Temporalizing Epistemic Default Logic+

Wiebe van der Hoek[a]
John-Jules Meyer[b]
Jan Treur[c]

1. Introduction

In [MH93a,b] a nonmonotonic logic was introduced, called Epistemic Default Logic (**EDL**). It is based on the metaphor of a meta-level architecture. It already has been established in [MH93a,b] how upward reflection can be formalized by a nonmonotonic entailment based on epistemic states, and the meta-level process by a (monotonic) epistemic logic. The meta-level reasoning can be viewed as the part of the reasoning pattern where it is determined what the possibilities are for default assumptions to be made, based on which information is available and (especially) which is not. The outcome at the meta-level concerns conclusions of the form Pφ, where φ is an object-level formula. In EDL, default conclusions are kept separate from the object level knowledge (they remain at the meta-level), by means of this explicit default operator P (just like in NML3, see [Doh91]). If one wants to draw further conclusions from them using object level knowledge this should be done at the meta-level. Compared to a meta-level architecture, what was still missing was the step where the default assumptions are actually made, i.e., where such formulas φ are added to the object level knowledge, in order to be able to reason further with them at the object level. Here we actually "jump (down) to conclusions". This is what should be achieved by the downward reflection step. In the current paper we will introduce a formalization of the downward reflection step in the reasoning pattern as well. Thus a formalization is obtained of the reasoning pattern as a whole consisting of a process of generating possible default assumptions and actually assuming them (a similar pattern as generated by the so-called BMS-architecture introduced in [TT91]).

The formalization of downward reflection is inspired by [Tre94, HMT94], where it is pointed out how temporal models can provide an adequate semantics for meta-level architectures in general, and [ET93, ET94] where these ideas have been worked out to obtain a linear time temporal semantics for default logic. The general idea is that conclusions derived at the meta-

a Utrecht University, Dept. of Computer Science, P.O. Box 80.089, 3508 TB Utrecht. Email: wiebe@cs.ruu.nl. Also partially supported by the Free University of Amsterdam.
b Like footnote a, but also partially supported by the University of Nijmegen. Email: jj@cs.ruu.nl.
c Free University of Amsterdam, Dept. of Mathematics and Computer Science, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands. Email: treur@cs.vu.nl.

level essentially are statements about the state of the object level reasoning at the next moment of time. So we can define downward reflection as a shift of time in a (reasoning) process that is described by temporal logic.

In this paper in the Sections 2 to 6 the logic EDL is presented. As compared with earlier publications ([MH91a, MH92a, MH93a,b]), the logic is slightly extended in order to cater for downward reflection in subsequent sections. In Section 7 we define a labeled branching time temporalization of this logic, in spirit of the approach of [FG92]. In Section 8 we define sceptical and credulous entailment relations based on temporal models.

2. Epistemic logic

To start with, our language consists of objective formulas together with epistemic formulas.

2.1. DEFINITION (objective and epistemic formulas). Let $\mathbf{P}$ be a set of propositional constants (atoms); $\mathbf{P} = \{p_k \mid k \in I\}$, where I is either a finite or countably infinite set. The set OBJ of objective formulas is the smallest set containing $\mathbf{P}$ and closed under the classical propositional connecitves. The formula $\perp$ is shorthand for $(p_k \wedge \neg p_k)$, for an arbitrary $p_k \in \mathbf{P}$. The set FORM of *epistemic formulas* φ, ψ,… is the smallest set containing OBJ, closed under the epistemic operator K, where $K\varphi$ means that φ is known. Moreover, we use $M\varphi$ as an abbreviation for $\neg K\neg\varphi$, meaning that φ is epistemically possible. For $\Gamma \subseteq$ FORM, we denote by Prop(Γ) the set of objective formulas in Γ.

Objective formulas are interpreted on ordinary valuations:

2.2. DEFINITION (propositional models, classical consequence). A *valuation* is a function s from $\mathbf{P}$ to $\{\mathbf{t}, \mathbf{f}\}$. The set of all valuations is denoted Mod($\mathbf{PC}$). By determining s' behaviour under the Boolean connectives, such a valuation is straightforwardly extended to a truth function on arbitrary objective formulas α (cf. Definition 2.5): we then say that s is a *(propositional model)* for α. For any *set* of valuations M and objective formula α, we write M $\vDash \alpha$ if any s $\in$ M is a model for α.

To interpret the whole set FORM of epistemic formulas, we need richer structures:

2.3. DEFINITION (simple S5-Kripke models). A (simple) S5-model $\mathbf{M}$ is a non-empty subset of Mod($\mathbf{PC}$). The elements of $\mathbf{M}$ are called worlds. We denote the class of simple S5-models by Mod($\mathbf{S5}$). Note that for any M $\subseteq$ Mod($\mathbf{PC}$) we may interpret arbitrary objective formulas, but in order to interpret epistemic formulas (cf. Definition 2.5) we have to conceive M as an S5-model $\mathbf{M}$.

The more general definition of an S5-model requires $\mathbf{M}$ to be of the format $\mathbf{M} = \langle W, \pi, R \rangle$, where, for each $w \in W$, $\pi(w)$ is a valuation (thus, different elements of W may verify the same objective formulas) and R is an equivalence relation (needed to interpret the operator K), but for our purposes we will assume S5-models to be simple in the sense as we defined above. One can show that our assumption does not change the validities of the logic (cf. Definition 2.6 below), but it facilitates a number of technical issues such as the definition of submodels and union of models. However, some caution is in order here: although the restriction to simple S5-models does not affect the notion of validity, some of the propositions on S5-models below *do* depend on the simplicity of the models, and do *not* hold for general S5-models.

The set of worlds in an S5-model represents a collection of alternative worlds that are considered (equally) possible on the basis of (lack of) knowledge. In the next section we shall use S5-models as representations of the reasoner's objective knowledge and epistemic meta-knowledge (i.e., what he knows that he knows or does not know).

2.4. REMARK (submodels and union of S5-models). Note that under our notion of S5-models, which are just sets of valuations, one easily defines a subset relation on S5-models, and an operation of union upon them.

2.5. DEFINITION (interpretation of epistemic formulas). Given the S5-model $\mathbf{M} \subseteq \mathrm{Mod}(\mathbf{PC})$, we define the relation $(\mathbf{M}, s) \vDash \varphi$, with $s \in \mathbf{M}$ by induction on the structure of the epistemic formula φ:

$$
\begin{aligned}
(\mathbf{M}, s) &\vDash p & &\Leftrightarrow s(p) = \mathbf{t} \text{ for } p \in \mathbf{P} \\
(\mathbf{M}, s) &\vDash \psi_1 \wedge \psi_2 & &\Leftrightarrow (\mathbf{M}, s) \vDash \psi_1 \text{ and } (\mathbf{M}, s) \vDash \psi_2 \\
(\mathbf{M}, s) &\vDash \neg\psi & &\Leftrightarrow (\mathbf{M}, s) \nvDash \psi \\
(\mathbf{M}, s) &\vDash K\psi & &\Leftrightarrow (\mathbf{M}, s') \vDash \psi \text{ for all } s' \in \mathbf{M} \\
(\mathbf{M}, s) &\vDash M\psi & &\Leftrightarrow (\mathbf{M}, s') \vDash \psi \text{ for some } s' \in \mathbf{M}.
\end{aligned}
$$

Note, that the definition of the *K*-operator states that φ is known precisely when φ holds in the whole set of epistemic alternatives; that for the M-operator states that φ is considered epistemically possible iff there is at least one epistemic alternative that satisfies φ. In the sequel, when we write $M \subseteq \mathrm{Mod}(\mathbf{PC})$ we want to interpret it as a collection of propositional formulas, whereas, if we write $\mathbf{M}$ for the same collection, we intend that $\mathbf{M} \in \mathrm{Mod}(\mathbf{S5})$, on which the epistemic formula $K\varphi$ has an interpretation. We then say that $\mathbf{M}$ is *based on* M.

2.6. DEFINITION (validity and satisfiability).
(i) φ is *valid in an S5-model* $\mathbf{M} \in \mathrm{Mod}(\mathbf{S5})$, denoted $\mathbf{M} \vDash \varphi$, if for all $m \in \mathbf{M}$: $(\mathbf{M}, m) \vDash \varphi$.
(ii) φ is *valid*, notation $\mathrm{Mod}(\mathbf{S5}) \vDash \varphi$, if $\mathbf{M} \vDash \varphi$ for all S5-models $\mathbf{M}$.

(iii)   φ is *satisfiable* if there is an S5-model **M** and a world m ∈ **M** such that (**M**, m) ⊨ φ.

Validity w.r.t. S5-models can be axiomatized by the system **S5**:

2.7. DEFINITION (system **S5**). The logic **S5** consists of the following:

Axioms:
(A1) All propositional tautologies
(A2) $(K\varphi \land K(\varphi \to \psi)) \to K\psi$ *Knowledge is closed under logical consequence.*
(A3) $K\varphi \to \varphi$ *Known facts are true.*
(A4) $K\varphi \to KK\varphi$ *One knows that one knows something.*
(A5) $\neg K\varphi \to K\neg K\varphi$ *One knows that one does not know something.*

Derivation rules:

(R1) $$\frac{\varphi \quad \varphi \to \psi}{\psi}$$ *Modus Ponens*

(R2) $$\frac{\varphi}{K\varphi}$$ *Necessitation*

That φ is a theorem derived by the system **S5** is denoted by **S5** ⊢ φ.

2.8. THEOREM (Soundness and completeness of **S5**). **S5** ⊢ φ ⟺ Mod(**S5**) ⊨ φ

2.9. DEFINITION. We say a formula ψ is in *normal form* if it is a disjunction of conjunctions of the form $\delta = \alpha \land K\beta_1 \land K\beta_2 \land \ldots \land K\beta_n \land M\gamma_1 \land M\gamma_2 \land \ldots \land M\gamma_k$, where α, β_i and γ_j (i ≤ n, j ≤ k) are all objective formulas.

