y)$. It is sufficient to prove $B(X) = B(K_B(X))$, because this condition implies $B(X) = B(Y)$.

Let $S \in B(X)$, by belief cut there is an extension $T \in B(K_B(X))$ such that $T \subseteq S$. By weak belief monotony there exists an $S_1 \in B(X)$ satisfying $S_1 \subseteq S$. Because the sets in $B(X)$ are pairwise non-inclusive we get $S = S_1$, which implies $T = S$, hence $S \in B(K_B(X))$.

Let $T \in B(K_B(X))$; by weak belief monotony there is an $S \in B(X)$ such that $S \subseteq T$. By the previous proved condition this implies $S \in B(K_B(X))$, hence by non-inclusiveness of B we get $T = S$. □

Further important impure properties of inference frames can be generalized to belief set frames.

Definition 13. Let (L, C_L, B) be a belief set frame.

1. B satisfies belief distribution if
$$(\forall XYS) \quad \big(S \in B\big(C_L(X) \cap C_L(Y)\big) \Rightarrow \big(S \in B(X) \text{ or } S \in B(Y)\big)\big).$$
2. B satisfies belief consistency preservation if
$$(\forall X) \quad \big(C_L(X) \neq L \Rightarrow B(X) \neq \{L\} \text{ and } B(X) \neq \emptyset\big).$$

In the last condition, both when $B(X) = \{L\}$ and when $B(X) = \emptyset$, the input can be considered 'nonmonotonically inconsistent'. Both possibilities occur in for instance default logic: there are default theories with just one inconsistent extension, and there are default theories without extensions.

The following proposition holds.

Proposition 14.

1. If B satisfies belief distribution then K_B satisfies distributivity.
2. If B satisfies belief consistency preservation then K_B satisfies consistency preservation.

Proof. 1. Suppose B satisfies belief distribution. Take any $S \in B(C_L(X) \cap C_L(Y))$, then $S \in B(X)$ or $S \in B(Y)$ so $\bigcap B(X) \subseteq S$ or $\bigcap B(Y) \subseteq S$. In both cases we have $\bigcap B(X) \cap \bigcap B(Y) \subseteq S$. It follows that $K_B(X) \cap K_B(Y) \subseteq \bigcap B(C_L(X) \cap C_L(Y))$.

2. Suppose $C_L(X) \neq L$, then $B(X) \neq \{L\}$ and $B(X) \neq \emptyset$ from which we immediately get $\bigcap B(X) \neq L$. $\square$

The semantics of a belief set is a set of models. Since there can be many belief sets we have to take into consideration functors associating to sets of assumptions sets of sets of models. Such functors are called *belief state operators*.

Definition 15.

1. A belief state operator Γ is a function $\Gamma : \mathcal{P}(L) \to \mathcal{P}(\mathcal{P}(M))$.
2. The tuple $(L, M, \models, \Gamma)$ is said to be a belief state frame.
3. Γ satisfies non-inclusiveness if $\forall K J \in \Gamma(X)$: $J \subseteq K \Rightarrow K = J$.
4. Γ satisfies inclusion if $(\forall X)$ $(\forall K \in \Gamma(X))$ $(K \subseteq \mathit{Mod}(X))$.
5. Γ satisfies left absorption, or L-invariance, if $\Gamma(X) = \Gamma(C_L(X))$ for all $X \subseteq L$.

For a given belief state operator Γ the following belief set operator B_Γ can be introduced: $B_\Gamma(X) = \{\mathit{Th}(K)\colon K \in \Gamma(X)\}$. The notion of a belief state operator is a generalization of the notion of a model operator.

The following examples summarize some types of belief state operators associated to belief set operators investigated in the literature.

Example 16 (Default logic, continued). Remember that we associated a belief set operator B_D with a set of defaults D. Then (L, Cn, B_D) is a deductive belief set frame. Then also the following belief state operator can be defined for $X \subseteq L$: $\Gamma_D(X) = \{\mathit{Mod}(E)\colon E \in \mathcal{E}(\Delta)\}$.

Example 17 (Poole systems, continued). Let $\Sigma = (D, E)$, $D \cup E \subseteq L$ be a Poole system and $Cons_\Sigma(X) = \{\delta\colon\ \delta \subseteq D$ and δ is a basis for $X\}$. Let $B_\Sigma(X) = \{Cn(X \cup \delta)\colon\ \delta \in Cons_\Sigma(X)\}$. A belief state operator Γ_Σ providing a semantics for B_Σ can be introduced by $\Gamma_\Sigma(X) = \{Mod(T)\colon\ T \in B_\Sigma(X)\}$. Obviously, Γ_Σ is Cn-invariant.