The following holds for the logic **S5**;

2.10. THEOREM ([HC68]) In **S5** every formula φ is equivalent to a formula ψ in normal form.

In particular, this theorem implies that every epistemic formula can be represented equivalently without nested epistemic modalities. In **S5** one thus can do without nestings.

3. Epistemic states and stable sets
In this paper we simply define an epistemic state as an S5-model. The idea behind this was already touched upon after Definition 2.3. The worlds in an S5-model represent the states of

the (real) world that the reasoner considers possible. Thus the S5-model as a whole delimits the ways the real world is like as far as the reasoner is concerned. In other words, it determines what he knows about the world and what he does not (what he has doubts about by considering contradictory possibilities). Therefore, an S5-model represents truly the epistemic state of the reasoner.

3.1. DEFINITION. An *epistemic state* is an S5-model $\mathbf{M} \in \text{Mod}(\mathbf{S5})$. The set M of worlds in $\mathbf{M}$ is the set of epistemic alternatives allowed by the epistemic state $\mathbf{M}$. For such a model $\mathbf{M}$, we define K($\mathbf{M}$) as the set of facts known in $\mathbf{M}$: K($\mathbf{M}$) = $\{\varphi \mid \mathbf{M} \vDash K\varphi\}$. We call K($\mathbf{M}$) the *theory of* $\mathbf{M}$ or *knowledge in* $\mathbf{M}$.

We mention here that the knowledge in $\mathbf{M}$ are exactly the validities in $\mathbf{M}$: K($\mathbf{M}$) = $\{\varphi \mid \mathbf{M} \vDash K\varphi\}$ = $\{\varphi \mid \mathbf{M} \vDash \varphi\}$. Cf. [MH94].

3.2. LEMMA. For any S5-models $\mathbf{M}_1$ and $\mathbf{M}_2$: $\mathbf{M}_1 \subseteq \mathbf{M}_2$ iff Prop(K($\mathbf{M}_2$)) $\subseteq$ Prop(K($\mathbf{M}_1$)).

3.3. REMARK. The converse relation of $\subseteq$ on Kripke models (Cf. Definition 2.4), will play an important role in the sequel. $\mathbf{M}_1 \supseteq \mathbf{M}_2$ means that the model $\mathbf{M}_2$, viewed as a representation of the knowledge of a reasoner, involves a *refinement* of the knowledge associated with model $\mathbf{M}_1$. This has to be understood as follows: in the model $\mathbf{M}_2$ less (or the same) worlds are considered possible by the reasoner as compared by the model $\mathbf{M}_1$. So, in the former case the reasoner has less doubts about the true nature of the world. By Lemma 3.2 it turns out that this means that with respect to model $\mathbf{M}_2$ the reasoner has at least the objective knowledge associated with model $\mathbf{M}_1$, and possibly more. So in a transition of $\mathbf{M}_1$ to $\mathbf{M}_2$ we may say that objective knowledge is gained by the reasoner. Thus the relation '$\supseteq$' acts as an information ordering on the set of S5-models. Finally we remark that Lemma 3.2 is a typical example of a property that holds for simple S5-models only; with respect to general S5-models we would only have the 'only if' part of the Lemma (cf. the discussion following Def. 2.3).

3.4. PROPOSITION (Moore [Moo85]).
(i)   The theory $\Sigma = $ K($\mathbf{M}$) of an epistemic state $\mathbf{M}$ is a so-called *stable set*, i.e., satisfies the following properties:
   (St 1)   all instances of propositional tautologies are elements of Σ;
   (St 2)   if $\varphi \in \Sigma$ and $\varphi \to \psi \in \Sigma$ then $\psi \in \Sigma$;
   (St 3) $\varphi \in \Sigma \Leftrightarrow K\varphi \in \Sigma$
   (St 4) $\varphi \notin \Sigma \Leftrightarrow \neg K\varphi \in \Sigma$
   (St 5) Σ is propositionally consistent.

(ii)   Every stable set Σ of epistemic formulas determines an S5-Kripke model $\mathbf{M}_\Sigma$ for which it holds that $\Sigma = K(\mathbf{M}_\Sigma)$. Moreover, if **P** is a finite set, then $\mathbf{M}_\Sigma$ is the unique S5-Kripke model with this property.

The (proof of) the following proposition, which determines stable sets in the realm of classical propositional logic (cf. St 1 & 5), is also to be found in [Moo85].

3.5. PROPOSITION. A stable set is uniquely determined by the objective formulas it contains.

3.6. REMARK. Thus stable sets act as the epistemic contents of an epistemic state (viz. an S5-model): a stable set $\Sigma = K(\mathbf{M})$ describes exactly the formulas known by the reasoner when he is is in epistemic state $\mathbf{M}$. From Proposition 3.4(i) we see that this knowledge is closed under classical propositional reasoning and under positive and negative introspection: if some formula is known also the fact that this is known is in its turn known by the reasoner, and if some formula is not known, the fact that it is not known is known itself. This reflects the (perhaps rather strong notion of) rationality of the reasoner. On the other hand, 3.4(ii) says that a stable set of epistemic formulas determines an S5-model, the epistemic state associated with (knowing) this stable set. Proposition 3.5 states that a stable set, and thus the associated epistemic state, is completely determined if one specifies exactly what *objective* knowledge is present at the reasoner. The rest of the known set of formulas (the stable set) then follows.

4. Entailment based on epistemic states: upward reflection

On the basis of epistemic states, Halpern & Moses define an entailment relation $\vdash\!\sim$ with which one can infer what is known, and, more importantly, what is unknown in such epistemic states.

Recall that with every set of valuations M we can associate its epistemic state $\mathbf{M}$ in a straightforward way (cf. Definition 2.5). Intuitively, $\mathbf{M}$ associates with M the epistemic state given by considering (only) the set M of classical propositional models to be the possible states of the world: this set, so to speak, gives a partial description of the real world. In general, more than just one classical model is considered possible so that only those facts are known that are true in all of them. We can exploit this idea further and define an entailment relation based on the premise that we *only know* some objective formula φ. To this end we need some additional notation:

4.1. DEFINITION. Given some objective formula φ, we define M_φ as the set of valuations satisfying φ, i.e., $M_\varphi = \{m \in \text{Mod}(\mathbf{PC}) \mid m \vDash \varphi\}$. The epistemic state that is associated with M_φ is denoted by $\mathbf{M}_\varphi$.

We have the following alternative characterizations of $\mathbf{M}_\varphi$:

4.2. PROPOSITION. $\mathbf{M}_\varphi = \bigcup \{\mathbf{M} \mid \mathbf{M} \vDash \varphi\} = \bigcup \{\mathbf{M} \mid \mathbf{M} \vDash K\varphi\}$.

Proposition 4.2 says that in order to get $\mathbf{M}_\varphi$, we can also consider all S5-models of φ and take their union to obtain one 'big' S5-model. The result immediately follows from the fact that we have simple S5-models, and from the fact that, for objective formulas φ and arbitrary $\mathbf{M} \in$ Mod($\mathbf{S5}$), $\mathbf{M} \vDash \varphi \Leftrightarrow$ (since φ is objective—adopting the view that $M \subseteq$ Mod($\mathbf{PC}$)) $M \vDash \varphi \Leftrightarrow M \subseteq M_\varphi$. So $\bigcup \{\mathbf{M} \mid \mathbf{M} \vDash \varphi\} = \bigcup \{M \mid M \subseteq M_\varphi\} = M_\varphi$. Thus $\mathbf{M}_\varphi = \bigcup \{\mathbf{M} \mid \mathbf{M} \vDash \varphi\}$.

Now we are ready to define an entailment relation based on this "big" model. Keeping in mind that '$\supseteq$' acts as an information order on models (Cf. 3.3), we define what are the consequences of knowing only the objective formula φ as those formulas that hold in the "big" model $\mathbf{M}_\varphi$:

4.3. DEFINITION (Nonmonotonic epistemic entailment). For $\varphi \in$ Prop(FORM), and $\psi \in$ FORM:

$$\varphi \mathrel{\vdash\mkern-7mu\sim} \psi \Leftrightarrow \psi \in K(\mathbf{M}_\varphi).$$

Informally, this means that ψ is entailed by φ, if ψ is contained in the theory (knowledge) of the "largest S5-model" $\mathbf{M}_\varphi$ of φ. Halpern & Moses showed in [HM84] that this "largest model" need not always be a model of φ itself if we allow φ to contain epistemic operators. However, in our case where we only use objective formulas φ, $\mathbf{M}_\varphi$ is always the largest model for φ. This is obvious from our construction of $\mathbf{M}_\varphi$, since $M_\varphi \vDash \varphi$ and therefore $\mathbf{M}_\varphi \vDash \varphi$ (since φ is objective). Moreover, in this case the theory $K(\mathbf{M}_\varphi)$ of this largest model is a stable set that contains φ and such that for all stable sets Σ containing φ it holds that Prop($K(\mathbf{M}_\varphi)$) $\subseteq$ Prop(Σ) (by Lemma 3.2 and Proposition 3.4), thus $K(\mathbf{M}_\varphi)$ is the "propositionally least" stable set that contains φ. So $\mathrel{\vdash\mkern-7mu\sim}$ can also be viewed as a *preferential entailment* relation in the sense of Shoham [Sho87, 88], where the preferred models of φ are the largest ones, viz. $\mathbf{M}_\varphi$, where the least objective knowledge is available.

We denote the mapping $\varphi \mapsto K(\mathbf{M}_\varphi)$ by κ: $\kappa(\varphi) = K(\mathbf{M}_\varphi)$, the stable set associated with knowing only φ. Alternatively viewed, $\kappa(\varphi)$ is the $\mathrel{\vdash\mkern-7mu\sim}$-closure of φ. Note that since $\kappa(\varphi) = K(\mathbf{M}_\varphi)$ is a stable set, it is also propositionally closed.