Example 18 (Generalized belief revision). Let $A \subseteq L$ be an arbitrary fixed consistent deductively closed set and $X \subseteq L$ an arbitrary set. Define $Cons(A, X) = \{Y\colon\ Y \subseteq A$, $Y \cup X$ is consistent and Y is maximal with this property$\}$. Let $B(X) = \{Cn(Y \cup X)\colon\ Y \in Cons(A, X)\}$. If $A \cup X$ is consistent then $B(X) = \{Cn(A \cup X)\}$. If $A \cup X$ is inconsistent then $B(X)$ contains all complete extensions of X. This can be shown using a generalization of results in [20]. To get belief set operators derived from A, subsets from $Cons(A, X)$ have to be selected. Let $S : \mathcal{P}(L) \to \mathcal{P}(\mathcal{P}(L))$ satisfying $S(X) \subseteq Cons(A, X)$ such that $S(X) \neq \emptyset$ if $Cons(A, X) \neq \emptyset$. Then the following belief set operators B_S can be introduced: $B_S(X) = \{Cn(Y \cup X)\colon\ Y \in S(X)\}$. Again, we may introduce a belief state operator Γ_S for B_S by defining $\Gamma_S(X) = \{Mod(T)\colon\ T \in B_S(X)\}$.

Example 19 (Stable generated models of logic programs). Generalized logic programs were introduced in [22]. A generalized logic program P is a set of open sequents, where an open sequent is an expression of the form $F_1, \ldots, F_m \Rightarrow G_1, \ldots, G_n$, where F_i, G_j are open first-order formulas. In [22] the notion of a stable generated model for generalized logic programs was proposed. Then the following system $(L_{\mathrm{seq}}, M, \models, \Gamma)$ is a belief state frame: L_{seq} is the set of open sequents, M the set of all Herbrand interpretations, $\models$ the classical satisfiability relation and $\Gamma(P) = \{\{I\} \mid I$ is a stable generated model of $P\}$.

5. Representation theorems

The methods described in section 2 can be generalized to the case of belief set operators and belief set frames. In particular, there is a canonical method to introduce a semantics for a given belief set frame.

Proposition 20. Let $\mathcal{F} = (L, C_L, B)$ be a belief set frame satisfying belief left absorption. Then there exists a belief state frame $\mathcal{SF} = (L, M, \models, \Gamma)$ such that $\mathcal{L} = (L, C_L)$ is complete with respect to $(L, M, \models)$ and $B = B_\Gamma$. If $\mathcal{F}$ is a deductive belief set frame then $\mathcal{SF}$ can be taken to be $\mathcal{L}$-invariant.

Proof. Let $\mathcal{F} = (L, C_L, B)$ be a belief set frame satisfying belief left absorption. We construct a belief state frame $\mathcal{SF} = (L, M, \models, \Gamma)$ such that $C_L = C^{\models}$ and $B = B_\Gamma$. Let $(L, M, \models)$ be the LT-semantics for (L, C_L), and define $\Gamma(X) = \{Mod^{\models}(T) \mid T \in B(X)\}$. Then $B_\Gamma = B$. $B_\Gamma(X) = \{Th(Mod^{\models}(T) \mid T \in B(X)\}$, and since $C_L(T) = T$ for $T \in B(X)$ it follows $Th(Mod^{\models}(T)) = T$, hence $B_\Gamma(X) = B(X)$.

Now assume that $\mathcal{F}$ is a deductive belief set frame. Then $C_L(X) = C_L(Y)$ implies $B(X) = B(Y)$. We show that the above defined belief state operator is invariant. Since $C_L(X) = C_L(C_L(X))$, and by congruence $B(X) = B(C_L(X))$,

$$\Gamma(X) = \{Mod^{\models}(T) \mid T \in B(X)\} = \Gamma\bigl(C_L(X)\bigr) = \{Mod^{\models}(T) \mid T \in B\bigl(C_L(X)\bigr)\}. \quad \square$$

The question arises whether a belief set operator B can be extended to a deductive belief set frame (L, C_L, B). Of course, there is the following trivial solution: $C_L(X) = X$, which cannot be considered as adequate. It is reasonable to assume that the desired logic for B should be as close as possible to K_B; i.e., C_L should be maximal below K_B with respect to the following partial ordering between inference operations C_1, C_2: $C_1 \leqslant C_2 \Leftrightarrow (\forall X \subseteq L)\ (C_1(X) \subseteq C_2(X))$.

Proposition 21. Let B be a belief set operator on L satisfying strong belief cumulativity. Then there exists a deductive system (L, C_L) such that following conditions are satisfied:

1. (L, C_L, B) is a deductive belief set frame.
2. If (L, C_1, B) is a deductive belief set frame then $C_1 \leqslant C_L$, i.e., C_L is the greatest deductive system for (L, B).

Proof. Since B is strongly cumulative the inference system (L, K_B) is cumulative. By the main result in [9] there exists a largest deductive operation $C_L \leqslant K_B$ such that (L, C_L, K_B) is a deductive inference frame. Since $\mathcal{BF} = (L, C_L, B)$ is a strong cumulative belief set frame it follows by proposition 12 that $\mathcal{BF}$ is a deductive belief set frame. $\mathcal{BF}$ satisfies the desired properties. $\square$

The semantical approach presented here can be summarized as follows. We start with a belief set operator B on a language L; in the next step we construct a belief set frame (L, C_L, B) such that the compact logic (L, C_L) satisfies additional properties, e.g., maximality. Then for (L, C_L, B) we may introduce the standard semantics indicated in proposition 20 (see figure 1).

Finally, we return to the connections between deductive frames and deductive belief set frames. Obviously, as mentioned before, deductive frames (L, C_L, C) can

Belief set system
(L, B)
$\Downarrow$
Deductive belief set frame
(L, C_L, B)
$\Downarrow$
Belief state frame
$(L, M, \models, \Gamma)$

Figure 1. Standard semantics of belief set operators.

be considered as a special case of belief set frames by taking $B_C(X) = \{C(X)\}$. On the other hand, for every deductive belief set frame (L, C_L, B) there exists exactly one deductive frame defined by the kernel K_B. The converse is not true: for a given deductive inference frame there can be many deductive belief set frames with the same kernel. Belief set frames can be understood as specializations of deductive inference frames and a deductive inference frame can be interpreted as an abstract representation of a family of deductive belief set frames. To make this view precise let $\mathcal{F} = (L, C_L, C)$ be a deductive inference frame and let $\Omega(\mathcal{F}) = \{B$: (L, C_L, B) is a consistency preserving deductive belief set frame such that $C = K_B\}$. The binary relation $\preceq$ between belief set operators in $\Omega(\mathcal{F})$ is defined as follows: $B_1 \preceq B_2$ if $(\forall X)$ $(B_1(X) \preceq B_2(X))$, and $B_1 \equiv B_2$ iff $B_1 \preceq B_2$ and $B_2 \preceq B_1$. Let $\mathbf{BF}(\mathcal{F}) = (\Omega(\mathcal{F}), \preceq)$ and $Max(X) = \{S$: S is a maximal consistent extension of $X\}$; $B \in \Omega(\mathcal{F})$ is said to be a *maximization operator* iff $(\forall X \subseteq L)$ $(B(X) \subseteq Max(C(X)))$.

Proposition 22. Let $\mathcal{F} = (L, C_L, C)$ be a deductive inference frame. Then $\mathbf{BF}(\mathcal{F}) = (\Omega(\mathcal{F}), \preceq)$ is a partial ordering.

Proof. Obviously, the relation $\preceq$ satisfies reflexivity and transitivity. We show anti-symmetry. Assume $B_1 \preceq B_2$ and $B_2 \preceq B_1$ for $B_1, B_2 \in \Omega(\mathcal{F})$. Let $U \in B_1(X)$, by assumption there is a $V \in B_2(X)$ such that $V \subseteq U$; since $B_1(X) \preceq B_2(X)$ there is a set $W \in B_1(X)$ satisfying $W \subseteq V$. Non-inclusiveness of $B_1(X)$ implies $U = V$, hence $U \in B_2(X)$. Analogously one shows $B_2(X) \subseteq B_1(X)$. $\square$

Proposition 23. Let $\mathcal{F} = (L_0, Cn, C)$ be a deductive inference frame over classical logic (L_0, Cn). The system $\mathbf{BF}(\mathcal{F})$ has a least element and a least maximization operator. A belief set operator $B \in \mathbf{BF}(\mathcal{F})$ is a maximal element with respect to $\preceq$ if and only if B is a maximization operator such that for every $X \subseteq L$ and $T \in B(X)$ the following condition $(*)$ $C(X) \neq \bigcap(B(X) - \{T\})$ is satisfied.