We give a few examples to show how the entailment $\mathrel{\vdash\mkern-7mu\sim}$ works: Let p and q be two distinct primitive propositions. Then:

$$p \mathrel{\mid\!\sim} K(p \vee q)$$

$$p \mathrel{\mid\!\sim} \neg Kq$$

$$p \mathrel{\mid\!\sim} Kp \wedge M\neg q$$

$$p \wedge q \mathrel{\mid\!\sim} K(p \wedge q) \wedge Kp \wedge Kq$$

$$p \vee q \mathrel{\mid\!\sim} K(p \vee q) \wedge M\neg p \wedge M\neg q$$

Obviously, the entailment relation $\mathrel{\mid\!\sim}$ is nonmonotonic, justifying the name we have given it. (For instance, we have $p \mathrel{\mid\!\sim} M\neg q$, while *not* $p \wedge q \mathrel{\mid\!\sim} M\neg q$; it even holds that $p \wedge q \mathrel{\mid\!\sim} \neg M\neg q$.)

The nonmonotonic epistemic entailment $\mathrel{\mid\!\sim}$ enables us to derive from an objective formula φ, characterizing the exact epistemic state of the reasoner (viz., technically, the epistemic state $\mathbf{M}_\varphi$), exactly what is known and, even more importantly, *what is unknown* in this epistemic state. This latter property renders the entailment relation context-sensitive and nonmonotonic, so that the relation $\mathrel{\mid\!\sim}$ goes beyond an entailment expressible in ordinary epistemic logic: with respect to its premise the relation $\mathrel{\mid\!\sim}$ involves a kind of epistemic closure.

Finally we state a property of that we shall need in the sequel:

4.4. PROPOSITION. The entailment relation $\mathrel{\mid\!\sim}$ enjoys the property:

$$\varphi \mathrel{\mid\!\sim} \psi_1 \ \& \ \varphi \mathrel{\mid\!\sim} \psi_2 \ \Rightarrow \ \varphi \mathrel{\mid\!\sim} \psi_1 \wedge \psi_2 \ .$$

PROOF. Directly from Definition 4.3 and the fact that the stable set $K(\mathbf{M}_\varphi)$ is closed under conjunction (which in its turn follows from the fact that a stable set is closed under propositional reasoning and contains the tautology $\psi_1 \rightarrow (\psi_2 \rightarrow (\psi_1 \wedge \psi_2))$). ∎

5. The epistemic preference logic S5P

The "upward reflection" entailment relation $\mathrel{\mid\!\sim}$ enables us to derive information about what is known and what is not known. In this section we show how we can use this information to perform default reasoning. To this end we extend our language with operators that indicate that something is a *default belief* and thus has a different epistemic status than a *certain fact*. In this way the proverbial "jump to conclusions" is not made directly in the logic, but a somewhat more cautious approach is taken. The "jump" itself will be part of a next operation, the "downward reflection", which will be discussed in the next section when we incorporate a temporal element into our approach.

Let I be a finite set of indexes. The logic **S5P** of epistemic default logic is an extension of the epistemic logic **S5** by means of special modal operators P_i denoting *default belief* (w.r.t. situation or frame of mind i), for $i \in I$, and also generalisations P_τ, for $\tau \subseteq I$.

Informally, $P_i\varphi$ is read as "φ is a default belief (within frame of reference i)". As we shall see below, a frame of reference (or mind) refers to a preferred subset of the whole set S of epistemic alternatives. This operator is very close to the PA (possible assumption) of [TT91] and the D (default) operator of [Doh91]. The generalisation $P_\tau\varphi$ is then read as a default belief with respect to the (intersection of the) frames of reference occurring in τ.

Formally, S5P-formulas are interpreted on Kripke-structures (called S5P-models) of the form $\mathbf{M} = \langle M, \{M_i \mid i \in I\}, \{R_i \mid i \in I\}\rangle$, where M is a collection of worlds, $M_i \subseteq M$ ($i \in I$) are sets ('frames') of preferred worlds, and $R_i = M \times M_i$ ($i \in I$). When writing $\mathbf{M}_1 \subseteq \mathbf{M}_2$, we always mean that the set of worlds of $\mathbf{M}_1$ is a subset of those of $\mathbf{M}_2$. Again we have identified worlds s and their truth assignments. We let Mod(**S5P**) denote the collection of Kripke-structures of the above form. Given an S5P-model $\mathbf{M}' = \langle M, \{M_i \mid i \in I\}, \{R_i \mid i \in I\}\rangle$, we call the S5-model $\mathbf{M}$ based on M the *S5-reduct* of $\mathbf{M}'$.

5.1. DEFINITION (interpretation of S5P-formulas). Given a model $\mathbf{M} = \langle M, \{M_i \mid i \in I\}, \{R_i \mid i \in I\}, \rangle$ we extend the truth definition on S5-models by the following clauses for the P_i- and P_τ-operators:

$(\mathbf{M}, m) \vDash P_i\varphi$ iff $(\mathbf{M}, m') \vDash \varphi$ for all m' with $R_i(m, m')$, for $i \in I$;

$(\mathbf{M}, m) \vDash P_\tau\varphi$ iff $(\mathbf{M}, m') \vDash \varphi$ for all m' with $R_\tau(m, m')$, where $R_\tau = \bigcap_{i \in \tau} R_i$ and $\tau \subseteq I$.

Thus the former clause states that $P_i\varphi$ is true if φ is a default belief w.r.t. subframe M_i, whereas the latter says that $P_\tau\varphi$ is true if φ is a default belief w.r.t. the intersection of the subframes M_i, $i \in \tau$. We will sometimes denote this intersection by M_τ. In the clause for P_τ, we assume that, for $\tau = \varnothing$, $\bigcap_{i \in \tau} R_i = M \times M$. So in this special case we get that the P_τ modality coincides with the knowledge operator K. Validity and satisfiability is defined analogously as before.

It is possible to axiomatise (the theory of) Mod(**S5P**) as follows (cf. [MH91a, MH93a]): take the **S5** system for the modality K (and dual M) and use K45 for the P-modalities (both the P_i and the P_τ), together with relating axioms, resulting in the system:

5.2. DEFINITION (system **S5P**). In the following, i ranges over I, and τ over subsets of I. Moreover, $\Box$ is a variable over $\{K, P_i, P_\tau \mid i \in I, \tau \subseteq I\}$; $\heartsuit$ and $\spadesuit$ range over $\{i, \tau \mid i \in I \; \tau \subseteq I\}$.

(B1)   All propositional tautologies;

(B2) $(\Box\varphi \wedge \Box(\varphi \to \psi)) \to \Box\psi$

(B3) $\Box\varphi \to \Box\Box\varphi$

(B4) $\neg\Box\varphi \to \Box\neg\Box\varphi$;

(B5) $K\varphi \to \varphi$;

(B6) $KP_\heartsuit\varphi \leftrightarrow P_\heartsuit\varphi$

(B7) $\neg P_\heartsuit\bot \to (P_\heartsuit P_\spadesuit\varphi \leftrightarrow P_\spadesuit\varphi)$;

(B8) $P_i\varphi \leftrightarrow P_{\{i\}}\varphi$

(B9) $P_\tau\varphi \to P_{\tau'}\varphi$ $\qquad\qquad \tau \subseteq \tau'$

(B10) $P_\varnothing\varphi \leftrightarrow K\varphi$.

(R1) Modus Ponens

(R2) Necessitation for K: $\vdash \varphi \Rightarrow\ \vdash K\varphi$.

Axiom (B1) says that we are dealing with an extension of propositional logic; (B2) says that all the operators K, P_i, and P_τ are 'normal'; (B3 and B4) express that the relations R, R_i and R_τ are transitive and Euclidean, respectively; (B5) says that R is reflexive; (B6) and (B7) help us to get rid of nested modalities: a nested modality is always referring to the frame corresponding to the innermost one. (B8) provides us with a bridge between the modalities with indices from I and those of P(I); it also shows that in fact we could do without the P_i's. (B9) says that if $\tau \subseteq \tau'$, then $R_{\tau'} \subseteq R_\tau$. Finally, (B10) is the syntactical counterpart of our definition that $R_\varnothing$ = R.

We call the resulting system **S5P**. In the sequel we will write $\Gamma \vdash_{\textbf{S5P}} \varphi$ or $\varphi \in$ Th$_{\textbf{S5P}}(\Gamma)$ to indicate that φ is an **S5P**-consequence of Γ. We mean this in the more liberal sense: it is allowed to derive φ from the assertions in Γ by means of the axioms and rules of the system **S5P**, *including the necessitation rule*. (So, in effect we consider the assertions in Γ as additional axioms: $\Gamma \vdash_{\textbf{S5P}} \varphi$ iff $\vdash_{\textbf{S5P}\cup\Gamma} \varphi$.)

As for **S5**, one can again prove a normal form theorem for **S5P** implying that all formulas can be represented equivalently as formulas without nestings.

5.3. THEOREM $\Gamma \vdash_{\textbf{S5P}} \varphi \Leftrightarrow$ (for all $\mathbf{M} \in$ Mod(**S5P**): $\mathbf{M} \vDash \Gamma \Rightarrow \mathbf{M} \vDash \varphi$)

PROOF. In the Appendix.

6. Epistemic Default Logic (EDL)

In the language of **S5P** we express defaults of the form $\varphi : \psi / \chi$ (using Reiter's notation, Cf. [Rei80]) as $\varphi \wedge M\psi \rightarrow P_i\chi$, for some $i \in I$. Here φ, ψ and χ are objective formulas. The reading of such a formula is "if φ is true and ψ is (considered) possible, then χ is preferred (within frame M_i)". Usually we consider cases where ψ is syntactically equal to χ (the so-called *normal* defaults).

By combining the formal apparatus of **S5P** with Halpern & Moses' nonmonotonic epistemic entailment we obtain a framework in which we can perform default reasoning. In this paper we call this framework Epistemic Default Logic (**EDL**).