Proof. Let $\mathcal{F} = (L_0, Cn, C)$; the least element $B_{\min}$ of $\Omega(\mathcal{F})$ is defined by $B_{\min}(X) = \{C(X)\}$, and the least maximization operator is determined by $B_{\max}(X) = Max(C(X))$. Now, let B be a maximal element. We firstly show that B is a maximization operator. Assume this is not the case. Then there is a belief set $T \in B(X)$ (for a certain set $X \subseteq L_0$), such that T is not maximal. We define a new operator B_1 as follows: $B_1(Y) = B(Y)$ for all $Y \neq X$, and $B_1(X) = (B(X) - \{T\}) \cup Max(T)$. It is easy to show that $B \preceq B_1$, but not $B_1 \preceq B$. Now we will show $(*)$. Suppose there exist $X \subseteq L_0$ and $T \in B(X)$ such that $C(X) = \bigcap(B(X) - \{T\})$. Then define B_1 by setting $B_1(Y) = B(Y)$ for all $Y \neq X$, and $B_1(X) = B(X) - \{T\}$. Then $B \preceq B_1 \in \mathbf{BF}(\mathcal{F})$, contradicting maximality of B. Conversely, assume that B is a maximization operator satisfying the condition $(*)$. Suppose B is not maximal. Then there is an operator $B_1 \in \mathbf{BF}(\mathcal{F})$, such that $B(X) \preceq B_1(X)$, but $B(X) \neq B_1(X)$. Since every $T \in B_1(X)$ is an extension of a belief set of $B(X)$ and every belief set in $B(X)$ is a maximal extension of $C(X)$ it holds that $B_1(X) \subseteq B(X)$. Hence, by condition $(*)$ it follows $\bigcap B_1(X) \neq C(X)$. This gives a contradiction. $\square$

A belief set operator $B \in \Omega(\mathcal{F})$ satisfies *C-congruence* iff $(\forall XY \subseteq L)(C(X) = C(Y) \Rightarrow B(X) = B(Y))$. The following observation is obvious.

Proposition 24. Let $\mathcal{F} = (L_0, Cn, C)$ be a cumulative deductive inference frame. Then every C-congruent belief set operator in $\Omega(\mathcal{F})$ satisfies belief cumulativity, i.e., weak belief monotony and belief cut. Furthermore, the least maximization operator satisfies C-congruence.

Remark. Concerning the structure of $\Omega(\mathcal{F})$ there is the following question. Let P be a property on belief set frames, and $\mathcal{F}$ is a cumulative deductive inference frame. Does there exist an element in $\Omega(\mathcal{F})$ which is maximal with respect to the property P? Examples of such properties are distributivity or strong belief cumulativity.

6. Selection operators

In the previous sections we concentrated on the multiple belief set view. The kernel of a belief set operator represents the most certain inferences the agent can make. But there is also another way in which the agent can handle the multiple views, and that is by selecting one (or a subset) of the possible views and focusing on this view. In the area of design, given some requirements a designing agent may have multiple (partial) descriptions of objects that do not contradict the requirements. It may have one of these descriptions (views) in focus, which it will try to complete. Here the selection indicates which view is in focus. On the other hand, for many nonmonotonic formalisms in which a theory can have multiple extensions (or expansions), a prioritized or stratified version exists, in which control knowledge (such as a preference ordering on the nonmonotonic rules) is used to designate one of the extensions as the most preferred one [2,8,24,36]. This focusing mechanism can be studied abstractly through *selective inference operations* for a given belief set operator which choose one of the sets of beliefs.

Definition 25. Let B be a belief set operator. A selective inference operation for B is an inference operation C such that $\forall X \subseteq L$: $C(X) \in B(X)$.

We consider a typical example of a selective inference operation for the belief set operator based on default logic.

Example 26 (Prioritized default logic [8]). Let D be a countable set of normal defaults, denoted by a/c, and let X be a set of formulas. According to example 4 we may define the belief set operator $B_D(X)$ collecting all Reiter-extensions of X with respect to D. If X is consistent then, since D contains only normal defaults, the set

$B_D(X)$ is non-empty. For every well-ordering $\ll$ of D we define a selective inference operation $C_\ll$ for B_D as follows. A default $\delta = a/c$ is said to be active in a set Z of formulas if $a \in Z$, $c \notin Z$, and $\neg c \notin Z$. Let a set X be given and define a sequence $\{E_i\colon\ i < \omega\}$ as follows. $E_0 = \mathit{Th}(X) = \{\phi\colon\ X \models \phi\}$,

$$E_{i+1} = \begin{cases} E_i, & \text{if no default is active in } E_i,\\ \mathit{Th}(E_i \cup \{c\}), & \text{otherwise, where } c \text{ is the consequent of the } \ll\text{-least default}\\ & \text{that is active in } E_i. \end{cases}$$

We define $C_\ll(X) = \bigcup_{i<\omega} E_i$. It can be shown that $C_\ll(X) \in B_D(X)$ [8]. The extension $\bigcup_{i<\omega} E_i$ is called the prioritized extension of (D, X) *generated by* $\ll$.