6.1. DEFINITION (default theory). A default theory Θ is a pair (W, Δ), where W is a finite, consistent set of objective (i.e. non-modal) formulas describing (necessary) facts about the world, and Δ is a finite set of defaults of the form $\varphi \wedge M\psi \rightarrow P_i\chi$, where φ, ψ and χ are again objective formulas, and $i \in I$. The sets W and Δ are to be considered as sets of axioms, and we may apply necessitation to the formulas in them.

In principle, it would be possible to use the index i of the *P*-operator in a completely arbitrary way, to be chosen by the knowledge engineer. However, in order to be able to treat the various defaults in a default theory separately from each other, in the sequel we shall assume all *P*-operators in a default theory to be distinct. This will allow for a generic way to look at the possibilities of combining default beliefs by using the P_τ-operators. So, for example rather than to look at a default theory with $\Delta = \{p \wedge Mq \rightarrow P_1q, r \wedge Ms \rightarrow P_1s\}$, in which for both defaults the same operator P_1 is used, we consider the set $\Delta' = \{p \wedge Mq \rightarrow P_1q, r \wedge Ms \rightarrow P_2s\}$, in which both defaults are represented by means of different operators P_1 and P_2, and consider the combined operator $P_{\{1, 2\}}$.

6.2. DEFINITION (default entailment). Let a default theory $\Theta = (W, \Delta)$ be given. Let W* be the conjunction of the formulas in W. Note that W is a finite set, and moreover that W only consists of objective formulas. Furthermore, let φ be an objective formula. Then we define the *default entailment relation* $\mathrel{|\!\sim}_\Theta$ w.r.t. default theory Θ as follows: (Recall that $\kappa(\varphi)$ stands for the $\mathrel{|\!\sim}$-closure of φ, where $\mathrel{|\!\sim}$ is the nonmonotonic epistemic entailment of section 4.)

$$\varphi \mathrel{|\!\sim}_\Theta \psi \Leftrightarrow_{\text{def}} \kappa(\varphi \wedge W^*) \cup \Delta \vdash_{\textbf{S5P}} \psi.$$

This definition states that, given a default theory Θ, ψ is a default consequence of φ iff ψ follows in the S5P-logic from the defaults together with what is implied by knowing only the conjunction of φ with the background information W.

Notice that the following equivalence holds, which states something about a modularity of the entailment relations.

6.3 PROPOSITION

$$\varphi \mathrel{\ventrightfilled}_\Theta \psi \qquad \Leftrightarrow \qquad \text{there exists a S5-formula } \psi' \in \text{FORM such that}$$
$$\varphi \wedge W^* \mathrel{\ventrightfilled} \psi' \text{ and } \{\psi'\} \cup \Delta \vdash_{\mathbf{S5P}} \psi$$

PROOF. This follows directly from the definition of $\mathrel{\ventrightfilled}_\Theta$ and Proposition 4.4, as is seen as follows. Suppose we have $\varphi \mathrel{\ventrightfilled}_\Theta \psi$, i.e, $\kappa(\varphi \wedge W^*) \cup \Delta \vdash_{\mathbf{S5P}} \psi$. The logic $\mathbf{S5P}$ is compact, hence there are $\alpha_1, \ldots, \alpha_k \in \kappa(\varphi \wedge W^*)$ and $\delta_1, \ldots, \delta_n \in \Delta$ such that $\alpha_1, \ldots, \alpha_k, \delta_1, \ldots, \delta_n \vdash_{\mathbf{S5P}} \psi$. By definition of κ, since for each $i \le k$, $\alpha_i \in \kappa(\varphi \wedge W^*)$ we have $\varphi \wedge W^* \mathrel{\ventrightfilled} \alpha_i$ for all $i \le k$. Let $\psi' = \alpha_1 \wedge \ldots \wedge \alpha_k$ then, by Proposition 4.4, ψ' satisfies the right-hand side of 6.3. $\blacksquare$

Instead of $\mathbf{true} \mathrel{\ventrightfilled}_\Theta \psi$, we simply write $\mathrel{\ventrightfilled}_\Theta \psi$. Note that, for $\Theta = (W, \Delta)$ we have $\varphi \mathrel{\ventrightfilled}_\Theta \psi$ iff $\mathrel{\ventrightfilled}_{\Theta'} \psi$, with $\Theta' = (W \cup \{\varphi\}, \Delta)$. Furthermore, if Γ is a finite set of epistemic formulas, and ψ an S5P-formula, then we define $\Gamma \mathrel{\ventrightfilled}_\Theta \psi$ as $\Gamma^* \mathrel{\ventrightfilled}_\Theta \psi$, where Γ^* stands for the conjunction of the formulas in Γ.

6.4. EXAMPLE (Tweety). Consider the following default theory $\Theta = (W, \Delta)$ with $W = \{p \to \neg f\}$ and $\Delta = \{b \wedge Mf \to Pf\}$, representing that penguins do not fly, and that by default birds fly. (For convenience we omit the subscript of the P-operator.) Now consider the following inferences):

(i). $b \mathrel{\ventrightfilled} b \wedge \neg K \neg f$ (1)
   $b \wedge \neg K \neg f \vdash_{\mathbf{S5P}} b \wedge Mf$ (2)
   $b \wedge Mf \vdash_{\mathbf{S5P}} Pf,$ (3)
   i.e., $b \mathrel{\ventrightfilled}_\Theta Pf,$

meaning that from the mere fact that Tweety is a bird, we conclude that Tweety is assumed to fly. (Here $\mathrel{\ventrightfilled}$ stands for Halpern & Moses' nonmonotonic epistemic entailment.) This must be contrasted to the inference:

(ii). $b \wedge p \mathrel{\ventrightfilled} Kp \vdash_{\mathbf{S5P}} K \neg f$ (1)
   $K \neg f \vdash_{\mathbf{S5P}} \neg Mf$ (2)
   $\neg Mf \nvdash_{\mathbf{S5P}} Pf,$ (3)
   i.e, $\mathbf{not}\ b \wedge p \mathrel{\ventrightfilled}_\Theta Pf,$

meaning that in case Tweety is a penguin, we cannot infer that Tweety is assumed to fly, but instead we can derive that we know for certain that Tweety does not fly: $b \wedge p \mathrel{\ventrightfilled}_\Theta K \neg f$.

7. A temporal formalization of downward reflection

In the previous section it has been described how upward reflection can be formalized by a nonmonotonic inference based on epistemic states, and the meta-level process by a (monotonic) epistemic logic. In the current section we will introduce a formalization of the downward reflection step in the reasoning pattern. The meta-level reasoning can be viewed as the part of the reasoning pattern where it is determined what the possibilities are for default assumptions to be made, based on which information is available and (especially) which is not. The outcome at the meta-level concerns conclusions of the form $P\varphi$, where φ is an object-level formula. What is still missing is the step where the default assumptions are actually made, i.e., where such formulas φ are added to the object level knowledge, in order to be able to reason further with them at the object level. Here we actually "jump (down) to conclusions". This is what should be achieved by the downward reflection step. Thus the reasoning pattern as a whole consists of a process of generating possible default assumptions and actually assuming them.

By these downward reflections at the object level a hypothetical world description is created and refined. This means that in principle not all knowledge available at the object level can be derived from the object level theory W: downward reflection creates an essential extension to the object level theory. Therefore it is excluded to model downward reflection according to reflection rules as sometimes can be found in the literature, e.g., "If at the meta-level it is provable that Provable(φ) then at the object level φ is provable" (e.g., see [Wey80]):

$$\frac{MT \vdash \text{Provable}(\varphi)}{OT \vdash \varphi}$$

A reflection rule like this can only be used in a correct manner if the meta-theory about provability gives a sincere axiomatisation of the object level proof system, and in that case by downward reflection nothing can be added to the object level that was not already derivable from the object level theory. In the above rule the meta-theory is only concerned with one fixed object level theory. Since we essentially extend the object level theory, and consequently want to move to another object level theory, an approach as taken in the rule above cannot serve our purposes here.

In fact, in our case a line of reasoning at the object level is modelled by inferences from *subsequently chosen theories* instead of inferences from one fixed theory, where in the choice of the next object-level theory the upward and downward reflections play a role. Another difference with the above "traditional" approach to meta-level reasoning is that we shall treat these reflection steps model-theoretically instead of proof-theoretically, as shifts from one

model to another, which results in a *temporal* "super"-model indicating how these shifts may take place over time. Actually, to keep in perfect line with the above, we should explain by means of these models how object knowledge (modelled by a set M of valuations) is reflected upwards to get a model for epistemic meta-knowledge (modelled by an S5-model **M**), how this meta-knowledge is then extended with meta-knowledge about default beliefs (modelled by S5P-models), and how, subsequently, this meta-knowledge is reflected downwards to object knowledge (modelled by a new set M' of valuations) again. However, representing such shifts in a temporal model would involve (sequences of) three different models (sets of valuations, S5-models and S5P-models), which would be quite involved and cumbersome. Therefore, in order not to complicate our temporal models too much, in our present setting we shall not represent the above three steps separately, but treat them "in one blow", so to speak, i.e., as one "super"-step. Now we are able to represent these "super"-steps as shifts from S5P-models to S5P-models, so that our temporal "super"-model only involves sequences of one kind of model, viz. S5P-models, of which the S5-reducts represent the reasoner's objective knowledge and epistemic meta-knowledge, and the frames ('P-') parts represent meta-knowledge in the form of default (or preferred) beliefs. We shall still refer to these supersteps as *downward reflection*, although as we stated above, apart from a reflection of the "meta-knowledge" in the form of default beliefs contained in the S5P-model at hand to the object level by converting some of these beliefs into object knowledge, it also includes an immediate upward reflection from the next object level theory (or rather model) resulting from this to the next meta-level theory as represented by an S5P-model again.