One may argue that the concept of a selective inference operation is already covered by the notion of a usual inference operation as discussed in section 2. Obviously, this is not the case because a selective inference operation is always connected with a belief set operator as a separate notion. As an example imagine an agent A which acts under incomplete information in a dynamic environment. It is important for A to have an appropriate basic space of different belief sets and an additional mechanism to choose and generate one of them to adapt his behavior to a particular situation.[2] In principle, this idea can also be realized by a suitable family of usual inference operations and a choice mechanism. To structure the connections between belief set operators and selective inference operations, we give the following definition:

Definition 27.

1. Let a belief set operator B be given. The family of selective inference operations for B, denoted by $\mathcal{C}_B$ is defined by
$$\mathcal{C}_B = \{C \mid C \text{ is a selective inference operation for } B\}.$$
2. Let $\mathcal{C}$ be a family of inference operations. Define the belief set operator $B_\mathcal{C}$ by $B_\mathcal{C}(X) = \{C(X) \mid C \in \mathcal{C}\}$.

A selective inference operation for a belief set operator will in general be more informative than the associated kernel: $K_B(X) \subseteq C(X)$. But even if the belief set operator is well-behaved, a selective inference operation can be badly behaved. The question arises whether a well-behaved selective inference operation always exists. That is, given a belief set operator B, the question is whether there exists a $C \in \mathcal{C}_B$ with certain structural properties. This is a very hard question. Sufficient conditions can be found, for instance for monotony: $\forall Y\ \exists T \in B(Y)\ \forall X \subseteq Y\ \forall S \in B(X)\colon\ S \subseteq T$. But this condition implies (in the presence of non-inclusiveness) that $B(X)$ is a singleton for all X. Necessary conditions are easier to find, but quite trivial. For a belief set

[2] It seems that this kind of non-determinism is an essential assumption for realizing intelligent behavior in a changing environment.

operator B and a selective inference operation C for B we have the following. If C satisfies cut then

$$(\forall X)\ \big(\exists S \in B(X)\big)\ (\forall Y)\ \big(X \subseteq Y \subseteq S \Rightarrow \big(\big(\exists T \in B(Y)\big)\ (T \subseteq S)\big)\big). \qquad (*1)$$

If C satisfies cautious monotony then

$$(\forall X)\ \big(\exists S \in B(X)\big)\ (\forall Y)\ \big(X \subseteq Y \subseteq S \Rightarrow \big(\big(\exists T \in B(Y)\big)\ (S \subseteq T)\big)\big). \qquad (*2)$$

If C satisfies cumulativity then

$$(\forall X)\ \big(\exists S \in B(X)\big)\ (\forall Y)\ \big(X \subseteq Y \subseteq S \Rightarrow \big(\big(\exists T \in B(Y)\big)\ (S = T)\big)\big). \qquad (*3)$$

If C satisfies monotony then

$$(\forall XY)\ \big(X \subseteq Y \Rightarrow \big(\big(\exists S \in B(X)\big)\ \big(\exists T \in B(Y)\big)\ (S \subseteq T)\big). \qquad (*4)$$

The preceding paragraph pertains to the situation when a belief set operator B is given, and we want to study $\mathcal{C}_B$. Questions about the second item in definition 27 are easier to answer. We will say a family $\mathcal{C}$ of inference operations satisfies one of the properties of cut, cautious monotony, cumulativity and monotony if all of the inference operations in $\mathcal{C}$ satisfy this property. Then we have:

Proposition 28. Let $\mathcal{C}$ be a family of inference operations.

(1) If $\mathcal{C}$ satisfies monotony then $B_{\mathcal{C}}$ satisfies belief monotony.

(2) If $\mathcal{C}$ satisfies cautious monotony then $B_{\mathcal{C}}$ satisfies weak belief monotony.

(3) If $\mathcal{C}$ satisfies cut, then $B_{\mathcal{C}}$ satisfies both belief transitivity and (strong) belief cut.

(4) If $\mathcal{C}$ satisfies cumulativity, then $B_{\mathcal{C}}$ satisfies strong belief cumulativity.

Proof. (1) Suppose $\mathcal{C}$ satisfies monotony, and suppose $X \preceq Y$. Take any $C(Y) \in B_{\mathcal{C}}(Y)$, then $C(X) \subseteq C(Y)$ and $C(X) \in B_{\mathcal{C}}(X)$. We have $B_{\mathcal{C}}(X) \preceq B_{\mathcal{C}}(Y)$.

(2) Suppose $X \preceq Y \preceq B_{\mathcal{C}}(X)$. Take a $C(Y) \in B_{\mathcal{C}}(Y)$, then $X \subseteq Y \subseteq C(X)$ (since $Y \preceq B_{\mathcal{C}}(X)$), so $C(X) \subseteq C(Y)$. It again follows that $B_{\mathcal{C}}(X) \preceq B_{\mathcal{C}}(Y)$.

(3) Suppose $\mathcal{C}$ satisfies cut. We only have to prove that $B_{\mathcal{C}}$ satisfies strong belief cut. So suppose $C(X) \in B_{\mathcal{C}}(X)$ and $X \subseteq Y \subseteq C(X)$. Then $C(Y) \subseteq C(X)$ and $C(Y) \in B_{\mathcal{C}}(Y)$.

(4) If $\mathcal{C}$ satisfies cumulativity, it satisfies cautious monotony and cut, so by (2) and (3), $B_{\mathcal{C}}$ satisfies weak belief monotony, belief cut and belief transitivity, hence it satisfies strong belief cumulativity. □

One way of defining selective inference operations for a given belief set operator is through *selection operators*. Given a set of views, such a selection operator selects one (or some) of them:

Definition 29. A selection operator is a function $s : \mathcal{P}(\mathcal{P}(L)) \to \mathcal{P}(\mathcal{P}(L))$ such that for all $\mathcal{A} \subseteq \mathcal{P}(L)$: $s(\mathcal{A}) \subseteq \mathcal{A}$, and $s(\mathcal{A}) \neq \emptyset$ if $\mathcal{A} \neq \emptyset$. A selection operator s is single-valued iff for all non-empty $\mathcal{A} \subseteq \mathcal{P}(L)$: *card*$(s(\mathcal{A})) = 1$. A single-valued selection operator s can be understood as a choice function $s : \mathcal{P}(\mathcal{P}(L)) \to \mathcal{P}(L)$ satisfying $s(\mathcal{A}) \in \mathcal{A}$ for all non-empty $\mathcal{A}$.

Using selection operators we can generate inference operations:

Definition 30. Let a belief set operator B and a selection operator s be given. We define the inference operation C_s^B by $C_s^B(X) = \bigcap s(B(X))$.

We will give some examples of belief set operators with selection operators.

Example 31 (Autoepistemic logic and parsimonious expansions). It is well known that in autoepistemic logic it may happen that the objective (i.e., non-modal) part of a stable expansion is contained in the objective part of another stable expansion. The easiest example is the theory $\{Lp \to p\}$, which has two stable expansions: the (unique) stable set with objective part *Cn*$(\emptyset)$, and the stable set with objective part *Cn*$(\{p\})$. Given a modal language L_m we can define the belief set operator B_{ael} which assigns to each set I of modal formulas the set of stable expansions of I. But an agent may want to keep only those expansions with a minimal objective part (these are called *parsimonious expansions* in [11]). We could define the selection operator s_p by $s_p(\mathcal{A}) = \{X \in \mathcal{A} \mid$ there is no $Y \in \mathcal{A}$ such that the objective part of Y is included in the objective part of $X\}$. Then $s_p(B_{\mathrm{ael}}(I))$ is the collection of parsimonious expansions of I, and $C_{s_p}^{B_{\mathrm{ael}}}$ gives the (skeptical) conclusions based on these expansions.

Example 32 (Prioritized default logic, continued). In example 26, a single extension was selected from the set of all extensions on the basis of a well-ordering $\ll$ on the set of defaults D. Often, the priority information will be partial, and we can select the extensions which comply with this partial information (see [8]). Given a partial ordering $<$ on D, we can define a selection operator that selects those extensions of (D, X) which are generated by a well-ordering $\ll$ that extends $<$ (meaning that $d_1 < d_2$ implies $d_1 \ll d_2$).

Example 33 (EKS: Ecological Knowledge System). Nature conservationists are interested in a number of so-called abiotic factors of terrains. These factors, examples of which are the moisture, acidity and nutrient value, give an indication of how healthy a terrain is. As these factors are difficult to measure directly, a sample of plant species growing on a terrain is taken. For each species, the experts have knowledge about the possible values of the abiotic factors of a terrain on which the species lives. So it may be known, for example, that a certain species can only live on medium to very acid terrains. Combining such knowledge for each of the plant species observed on a terrain leads to conclusions about the abiotic factors of the terrain. During the development of a knowledge-based system, EKS, to automate this classification process, however, it turned out that the samples of species taken were often incompatible. That is, there was at least one abiotic factor for which no value could be found that was permissible for all species. This is not due to errors in the knowledge of abiotic factors needed by species to live, but due to other effects. For example, a terrain may lie on the transition of a dry and a wet piece of land. Some of the observed species may occur on the drier, and others on the wetter side. This can also be due to the presence of ponds in an otherwise dry terrain. Also transitions of a terrain over time, or vertical inhomogeneity may be causes.