In [GTG93] such a shift between theories is formalized by using an explicit parameter referring to the specific theory (called 'context' in their terms) that is concerned, and by specifying relations between theories. In their case reflection rules ('bridge rules' in their terms) may have the form:

$$\frac{MT \vdash Provable(OT', \varphi)}{OT' \vdash \varphi}$$

Here it is assumed that at the meta-level, knowledge is available to derive conclusions about provability relations concerning a variety of object level theories OT. So, if at the object level from a (current) theory OT some conclusions have been derived, and these conclusions have been transformed to the meta-level, then the meta-level may derive conclusions about provability from another object level theory OT'. Subsequently one can continue the object level reasoning from this new object level theory OT'. The shift from OT to OT' is introduced by use of the above reflection rule.

As we said above, in the approach as adopted here we give a *temporal* interpretation to these shifts between theories. This can be accomplished by formalizing downward reflection by *temporal logic* (as in [Tre94]). In a simplified case, where no branching is taken into account, the following temporal axioms can be used to formalize downward reflection:

$$P\varphi \to X\varphi$$

for every objective formula φ. Here X is the temporal operator asserting that in the next (epistemic) state its argument is true.

In the general case we want to take into account branching and the role to be played by an index τ in $P_\tau\varphi$. We will use this index τ to label branches in the set of time points. By combining EDL with the temporal logic obtained in this manner we obtain a formalization of the whole reasoning pattern.

We start (following [FG92]) by defining the temporalized models associated to any class of models and apply it to the classes of models as previously discussed.

In contrast to the reference as mentioned we use labeled flows of time. We use one fixed set L of labels, viz. $L = 2^I$, the powerset of the index set I. However, in most definitions we do not use this fact, but only refer to (elements τ of) L.

7.1 Flows of time

7.1. DEFINITION (discrete labeled flow of time).

Suppose L is a set of labels. A *(discrete) labeled flow of time* (or *lft*) , labeled by L is a pair $\mathbf{T} =$ (T, $(<_\tau)_{\tau \in L}$) consisting of a nonempty (countable) set T of time points and a collection of binary relations $<_\tau$ on T. Here for s, t in T and τ in L the expression s $<_\tau$ t denotes that t is a (immediate) *successor* of s with respect to an arc labeled by τ. Sometimes it is convenient to leave the indices out of consideration and use just the binary relation s $<$ t denoting that s $<_\tau$ t for some τ (for some label τ they are connected). Thus we have that $< = \cup_{\tau \in L} <_\tau$. We also use the (nonreflexive) transitive closure « of this binary relation: « $= <^+$.

We will make additional assumptions on the flow of time; for instance that it describes a discrete tree structure, with one root and in which time branches in the direction of the future.

7.2. DEFINITION (labeled time tree)

An lft $\mathbf{T} = $ (T, $(<_\tau)_{\tau \in L}$) is called a *labeled time tree* (*ltt*) if the following additional conditions are satisfied (recall that $< = \cup_\tau <_\tau$):

(i) the graph (T, $<$) is a directed rooted tree.

(ii) *(successor existence)*

        Every time point has at least one successor:

        for all $s \in T$ there exists a τ and a $t \in T$ such that $s <_\tau t$

(iii) *(label-deterministic)*

        For every label τ there is at most one τ-successor:

        for all $s, t, t' \in T$ it holds: $s <_\tau t$, $s <_\tau t' \Rightarrow t = t'$

There are still some additional properties that sometimes are required.

7.3. DEFINITION (sub-lft and (maximal) branch)

a)   An lft $(T', (<'_\tau)_{\tau \in L})$ is called a *sub-lft* of an lft $(T, (<_\tau)_{\tau \in L})$ if $T' \subseteq T$ and for all τ it holds $<'_\tau = <_\tau \cap T' \times T'$. It is also called the sub-lft of $(T, (<_\tau)_{\tau \in L})$ *defined by* T', or the *restriction of* $(T, (<_\tau)_{\tau \in L})$ to T'.

b)   A *branch* in an lft $\mathbf{T}$ is a sub-lft $\mathbf{B} = (T', (<'_\tau)_{\tau \in L})$ of $\mathbf{T}$ such that:

    (i) $\ll' = \ll \cap T' \times T'$ is a total ordering on $T' \times T'$

    (ii)   Every t' in T' with a $<$-successor in T also has a $<$-successor in T':

        for all $t' \in T'$, $t \in T : t' < t \Rightarrow$ there is a $s' \in T' : t' < s'$

    (iii)   Every element of T that is in between elements of T' is itself in T':

        for all $s' \in T'$, $t \in T$, $u' \in T' : s' \ll t \ll u' \Rightarrow t \in T'$

c)   A branch is called *maximal* if every t' in T' with a τ-predecessor in T also has a τ'-predecessor in T' for some τ': for all $t \in T$, $t' \in T' : t <_\tau t'$, there is a τ' and an $s' \in T' : s' <_{\tau'} t'$. (Note that if T is a labeled time tree, this condition can be simplified to: the root r of T is also the root of T'.)

We now immediately have the following:

7.4. PROPOSITION Any branch of an ltt $\mathbf{T}$ is an ltt.

7.5. DEFINITION

a) An lft is called *successor-complete* if for every label τ every time point has at least one τ-successor: for all s and τ there exists a t such that $s <_\tau t$.

b) A *path* is a finite sequence of successors: $s_0, ..., s_n$ such that: $s_i < s_{i+1}$ for all $0 \le i \le n-1$. We call s_0 the starting point and s_n the end point of the path.

7.6. DEFINITION (time stamps). Given an ltt $(T, (<_\tau)_{\tau \in L})$, a mapping $|\cdot| : T \to \mathbb{N}$ is called a *time stamp mapping* if for the root r it holds that $|r| = 0$, and for all time points s, t it holds $s < t \Rightarrow |t| = |s| + 1$. (Note that this time stamp mapping is unique.)

Note that an ltt is *infinitely deep*, i.e., for every $k \in \mathbb{N}$ there is a time point $t \in T$ with $|t| = k$. This is a direct consequence of the following proposition:

7.7. PROPOSITION

If $\mathbb{B}$ is a maximal branch in an ltt, then any time stamp mapping is an isomorphism between $\mathbb{B}$ and $\mathbb{N}$.

7.2 Temporal models

We first define our temporal formulas:

7.8. DEFINITION (temporal formulas).

a) Given a logic $\mathbf{L}$, *temporal formulas over* (the language of) $\mathbf{L}$ are defined as follows:

   i if φ is a formula of $\mathbf{L}$ then $C\varphi$ is a temporal formula (also called a *temporal atom*)

   ii if φ and ψ are temporal formulas, then so are:

   $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \rightarrow \psi$, $X_{\exists,\tau}\varphi$, $X_{\exists}\varphi$, $X_{\forall,\tau}\varphi$, $X_{\forall}\varphi$, $F_{\exists}\varphi$, $F_{\forall}\varphi$, $G_{\exists}\varphi$, $G_{\forall}\varphi$.

b) Below, our main concern will be temporal formulas over $\mathbf{S5P}$; we will also refer to them as TEDL-formulas (temporal epistemic default logic formulas).

The above temporal operators are fairly standard in branching-time temporal logic: X refers to next-time, F to sometimes in the future and G to always in the future; the subscripts refer to whether one considers only some ($\exists$) possible path (possibly with a fixed label τ) or all ($\forall$) of these. The C-operator is the least standard: it means 'currently'. Usually in temporal logic this operator is not really useful, since it just states that its argument, say φ, holds in the current state, which is normally represented by the formula φ itself, without a C in front. Here, however, we have a logic where we mix temporal and epistemic logic, and the C-operator acts as a kind of separator between the epistemic and temporal part: in $C\varphi$, its argument φ is an epistemic formula, while $C\varphi$ itself is a temporal formula. This also facilitates the semantic definition below.

7.9. DEFINITION (temporal models)

a) Let MOD be a class of models, and $\mathbb{T} = (T, (<_{\tau})_{\tau \in L})$ a labeled flow of time.
   A *temporal* MOD-*model over* $\mathbb{T}$ is a mapping $\_ : T \rightarrow$ MOD. If $\_$ is a temporal MOD-model for any class MOD we call $\_$ a *temporal model*.
   For $t \in T$ we sometimes denote $\_(t)$ (the snapshot at time point t) by $\_t$. The temporal model can alternatively be denoted by $(\_t)_{t \in T}$.

b) If we apply a) to the classes of models Mod($\mathbf{PC}$), Mod($\mathbf{S5}$) and Mod($\mathbf{S5P}$), we call these temporalized models *temporal PC-models*, *temporal S5-models* and *temporal S5P-models over* $\mathbb{T}$, respectively.

c) Given an lft $\mathbb{T}$, the temporal formulas are interpreted on MOD-models over $\mathbb{T}$ as follows:

    i   conjunction and implication are defined as expected; moreover

        $\_\_$, $s \vDash \neg\varphi$ iff not $\_\_$, $s \vDash \varphi$;

   ii The temporal operators are interpreted as follows:

      1) $C\varphi$ means that in the current state φ is true, i.e.

           $\_\_$, $s \vDash C\varphi$ iff $\_\__s \vDash \varphi$

      2) $X_{\exists,\tau}\varphi$ means that φ is true in some τ-successor state i.e.,

           $\_\_$, $s \vDash X_{\exists,\tau}\varphi$ iff there exists a time point t with $s <_\tau t$ such that $\_\_$, $t \vDash \varphi$

      3) $X_\exists\varphi$ means that there is a τ with some τ-successor in which φ is true.

           $\_\_$, $s \vDash X_\exists\varphi$ iff there exists a time point t with $s < t$ such that $\_\_$, $t \vDash \varphi$

      4) $X_{\forall,\tau}\varphi$, meaning that φ is true in all τ-successor states, i.e.,

           $\_\_$, $s \vDash X_{\forall,\tau}\varphi$ iff for all time points t with $s <_\tau t$ it holds $\_\_$, $t \vDash \varphi$

      5) $X_\forall\varphi$ means that φ is true in all immediate successors:

           $\_\_$, $s \vDash X_\forall\varphi$ iff for all time points t with $s < t$ it holds $\_\_$, $t \vDash \varphi$

      6) $F_\exists\varphi$ means that φ is true in some future state, i.e.,

           $\_\_$, $s \vDash F_\exists\varphi$ iff there exists a time point t with $s \ll t$ such that $\_\_$, $t \vDash \varphi$

      7) $F_\forall\varphi$, means that for all future paths there is a time point where φ is true, i.e.,

           $\_\_$, $s \vDash F_\forall\varphi$ iff for all branches $\mathbb{B}$ starting in s there is a t in $\mathbb{B}$ with $\_\_$, $t \vDash \varphi$.

      8) $G_\exists\varphi$ means that φ is true along some future path, i.e.,

           $\_\_$, $s \vDash G_\exists\varphi$ iff there exists a branch $\mathbb{B}$ starting in s with $\_\_$, $t \vDash \varphi$ for all t in $\mathbb{B}$.

      9) $G_\forall\varphi$, means that φ is true all future states i.e.

           $\_\_$, $s \vDash G_\forall\varphi$ iff for all time points t with $s \ll t$ it holds $\_\_$, $t \vDash \varphi$.

Note that the operator C enforces a shift in the evaluation of formulas; taking us from a temporal model $\_\_$ and a time point t to an S5P-model $\_\__t$.

During the reasoning process we assume to gradually extend the information we have at the object level, and consequently to shrink the set of possible worlds by means of reflecting default beliefs downwards to object knowledge. In terms of temporal S5-models we can formulate this property as follows:

7.10. DEFINITION. A temporal S5P-model *obeys downward reflection*, if the following holds for any s and τ :

the frame M_τ in $\_\__s$ is non-empty $\Leftrightarrow$ there is a t with $s <_\tau t$ and for all such t

                                  the set of worlds of $\_\__t$ equals M_τ

The above property expresses that the possible worlds in frame M_τ (representing the P_τ-default beliefs) are taken to be the whole set of possible worlds (representing the objective knowledge) in some (τ-) successor epistemic state (or, rather, S5P-model).

Finally, we are ready to zoom in into the models we like to consider here, the temporal epistemic default logic models.

7.11. DEFINITION (TEDL-models)

A TEDL-model __ is a temporal S5P-model over an lft **T** such that:

1)   **T** is a labeled time tree;

2)   For every time point s, there is exactly one t with $s <_\varnothing t$;

3)   __ obeys downward reflection.

The following notion is a crucial one for the sequel:

7.12. DEFINITION (conservativity and limit models). Let $\mathbf{T} = (T, (<_\tau)_{\tau \in L})$ be a labeled flow of time. A temporal PC-model __ over **T** is *conservative* if for every two time points s and t with s $< t$ it holds that $\_\_(s) \supseteq \_\_(t)$.

Suppose __ is a conservative temporal PC-model. The intersection of the models __(s) for all s in a given (maximal) branch $\mathbf{B} = (T', (<`_\tau)_{\tau \in L})$ of the lft **T** is called the *limit model* of the branch, denoted $\lim_{\mathbf{B}} \_\_$. The set of limit models for all (maximal) branches is called the *set of limit models* of __. These definitions straightforwardly extend to S5- and S5P-models, by identifying **M** with its set of worlds, M.

7.13. THEOREM. TEDL-models are conservative.

PROOF. Suppose s $< t$, that is, for some label τ, $s <_\tau t$. Since TEDL-models obey downward reflection and the underlying lft is an ltt, this means that *the* (now unique) set of worlds M_t of $\_\_t$ is the frame M_τ of $\_\_s$. Since $M_\tau \subseteq M_s$, the set of worlds of $\_\_s$, this gives $\_\_t \subseteq \_\_s$, which proves the theorem. ∎

Since we know that TEDL-models are conservative, we have that once we have obtained that somewhere in a time point s along some path of such a model some objective formula is known, it remains known in all successor points of s. (This is in fact the rational behind the name "conservative models".) We can make this more precise if we introduce so-called *cko-formulas*, which are formulas of the form $CK\varphi$, where φ is objective. In the sequel we will denote cko-formulas by α. Then we have that:

7.14. PROPOSITION. Let __ be a TEDL-model, and α a cko-formula.

If, for some s, $\_\_, s \vDash \alpha$, then $\_\_, t \vDash \alpha$ for all t with s « t.

PROOF. Suppose $\_\_, s \vDash \alpha$, for $\alpha = CK\varphi$. We show that for t with s $< t$ we have that $\_\_, t \vDash \alpha$. (Then we can finish the proof by induction on the number of time point between s and t.) By Theorem 7.13, we have that $\_\_t \subseteq \_\_s$. This yields: $\_\_, s \vDash \alpha \Leftrightarrow \_\_, s \vDash CK\varphi \Leftrightarrow \_\_s \vDash$

$K\varphi \Leftrightarrow$ ___$_s$, $m \vDash \varphi$ for all m in ___$_s$. Since φ is objective and ___$_t \subseteq$ ___$_s$ we then must have $\mathbf{M}_t$, $m \vDash \varphi$ for all m in ___$_t$ and hence ___$_t \vDash K\varphi$. Finally, we have the equivalences ___$_t \vDash K\varphi \Leftrightarrow$ ___, $t \vDash CK\varphi \Leftrightarrow$ ___, $s \vDash \alpha$. ∎

Furthermore, TEDL-models enjoy a number of properties that can be expressed as validities in our logic:

7.15. THEOREM. TEDL-models satisfy the following validities:

T0   All the operators of $\{X_{\forall,\tau}, X_\forall, F_\forall, G_\forall\}$ satisfy the K-axiom (C too) and generalisation;

T1 $\vdash_{\mathbf{S5P}} \varphi \Rightarrow \vdash_{\mathrm{TEDL}} C\varphi$ (introduction of C)

T2 $\neg X_\forall \bot$ (successor existence)

T3 $X_{\exists,\tau}\varphi \to X_{\forall,\tau}\varphi$ (label-deterministic)

T3' $X_{\exists,\varnothing}\varphi \leftrightarrow X_{\forall,\varnothing}\varphi$ ($\varnothing$-successor existence & label-deterministic)

T4 $X_{\forall,\tau}\varphi \leftrightarrow \neg X_{\exists,\tau}\neg\varphi$ (duality)

T5 $X_\forall\varphi \leftrightarrow \neg X_\exists\neg\varphi$ (duality)

T6 $X_\forall\varphi \leftrightarrow \bigwedge_{\tau \subseteq I} X_{\forall,\tau}\varphi$ ($<$ is union of $<_\tau$)

T7 $X_\exists\varphi \leftrightarrow \bigvee_{\tau \subseteq I} X_{\exists,\tau}\varphi$ (dual of T6)

T8 $C(\neg P_\tau \bot \wedge P_\tau\varphi) \leftrightarrow X_{\exists,\tau}CK\varphi$, if φ is objective (allowing downward reflection)

T8' $CK\varphi \leftrightarrow X_{\exists,\varnothing}CK\varphi$, if φ is objective (trivial downward reflection)

T9 $(C\varphi \to X_\forall C\varphi) \wedge (CK\varphi \to X_\forall CK\varphi)$, if φ is objective (conservativity)

T10 $G_\forall\varphi \to X_\forall\varphi$ ($< \subseteq \ll$)

T11 $G_\forall\varphi \to X_\forall G_\forall\varphi$ (since $\ll$ is transitive closure of $<$)

T12 $G_\forall(\varphi \to X_\forall\varphi) \to (X_\forall\varphi \to G_\forall\varphi)$ (induction)

T13 $CK\varphi \to G_\forall CK\varphi$ (from conservativity and induction)

7.17. REMARK. The Theorem above says that the formulas T1 - T13 are at least *sound*; until now we have not been concerned by designing a logic that is complete for TEDL-models.

8. TEDL models of default theories and entailment relations

We could formulate definitions of entailment of objective formulae related to any model, or any model based on the standard tree. But it may well happen that there are branches in such models, for instance labeled by the empty set only, that contain no additional information as compared to the background knowledge. It is not realistic to base entailment on such informationally poor branches in a model. Therefore we define:

8.1. DEFINITION (informationally maximal)
We define for TEDL-models ___$_1$ and ___$_2$ over the same flow of time that ___$_2$ is *informationally larger* than ___$_1$, ___$_1 \leq$ ___$_2$, if for all t it holds ___$_2(t) \subseteq$ ___$_1(t)$.

We call $\_\__1$ *informationally maximal* in a class M of models if it is itself the only model that is informationally larger.

For a given model $\_\_$ we will apply this definition to the set B of all maximal branches, with $\mathbf{T} = (T, (<_\tau)_{\tau \in L})$ as flow of time.

8.2. DEFINITION (regular model)
A TEDL-model $\_\_$ is called *regular* if all maximal branches are informationally maximal in the class of maximal branches. The submodel based on all time points t included in at least one maximal, informationally maximal branch is called the *regular core* of $\_\_$, denoted by $\mathrm{reg}(\_\_)$.

8.3. REMARK.
In general the regular core will not be label-complete, because branches may be cut off. The idea behind taking informationally complete branches is that we want to maximalise the effect of applying defaults in order to obtain as much (default) knowledge as possible.

8.4. DEFINITION.
Let $\_\_$ be a TEDL-model over $\mathbf{T} = (T, (<_\tau)_{\tau \in L})$.
We define for $k \in \mathbf{N}$
$$\_\_^{(k)} = \bigcup\nolimits_{t \in \mathrm{reg}(\_\_),\, |t| = k} \_\_'(t),$$
$$\_\_^{\omega} = \bigcap\nolimits_{k \in \mathbf{N}} \_\_^{(k)},$$
where $\_\_'(t)$ stands for the S5-reduct of the S5P-model $\_\_(t)$.