If the sample of species is incompatible, one can consider maximal compatible subsets of the sample. Each of these subsets defines a possible view on the terrain, with possible values for the abiotic factors. This gives rise to a belief set operator B_{EKS} that assigns to each sample of species, the set of maximal compatible subsets. The knowledge-based system, EKS, implements this operator. The user can input the species found in the sample, and the system presents the maximal compatible subsets. After that, the user can select one of these possible views on the terrain. The (ecologist) user makes this selection using additional knowledge (for instance about the history of the terrain, or about vertical inhomogeneity). This selection process can be formalized by a (single-valued) selection operator s_{user}. The final conclusions of the system contain the possible values of the abiotic factors for the chosen subset. It is intended to also automate this selection process. Presentation of the maximal compatible subsets was much appreciated by the users of the system, and helps them to classify the terrain. Separation of the generation of possibilities and the selection was a crucial step in the development of the system. It also allows different users to select different sets (from the possibilities generated by the system) and argue about which one is the right one. Thus, one could distinguish different selection functions $s_A, s_B, \ldots$ for the same belief set operator, corresponding to the choice of different users $A, B, \ldots$. The interested reader is referred to [5] for more information on the system EKS, and to [6] for the formalization of the reasoning task of the system (in terms of a belief set operator and selection function).

Single-valued selection operators generate *selective* inference operations. A first observation about when a selective inference operation can be generated by a single-valued selection operator:

Proposition 34. Let a selective inference operation C for a belief set operator B be given. Then $C = C_s^B$ for some single-valued selection operator s iff

$$(\forall X \forall Y)\ \big(B(X) = B(Y) \Rightarrow C(X) = C(Y)\big).$$

Proof. Define s as follows: for $\mathcal{A} \subseteq \mathcal{P}(L)$, if $\mathcal{A} = B(X)$ for some $X \subseteq L$, then $s(\mathcal{A}) = \{C(X)\}$, and if not, then s selects any set from $\mathcal{A}$ (and $s(\emptyset) = \emptyset$). The requirement ensures that s is well-defined, and it is easy to see that s is a single-valued selection operator. For any $X \subseteq L$ we have $C_s^B(X) = \bigcap s(B(X)) = \bigcap\{C(X)\} = C(X)$. The other direction is trivial. $\square$

We can study properties of selection operators and the relation with properties of belief state operators and selective inference operations. Although a full treatment is beyond the scope of this paper, we will give an example.

Definition 35. A selection operator s satisfies selection monotony if for all belief set families $\mathcal{A}$, $\mathcal{B}$ we have $\mathcal{A} \preceq \mathcal{B} \Rightarrow s(\mathcal{A}) \preceq s(\mathcal{B})$.

Then we have the following:

Proposition 36.

1. Let a belief set operator B and a selection operator s be given. If B satisfies belief monotony and s satisfies selection monotony then C_s^B satisfies monotony.
2. Let a single-valued selection operator s be given. If for any belief set operator B which satisfies belief monotony, C_s^B satisfies monotony, then s satisfies selection monotony.

Proof. 1. If $X \subseteq Y$ then $B(X) \preceq B(Y)$ (belief monotony) so $s(B(X)) \preceq s(B(Y))$ (selection monotony) so $C_s^B(X) \subseteq C_s(Y)$.

2. Suppose we have two belief set families $\mathcal{A} \preceq \mathcal{B}$. Define a belief set operator B by $B(\emptyset) = \mathcal{A}$ and $B(X) = \mathcal{B}$ for $X \neq \emptyset$. It is easy to see that B satisfies belief monotony. Then as $\emptyset \subseteq L$, we must have $C_s^B(\emptyset) \subseteq C_s^B(L)$, and as s is single-valued this means that $s(B(\emptyset)) \preceq s(B(L))$ or $s(\mathcal{A}) \preceq s(\mathcal{B})$. $\square$

The problem with selection operators is that they are blind to the initial facts: if $B(X) = B(Y)$, then we may sometimes want to make a different selection from $B(X)$ than from $B(Y)$. One option would be to define selection operators s_X with an index for the initial facts. An inference operation $C_s^B(X)$ could then be defined by $C_s^B(X) = \bigcap s_X(B(X))$. This would yield results similar to the construction of $B_{\mathcal{C}}$ defined earlier.