8.5. PROPOSITION.
Let $\_\_$ be a TEDL-model.
a) For $k \leq k'$ it holds $\_\_^{(k')} \subseteq \_\_^{(k)}$
b) $\_\_^{\omega} = \bigcup\nolimits_{\mathbf{B} \text{ maximal branch of } \mathrm{reg}(\_\_)} \lim_{\mathbf{B}} \_\_$
PROOF. a) follows directly from Theorem 7.13, b) follows from the distributivity of intersection over union (and vice versa). ∎

8.6. DEFINITION (sceptical entailment)
Let $\_\_$ be a TEDL-model and α a cko-formula. We define the *sceptical entailment relation* by:
$\_\_ \approx_{\mathrm{scep}} \alpha \Leftrightarrow$ for every maximal branch $\mathbf{B}$ in $\mathrm{reg}(\_\_)$ there is a t in $\mathbf{B}$ such that $\_\_, t \vDash \alpha$.

An immediate consequence of this definition is:

8.7. PROPOSITION.
If, for some $k \in \mathbf{N}$, $\_\_^{(k)} \vDash K\varphi$, then $\_\_ \approx_{\mathrm{scep}} CK\varphi$.

PROOF. Suppose $\underline{\quad}^{(k)} \vDash K\varphi$, for some $k \in \mathbf{N}$. This means that $\underline{\quad}'(t) \vDash K\varphi$ for all S5-reducts $\underline{\quad}'(t)$ with t in reg($\underline{\quad}$) and $|t| = k$. So for every maximal branch $\mathbf{B}$ in reg($\underline{\quad}$) we can find a t in $\mathbf{B}$ such that $\underline{\quad}, t \vDash K\varphi$ (viz. take t in $\mathbf{B}$ such that $|t| = k$. By Proposition 7.7 we can always do this). Hence, $\underline{\quad} \mathrel{\vDash}_{\text{scep}} CK\varphi$. ∎

8.8. PROPOSITION.

Let $\underline{\quad}$ be a TEDL-model with root r. Let $\alpha = CK\varphi$ be a cko-formula.

The following are equivalent:

(i) $\underline{\quad} \mathrel{\vDash}_{\text{scep}} \alpha$

(ii)   reg($\underline{\quad}$), $r \vDash F_\forall \alpha$

(iii) $\underline{\quad}^\omega \vDash K\varphi$

(iv) $\lim_{\mathbf{B}} \underline{\quad} \vDash K\varphi$ for every maximal branch $\mathbf{B}$ of the regular core of $\underline{\quad}$.

PROOF. Clearly (i) and (ii) are equivalent. By Prop. 8.5 also (iii) and (iv) are equivalent. We now show the equivalence of (i) and (iv). First we prove (i) $\Rightarrow$ (iv): Suppose $\underline{\quad} \mathrel{\vDash}_{\text{scep}} CK\varphi$, i.e. for every maximal branch $\mathbf{B}$ in reg($\underline{\quad}$) there is a t in $\mathbf{B}$ such that $\underline{\quad}, t \vDash CK\varphi$. Now consider a maximal branch $\mathbf{B}$ in reg($\underline{\quad}$). Then there is a t in $\mathbf{B}$ with $\underline{\quad}, t \vDash CK\varphi$, i.e. $\underline{\quad}(t) \vDash K\varphi$. By the conservativity of $\underline{\quad}$ we have that $\underline{\quad}(u) \vDash K\varphi$ for all $u \gg t$, i.e., all worlds of the models $\underline{\quad}(u)$ with $u \gg t$ satisfy φ. But then also all worlds of the models $\underline{\quad}(u)$ with $u \gg t$ and u in $\mathbf{B}$ satisfy φ. Consequently, $\lim_{\mathbf{B}} \underline{\quad} = \bigcap_{u \text{ in } \mathbf{B}} \underline{\quad}(u) \vDash K\varphi$.

Next we prove (iv) $\Rightarrow$ (i): Suppose $\lim_{\mathbf{B}} \underline{\quad} \vDash K\varphi$ for every maximal branch $\mathbf{B}$ of the regular core of $\underline{\quad}$. We have to prove that for every maximal branch $\mathbf{B}$ in reg($\underline{\quad}$) there is a t in $\mathbf{B}$ such that $\underline{\quad}, t \vDash CK\varphi$. Take some maximal branch $\mathbf{B}$ in reg($\underline{\quad}$). Then we know that $\lim_{\mathbf{B}} \underline{\quad} \vDash K\varphi$, i.e., $\bigcap_{u \text{ in } \mathbf{B}} \underline{\quad}(u) \vDash K\varphi$. By the conservativity of the model $\underline{\quad}$ (Theorem 7.13) we know that the sequence $\langle \underline{\quad}(u) \mid u \text{ in } \mathbf{B}\rangle$ is monotonically decreasing (with respect to $\subseteq$). Let $\mathbf{P}_0 \subseteq \mathbf{P}$ the set of propositional atoms occurring in the formula α (or φ). Clearly, $\mathbf{P}_0$ is finite. Now, for any S5-model $\mathbf{M}$ with set S of worlds (truth assignment functions) we let $\mathbf{M}_0$ denote the model with set S_0 of worlds: $S_0 = \{t \mid t = s \mid_{\mathbf{P}_0} \text{ for some } s \in S\}$. Clearly, $\mathbf{M} \vDash K\varphi$ iff $\mathbf{M}_0 \vDash K\varphi$. Now consider the sequence $\langle \underline{\quad}(u)_0 \mid u \text{ in } \mathbf{B}\rangle$. This is a monotonically decreasing sequence with intersection $\bigcap_{u \text{ in } \mathbf{B}} \underline{\quad}(u)_0$. Since $\mathbf{P}_0$ is finite, and we identified worlds with truth assignment functions, we have that all the models $\underline{\quad}(u)_0$ contain only a *finite* number of worlds. Together with the fact that the sequence is monotonically decreasing this yields that the sequence must be stable from some point s in $\mathbf{B}$ on: $\underline{\quad}(u)_0 = \underline{\quad}(s)_0$ for all $u \gg s$. But then obviously the intersection $\bigcap_{u \text{ in } \mathbf{B}} \underline{\quad}(u)_0 = \underline{\quad}(s)_0$. So now $\underline{\quad}(s)_0 \vDash K\varphi$, and thus $\underline{\quad}(s) \vDash K\varphi$, i.e. $\underline{\quad}, s \vDash CK\varphi$. ∎

For our definition of credulous entailment we can be less restrictive. Especially, too less information in one branch can always be overcome by another, informationally larger branch.

8.9. DEFINITION (credulous entailment)

Let __ be a TEDL-model. We define:

$$\underline{\quad} \approx^{k}_{cred} CK\varphi \qquad \Leftrightarrow \qquad \text{there exists an s with } |s| = k \text{ and } \underline{\quad}(s) \vDash K\varphi$$

$$\underline{\quad} \approx_{cred} CK\varphi \qquad \Leftrightarrow \qquad \text{there exists an s with } \underline{\quad}(s) \vDash K\varphi$$

8.10. PROPOSITION.

Let __ be a TEDL-model, and $\alpha = CK\varphi$ a cko-formula.

The following are equivalent:

(i) $\underline{\quad} \approx_{cred} \alpha$

(ii) $\underline{\quad} \approx^{k}_{cred} \alpha$ for some $k \in \mathbb{N}$.

(iii) $\underline{\quad}, r \vDash F_{\exists}\alpha$

(iv) $\lim_{\mathbf{B}} \underline{\quad} \vDash K\varphi$ for some maximal branch $\mathbf{B}$

PROOF. Since clearly (i), (ii) and (iii) are equivalent, we concentrate on the equivalence of (ii) and (iv): we first prove (ii) $\Rightarrow$ (iv): Let $\underline{\quad} \approx^{k}_{cred} CK\varphi$ for some $k \in \mathbb{N}$. This means that there exists an s with $|s| = k$ and $\underline{\quad}(s) \vDash K\varphi$, i.e., $\underline{\quad}, s \vDash CK\varphi$. By Prop. 7.14, we then have that $\underline{\quad}, t \vDash CK\varphi$ for all t with $s \ll t$. Now consider any maximal branch $\mathbf{B}$ through s. Clearly, this branch has the property that the models $\underline{\quad}(t)$ with $s \ll t$ all satisfy $K\varphi$, i.e., φ holds in every world of every model $\underline{\quad}(t)$. But then φ holds in the intersection of these models as well. This proves $\lim_{\mathbf{B}} \underline{\quad} \vDash K\varphi$.

Next we prove (iv) $\Rightarrow$ (ii): Suppose that $\lim_{\mathbf{B}} \underline{\quad} \vDash K\varphi$ for some maximal branch $\mathbf{B}$. This means that $K\varphi$ holds in the intersection of the models $\underline{\quad}(t)$ along $\mathbf{B}$. By the conservativity of the model __ (Theorem 7.13) we know that the sequence $< \underline{\quad}(t) \mid t$ in $\mathbf{B}>$ is monotonically decreasing (with respect to $\subseteq$). Let $\mathbf{P}_0 \subseteq \mathbf{P}$ the set of propositional atoms occurring in the formula α (or φ). Clearly, $\mathbf{P}_0$ is finite. As above, we denote, for any S5-model $\mathbf{M}$ with set S of worlds (truth assignment functions), the model with set S_0 of worlds by $\mathbf{M}_0$ where $S_0 = \{t \mid t = s \mid_{\mathbf{P}_0} \text{ for some } s \in S\}$. Again, $\mathbf{M} \vDash K\varphi$ iff $\mathbf{M}_0 \vDash K\varphi$. Now consider the sequence $< \underline{\quad}(u)_0 \mid u$ in $\mathbf{B}>$. This is a monotonically decreasing sequence with intersection $\bigcap_{u \text{ in } \mathbf{B}} \underline{\quad}(u)_0$. Since $\mathbf{P}_0$ is finite, and we identified worlds with truth assignment functions, we have that all the models $\underline{\quad}(t)_0$ contain only a finite number of worlds. Together with the fact that the sequence is monotonically decreasing this yields that the sequence must be stable from some point s on: $\underline{\quad}(t)_0 = \underline{\quad}(s)_0$ for all $t \gg s$. But then obviously the intersection $\bigcap_{t \text{ in } \mathbf{B}} \underline{\quad}(t)_0 = \underline{\quad}(s)_0$. Since the intersection satisfies $K\varphi$, we obtain that $\underline{\quad}(s)_0 \vDash K\varphi$, and thus $\underline{\quad}(s) \vDash K\varphi$. Hence $\underline{\quad} \approx_{cred} CK\varphi$. ∎

We can now associate TEDL-models with default theories as follows.