Example 37 (Contraction functions). In [1], eight rationality postulates are given for contraction functions. A contraction function $\dot{-}$ is a function that given a belief set K (satisfying *Cn*$(K) = K$) and a formula φ yields a new belief set $K \dot{-} \varphi$ which is meant to be the result of 'removing' φ from K. The function $\dot{-}$ should satisfy the following conditions:

1. For any sentence ϕ and any belief set K the set $K \dot{-} \phi$ is belief set.
2. $K \dot{-} \phi \subseteq K$.
3. If $\phi \notin K$, then $K = K \dot{-} \phi$.

4. If $\not\models \phi$ then $\phi \notin K \dot{-} \phi$.
5. $K \subseteq (K \dot{-} \phi) + \phi$.
6. If $\models \phi \leftrightarrow \psi$, then $K \dot{-} \phi = K \dot{-} \psi$.
7. $K \dot{-} \phi \cap K \dot{-} \psi \subseteq K \dot{-} (\phi \wedge \psi)$.
8. If $\phi \notin K \dot{-} \phi \wedge \psi$, then $K \dot{-} \phi \wedge \psi \subseteq K \dot{-} \psi$.

Call a selection operator s_X *invariant* if $s_X = s_{Cn(X)}$ for all X. Then a result from [1] can be given in our terms:

- A contraction function $\dot{-}$ satisfies postulates 1–6 iff $X \dot{-} \varphi = C_s^{B_{-\varphi}}(X)$ for some invariant s, where $B_{-\varphi}$ is as defined in example 5. Furthermore, if we put extra conditions on the selection operator – intuitively, that it picks maximal elements from $B_{-\varphi}$ given some transitive and reflexive order – then this result can be strengthened in the sense that all rationality postulates hold.

Remark. The considerations in sections 5 and 6 reflect certain aspects of knowledge dynamics [33]. Let X_0 be a deductively closed set representing the knowledge at a certain time point. X_0 can be extended by a combined application of a belief set operator B_0 whose kernel is X_0 and a generalized selection operator s_0. The new knowledge stage X_1 is defined by $X_1 = \bigcap s_0(B_0(X_0))$. The forming of belief sets for a knowledge base can be understood as theory formation or hypothesis building; after new observations are performed those belief sets are left out which contradict the observations.

7. Conclusions and future research

In research on nonmonotonic reasoning often an ambivalent or negative attitude is taken towards the phenomenon of multiple (belief) extensions. Of course, from the classical viewpoint it may be considered disturbing when a reasoning process may have alternative sets of outcomes, often mutually inconsistent. A number of approaches try to avoid the issue by adding additional control knowledge to decide which extension is intended, thus obtaining a parameterization of the possible sets of outcomes of the reasoning by the chosen control knowledge: for each control knowledge base a unique outcome (e.g, [8,36]). Another approach to avoid the multiple extension issue is to concentrate on the intersection of them: the sceptical approach. A number of results have been developed on nonmonotonic inference operations that are a useful formalization of this approach (e.g, [25]). However, in the sceptical approach the remaining conclusions may be very limited, insufficient for an agent to act under incomplete information in a dynamic environment.

In the current paper we address the multiple extension issue in an explicit manner by introducing nonmonotonic (multiple) belief set operators and their semantical counterpart: belief state operators. Many properties and results on nonmonotonic inference operations (and model operators) turn out to be generalizable to this notion.

Introducing alternative belief sets that can serve as the outcomes of a nonmonotonic reasoning process, the question becomes how to formalize the process of committing to one belief set. To this end in the current paper selection operators are introduced that formalize this process. The specification of a selection operator expresses the strategic (control) knowledge used by an agent to choose between the different alternatives.

Agents often construct belief sets to which they commit in a step by step manner, using some kind of inference rules. Specification of such a nonmonotonic reasoning process is easier to obtain if the reasoning patterns leading to the outcomes are specified instead of (only) the outcomes of the reasoning. In related and future research the notion of a trace for a nonmonotonic reasoning process is taken as a point of attention, and we have used a temporal epistemic logic to specify such traces (e.g., [17], following the line of [15,16]). Of course, a set of (multiple) reasoning traces generates a belief set operator by considering only the start and endpoints of the traces. The dualism between multiple outcomes and multiple reasoning traces of a nonmonotonic reasoning process is also studied in the context of default logic, leading to a representation theory: for which set of outcomes can a default theory be found with these outcomes (see [14,30]).

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