8.11. DEFINITION (TEDL-model of a default theory).

Let $\Theta = (W, \Delta)$ be a default theory. Then we define a TEDL-model of Θ as a TEDL-model $\_\_^\Theta$ such that:

(i) (basis: the root) $\_\_^\Theta_r$ is an S5P-model such that (a) the S5-reduct of $\_\_^\Theta_r$ is the S5-model $\mathbf{M}_{W^*}$, as defined in Section 4, and (b) $\_\_^\Theta_r$ satisfies the set of defaults, i.e., $\_\_^\Theta_r \vDash \Delta$.

(ii) (induction step) Suppose that we are given an S5P-model at snapshot $\_\_^\Theta_s$. Then we have that for a(n S5P-) model $\_\_^\Theta_t$ with $s <_\tau t$, it holds that: (a) the S5-reduct of $\_\_^\Theta_t$ is the S5-model $\mathbf{M}_\tau$ as it appeared as a frame in $\_\_^\Theta_s$, and (b) $\_\_^\Theta_t$ satisfies the set of defaults again, i.e., $\_\_^\Theta_t \vDash \Delta$.

Note that, in general, there are multiple TEDL-models of a default theory Θ. Furthermore, note that clause (ii)(a) reflects the downward reflection operation with respect to the P_τ-defaults.

This definition enables us to finally give the definitions of sceptical and credulous entailment from a default theory.

8.12. DEFINITION (entailment from a default theory).
Let $\Theta = (W, \Delta)$ be a default theory, and φ an objective formula. Then:

$\Theta \approx_{scep} \varphi$ iff   for all models $\_\_$ of Θ it holds $\_\_ \approx_{scep} CK\varphi$

$\Theta \approx_{cred} \varphi$ iff   for all models $\_\_$ of Θ it holds $\_\_ \approx_{cred} CK\varphi$

Of course, we have that what is entailed sceptically, also is entailed credulously:

8.13. PROPOSITION. For default theory Θ and objective formula φ: $\Theta \approx_{scep} \varphi \Rightarrow \Theta \approx_{cred} \varphi$

Sceptical and credulous entailment are not the only approaches to define semantics for default logic. Sceptical entailment can be considered a lower bound of the conclusions that can be drawn, credulous entailment as an upper bound. A more differentiated approach bases semantics of the default theory on the set of all alternative extensions that actually can be constructed out of a default theory using the defaults as (non-monotonic, context-dependent) inference rules. The original approach by Reiter [Rei80] defines the extensions of a default theory by means of a pseudo-construction or fixed point definition; we will call these extensions *Reiter extensions*.

In Constructive Default Logic, introduced in [TT92], the notion of Reiter extension is replaced by the weaker notion of *constructive extension*. In [TT92] it was proven that the Reiter extensions of a default theory are among the constructive extensions and they were characterised. Constructive extensions give a broader perspective on the semantics of a default theory. The formalisation in terms of branching time models introduced in the current paper, in

the first place is the temporal formalisation of the construction process of a constructive extension. Just like in the case of Constructive Default Logic, in the applicability condition of a default $\varphi \wedge M\psi \rightarrow P_i\chi$ at time point s, the justification $M\psi$ only refers to the current state of information (and not to the whole (future) context of the reasoning pattern as in Reiter's case is required). Because of this similarity for every (maximal) branch __ in a TEDL-model __ of a default theory $\Theta = (W, \Delta)$, the theory of its limit $Th(\lim\_\_\_)$ is a constructive extension of Θ and the branch itself represents a construction chain of this constructive extension.

As the Reiter extensions are among the constructive extensions, a natural question is how in a TEDL-model of a default theory the branches that lead to a Reiter extension can be characterised. Based on variants of the notions self-defeatingness and exhaustiveness introduced in [TT92], it is not difficult to find such a characterisation.

8.14. DEFINITION
Let __ be a TEDL-model of the default theory $\Theta = (W, \Delta)$ and __ a branch in __.
The branch __ is called *self-defeating* if there is a time point s in __ and a default
$\varphi \wedge M\psi \rightarrow P_i\chi$ in D such that the following holds:
for all time points t, t' in __ and labels τ with $t <_\tau t'$ and every default $\varphi' \wedge M\psi' \rightarrow P_j\chi$ in D with j in τ, there is some time point u in __ with $\_\_u \models \neg\psi'$.
Otherwise the branch __ is called *non-self-defeating*.

The notion of non-self-defeatingness quarantees that any default conclusion that is drawn will remain justified in the strong Reiter sense during the whole process. However, this is only one side of the characterisation. It may well be the case that the defaults applied in branch __ are justified in the strong sense, but not all applicable defaults actually have been applied in __. Therefore the notion of exhaustiveness is required:

8.15. DEFINITION
Let __ be a TEDL-model of the default theory $\Theta = (W, \Delta)$ and __ a branch in __.
The branch __ is called *exhaustive* if for every time point s in __ and every default
$\varphi \wedge M\psi \rightarrow P_i\chi$ in D such that $\_\_s \models \varphi$ and for all time points t in __ it holds $\_\_t \not\models \neg\psi$,
then there is a time point s' in __ and a label τ with $s <_\tau s'$ and i is in τ.

Based on these two conditions we can characterise the branches leading to Reiter extensions:

8.16. THEOREM
Let __ be a TEDL-model of the default theory $\Theta = (W, \Delta)$. The following are equivalent:
(i) E is a Reiter extension of Θ
(ii) there exists a non-self-defeating and exhaustive branch __ in __ with $E = Th(\lim\_\_\_)$.

The proof is similar to the proof of the main result in [TT92]. The following example default theory illustrates our approach:
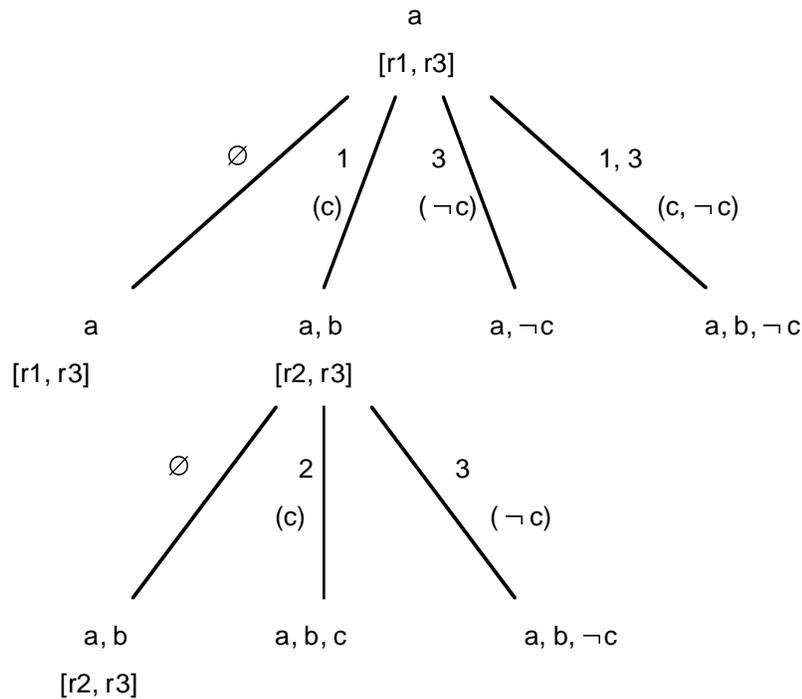
W

    a

D

    r1: $a \wedge Mc \rightarrow P_1 b$
    r2: $b \wedge Mc \rightarrow P_2 c$
    r3: $a \wedge M\neg c \rightarrow P_3 \neg c$

In Figure 1 the different reasoning paths are depicted. At each node the literals that are true at the object level are depicted, and, between [..], the applicable default rules. At each arc the label is depicted and, between (..), the justification it is based on.

Figure 1   **Reasoning paths for an example default theory**

The branch leading to a, b, c is non-self defeating and exhaustive, so it leads to the Reiter extension Cn({a, b, c}). The same holds for the branch leading to a, ¬ c. The two branches leading to a, b, ¬c, however, are self-defeating: the occurrence of ¬ c leads to the defeat of the earlier justification c. Therefore Cn({a, b, ¬ c}) is a constructive extension, but not a Reiter extension.

9. Conclusions

In [MH93a,b] an Epistemic Default Logic (**EDL**) was introduced inspired by the notion of meta-level architecture that also was the basis for the approach introduced in [TT91]. In EDL drawing a default conclusion has no other semantics than that of adding a modal formula to the meta-level. No downward reflection takes place to be able to reason with the default conclusions at the object level (by means of which default assumptions actually can be made). In [TT91] downward reflection takes place, but no semantical formalization was given: it was defined only in a procedural manner.

In principle downward reflection disturbs the object level semantics, since facts are added that are not logically entailed by the available knowledge. Adding a temporal dimension (in the spirit of [FG92]) to **EDL** enabled us to obtain formal semantics of drawing a default conclusion in a dynamic sense: as a transition from the current object level theory to a next one (where the default conclusion has been added).

The approach as developed provides a formalisation of default reasoning using labeled branching time temporal models. Within these models various types of semantics can be defined: semantics based on sceptical or credulous entailment, and semantics based on constructive extensions or Reiter extensions.

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Acknowledgements

Appendix

In this appendix, we provide a proof for theorem 5.3. First, we introduce an alternative class of Kripke models.

Definition A1. We define the class A(**S5P**) of alternative models for **S5P**. Elements $\mathbf{M}$ of A(**S5P**) are tuples $\mathbf{M} = \langle S, \pi, R_k, R_1, \ldots,$

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